4D inversion of L1 and L2 norm minimizations

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Abstract

A new 4-dimensional (4D) inversion algorithm is developed so that any of data misfits and model roughness in the space and time domains can be selectively minimized either in terms of L1 norm or in L2 norm. This study is motivated by the experiences that a 4D inversion adopting full L2 norm minimization may sometimes result in a model too smoothly varying with time. It is further encouraged by the realization that a particular criterion of either L1 or L2 norm cannot universally be the optimal approach for accurately reconstructing the subsurface condition. Along with this development of the algorithm, we try to overcome the difficulties of jointly choosing two optimal regularization parameters in the space and time domains. To achieve this, we devise automatic determination methods not only of the Lagrangian multipliers for the space-domain smoothness constraint but also of the regularization parameter for penalizing the model roughness along the time axis. Both kinds of the regularization parameters are actively updated as data misfits and model roughness vary at each iteration step. We conducted inversion experiments using synthetic and field monitoring data to test the proposed algorithms and further to compare the performance of L1 norm and L2 norm minimizations. Both of the synthetic and field data experiments proved that the automatic determination method developed in this study is very effective for calculating the ground changes that are closer to the ground truth than the approaches of using pre-determined parameter values. Synthetic data examples showed that L1 norm minimization of the time-domain roughness could cure the problem of unnecessary smooth model changes when the subsurface changes are locally confined, but at the same time, the L2 norm approach would be more reasonable when the changes are expected widespread.

Keywords: 4D, inversion; L1 norm; resistivity monitoring; geophysical monitoring.

Introduction

DC resistivity monitoring has widely been applied to many environmental and engineering problems (e.g., DAILY and RAMIREZ, 1995) and its application has recently been extended to the geological disaster mitigation (e.g., SUPPER et al., 2009). The ground condition changes are quantified and visualized by the difference of a pair of time-lapse images which frequently boosts up the artefacts; these artefacts amplified in the difference images may contribute to a misinterpretation of the ground condition change. KIM et al. (2009) proposed a four dimensional (4D) inversion algorithm where time dimension is included into inversion. The regularization in both the space domain and the time domain effectively reduce inversion artefacts. KARAOULIS at al. (2011) noted that the time regularization sometimes makes the inverted results too smooth in the time domain and proposed the 4D Active Time Constrained (4D-ATC) inversion where the time regularization is allowed to vary depending on the degree of resistivity changes in the space-time domain.

Most of the studies on the inversion of electric monitoring data have adopted the L2 norm minimization of penalty values. It is known that the L2 norm minimization assumes a normal distribution, while a Laplace or exponential distribution is the basic assumption for the L1 norm inversion (MENKE, 1984). Therefore, a particular criterion of either L1 or L2 norm cannot universally be the optimal approach for accurately reconstructing the subsurface condition. We should flexibly select either L1 or L2 norm according to the behaviours of monitoring data and inverse model parameters in the 4D space. To address this, we develop a new inversion algorithm where any of the data misfits and two kinds of model roughness in the space and time domains can be selectively minimized either in terms of L1 norm or in L2 norm. Together with this, we note that it is very difficult to simultaneously determine regularization parameters which optimally control the two smoothness constraints both in the space and time domains. To solve the difficulties, we devise methods to automatically determine the regularization parameters that are actively updated as the data and model roughness vary at each iteration step. The newly developed methods are compared and their performances are demonstrated via synthetic examples as well as field data application for landslide monitoring.

4D inversion algorithm based on either L1 or L2 norm minimization

The 4D inversion defines the many subsurface models sampled in each monitoring time-laps as a single model vector in the space-time domain and the entire monitoring data sets as well:

$$\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_o\} \text{ and} \quad (1a)$$

$$\mathbf{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_i, \ldots, \mathbf{d}_o\}, \quad (1b)$$

where o is the number of time-lapses. Owing to these definitions, the regularizations can be introduced both in the space domain and in the time domain. Accordingly, the objective function is expressed as

$$\Phi(\mathbf{P}+\Delta\mathbf{P}) = \Xi\,(\mathbf{P}+\Delta\mathbf{P}) + \Psi\,(\mathbf{P}+\Delta\mathbf{P}) + \Gamma\,(\mathbf{P}+\Delta\mathbf{P}). \quad (2)$$

P is a starting model or a model calculated in the previous iteration. $\Delta\mathbf{P}$ is the unknown model perturbation vector, i.e., $\Delta\mathbf{P} = \mathbf{P}_{j+1} - \mathbf{P}_j$, where j is the iteration number. Ξ, Ψ and Γ are the penalty functions of the data misfit, the model roughness in the space and that in the time domains, respectively. Any of these three functions are defined in either L1 or L2 norm which is to be minimized or penalized through an inversion process in a trade-off manner. Each penalty function is a measure of data misfit or model roughness and is quantified as its L1 norm or L2 norm:

$$\Xi(\mathbf{P}) = \left\|\mathbf{W}\mathbf{E}(\mathbf{P})\right\|^p,\ \Psi(\Delta\mathbf{P}) = \left\|\sqrt{\boldsymbol{\Lambda}}\mathbf{C}^s\Delta\mathbf{P}\right\|^p, \text{ and } \Gamma(\mathbf{P}) = \left\|\sqrt{\alpha\mathbf{A}}\mathbf{C}^t\mathbf{P}\right\|^p, \text{ where } p = 1 \text{ or } 2. \quad (3)$$

E is the data misfit between the field and theoretically calculated data. The superscripts t and s imply space and time domains, respectively. **W** is a diagonal matrix of data weighting factors. $\mathbf{C}^s$ is a second order differential operator in the space domain. $\mathbf{C}^t$ is a difference operator to calculate the model roughness in the time domain.

The active constraint balancing (ACB) method (YI et al., 2003) is adopted to balance the smoothness constraints in the space domain; thus, the regularization parameter for controlling the contribution of the roughness term Γ is expressed as a diagonal matrix, $\mathbf{\Lambda} = \mathrm{diag}\ (\lambda_i)$, in the above equations. The model smoothness along the time axis is controlled by another regularization parameter α in the roughness term Ψ. These two different kinds of regularization parameters should optimally be chosen, which will be discussed later.
The time domain model roughness Γ includes a diagonal matrix, **A** = diag (a_i), which is called as the cross-time weighting matrix. It is introduced mainly to alleviate the problem that the 4D inversion of L2 norm minimization may result in an inverse 4D model too smoothly varying in the time domain (KARAOULIS et al., 2011). Another function of the matrix is to reduce the problem that the model parameters having less resolving power are more likely to be contaminated by inversion artefacts. Its diagonal elements (cross-time weighting factors) are automatically calculated so that a lower weighting factor can be assigned to an inverse model parameter which is more quickly changing with time, and vice versa. It should be noted that the average of the upper and lower bounds of the weighting factors is set as one on logarithmic scale. By doing this, the contribution of the time-domain roughness is systematically controlled by adjusting the regularization parameter α.
To numerically implement the L1 norm minimization of a particular penalty term, we adopt an algorithm of the iteratively reweighted least-squares inversion (FARQUHARSON and OLDENBURG, 1998). A merit of the algorithm is to easily implement the L1 norm minimization within the framework of commonly used least-squares inversion. By adopting the reweighting algorithm, the partial derivative of a penalty term with respect to the model perturbation vector is expressed in a similar form of L2 norm minimization but includes additional term of a reweighting matrix, **R** = diag (r_i). For example, the derivative of the L1 norm model roughness in the time domain, i.e., Γ, is expressed as following:

$$\frac{1}{2}\frac{\partial\Gamma}{\partial\Delta\mathbf{P}} \cong (\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)^T\mathbf{R}_t(\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)(\mathbf{P}+\Delta\mathbf{p}) \text{, and} \tag{4}$$

$$r_{i,i}^t = [\sqrt{\alpha a_i}\sum_{k=1}^{m\times o} C_{i,k}^t P_k + \varepsilon^2]^{-1/2}, \tag{5}$$

where the superscript T means the transpose of a matrix and ε is a small constant to avoid the division by zero. Note that the updated model and data misfits are necessary to calculate the reweighting matrices but they are actually not calculated yet. FARQUHARSON and OLDENBURG (1998) once calculated the model using a least-squares inversion and again used it for computing the reweighting matrices as an approximation of the updated model for L1 norm inversion. In this study, we simply use the solution at the previous iteration as equation (5) implies.
In such ways, we have the following normal equation for minimizing the L1 norms of all penalty terms:

$$\begin{aligned}&\{(\mathbf{WJ})^T\mathbf{R}^d\mathbf{WJ} + (\sqrt{\mathbf{\Lambda}}\mathbf{C}^s)^T\mathbf{R}^s(\sqrt{\mathbf{\Lambda}}\mathbf{C}^s) + (\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)^T\mathbf{R}^t(\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)\}\Delta\mathbf{P}\\ &= (\mathbf{WJ})^T\mathbf{R}^d\mathbf{WE} - (\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)^T\mathbf{R}_t(\sqrt{\alpha\mathbf{A}}\mathbf{C}^t)\mathbf{P}\end{aligned}, \tag{6}$$

where **J** is the partial derivatives of the data (apparent resistivity) with respect to the model parameters (resistivity), Jacobian matrix. $\mathbf{R}^d$, $\mathbf{R}^s$ and $\mathbf{R}^t$ are the reweighting matrices for the data misfit functional and the two model roughness in the space and time domains, respectively. Note that we can have a normal equation of a mixed version of L1 and L2 norm minimizations of penalty measures, when a particular reweighting matrix is selectively replaced with a unit matrix. Solving the normal equation iteratively results in the final inverted subsurface model in the space-time domain.

Automatic determination of regularization parameters in the space and time domains

Our inversion algorithm includes two smoothness constraints in the space and time domains. Correspondingly, two different kinds of regularization parameters should optimally be chosen, but it is not easy since two constraints both in the space and time domains would be cross-related in an actual inversion process. Furthermore, the developed algorithm allows us to define the measure of each roughness in either L1 or L2 norm as we want. This in turn gives rise to more difficulties in selecting optimal Lagrangian multiplier values, since it is practically impossible to choose regularization parameter to be universally optimal for any arbitrary combination of penalty measures. These difficulties lead us to devise methods to automatically determine two different regularization parameters. The proposed methods are to calculate the parameter values by using of the relative value of each penalty measure with respect to the data misfit measure as follows.

The space-domain parameters, i.e., $\mathbf{\Lambda}$ = diag (λ_i), are computed so that the space-domain model roughness Ψ will be a constant fraction of the data misfit measure Ξ; the fraction value is given by the user and specified as a percentage with respect to the data misfit value Ξ. This is a practical implementation for maintaining the relative contribution of the space-domain model roughness with respect to the data misfits at a certain fixed level throughout the entire inversion process.

For automatically determining the time-domain parameter, we newly introduce a concept of the time-domain data roughness which is defined using the time differences of data misfits. This is introduced to realize that the amount of data misfits at each time-lapse should be almost constant along the time axis if an inverted 4D model would appropriately mimic the true subsurface model changes. The time domain regularization parameter α is computed to be inversely proportional to the relative value of the newly introduced data roughness with respect to the data misfit value. Actual computation of the parameter value includes the relative amounts of the model roughness in the time domain with respect to that in the space domain as well.

The devised method demands to optimally choose two predetermined proportional constants, but it is much more effective compared to regular approaches using pre-determined parameter values. Once optimal constants are chosen, then the value would be optimal to any combinations of penalty measures, since the penalty values themselves are used to compute the parameter. Furthermore, the regularization parameters are actively computed as the data and model roughness vary at each iteration step.

Numerical example

We conducted numerical experiments of surface resistivity monitoring firstly to verify the performance of the proposed algorithm and secondly to compare the performances of the L1 norm and L2 norm minimizations particularly of the time-domain model roughness. For convenience of discussion, let us denote “L1” and “L2” for the L1 and L2 norm minimization respectively. Similarly, “D”, “S” and “T” denote the data misfit, the model roughness in the space domain and that in the time domain, respectively. For example, L2D_L2S_L1T means the 4D inversion of minimizing the L2 norm of both the data misfit and the model roughness in the space domain, and L1 norm in the time domain. In model studies discussed here, the adopted array was dipole-dipole and there were eight monitoring surveys. Synthetic data were calculated using the 2.5D finite element modeling code. Random electrical noise of 1 mV/A pea-to-peak amplitude was added to the synthetic potential difference data to simulate the field data.

Figure 1 illustrates inversion experiments using a subsurface model where the resistivity changes are spatio-temporally localized as shown in Figure 1a. The automatic determination method always resulted in the difference images much closer to the ground truth than the usual methods of using pre-determined parameter values. Comparing the L2 and L1 norm minimizations of the time-domain roughness, the L1 norm minimization is much superior to the L2 one in this test model experiments. The cross-time weighting positively affected the inversion results when adopting the L2 norm of the time-domain roughness (see Figure 1c and 1d), but negatively when the L1 norm was selected. Applying the cross-time weighting can be regarded as an attempt to partly introduce a L1 norm minimization concept into L2 norm inversion. The experiments using this type model seem to conclude that the L1 norm of the time-domain model roughness would be a way to achieve the best inversion results. We can find a good example of field application in Kim et al. (2010).

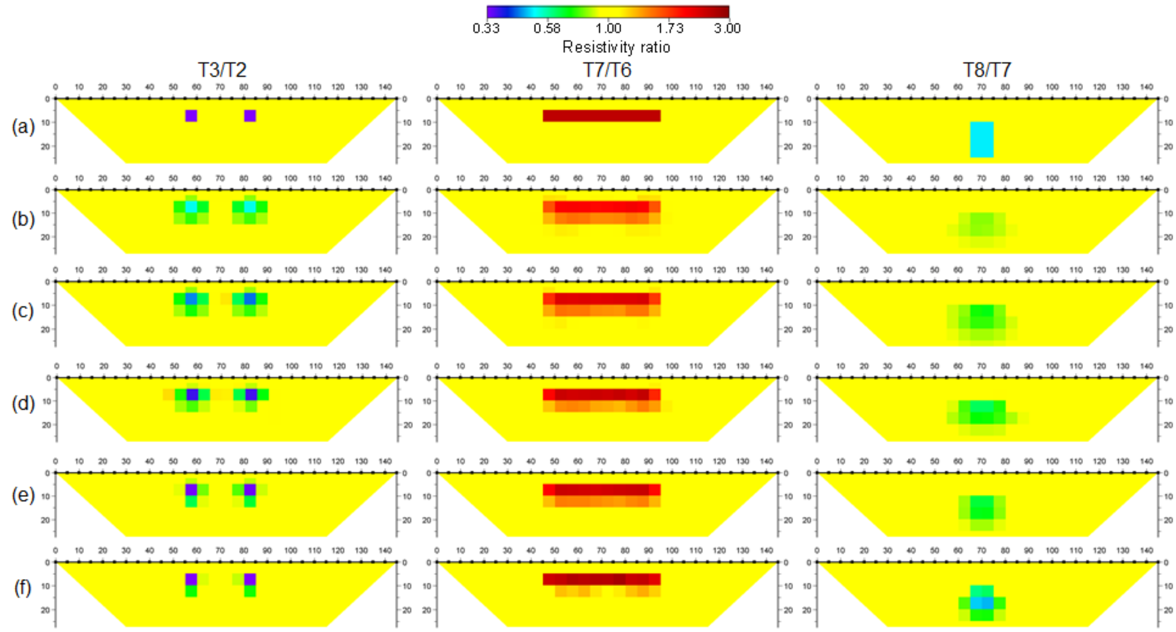

Fig. 1: Inversion experiments based on a model where the resistivity is changing very locally not only in the space domain but also in the time. (a) The true model changes between two sequential time steps. (b)-(d) are the results of L2D_L2S_L1T approach adopting (b) α = 0.1, and the automatic determination of regularization parameters (c) without and (d) with the cross-time weighting. (e) and (f) are those of L2D_L2S_L1T. (e) α = 0.1. (f) Automatic determination of regularization parameters without the cross-time weighting.

Figure 2 shows another test model in the case that the resistivity throughout the entire modeled region is always changing during the whole monitoring period. The characteristics of the assumed resistivity changes are completely different from those of the previous scenario that the resistivity changes are confined only at several particular 4D coordinates. As shown in Figure 3, the reconstructed difference images in this test case are relatively less accurate compared to the previous numerical experiments. In particular, discrepancies from the ground truth are more pronounced in deep depths, which are mainly due to the lower resolving power of deeper region.

The most important conclusion in this test case is that the L2 norm of the time-domain roughness is far better than the L1 norm approach; this is completely opposite to the previous experiments.

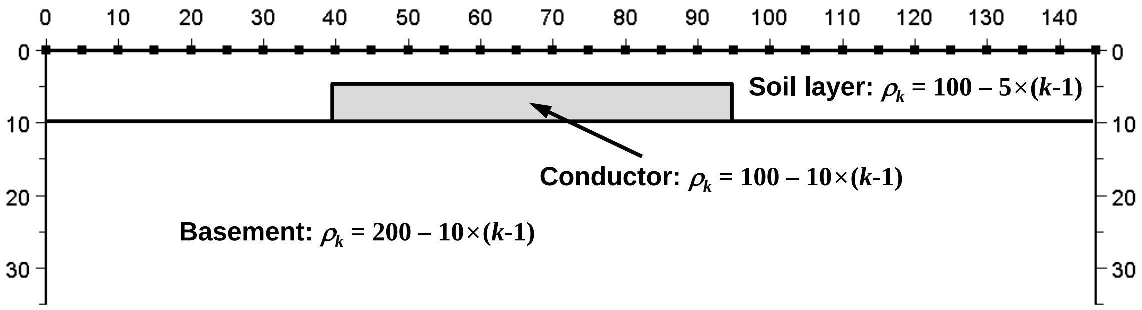

Fig. 2: A test model in which the resistivity throughout the entire modelled region is decreasing with time. ρ_k is the resistivity of each zone in Ωm at the monitoring time step k.

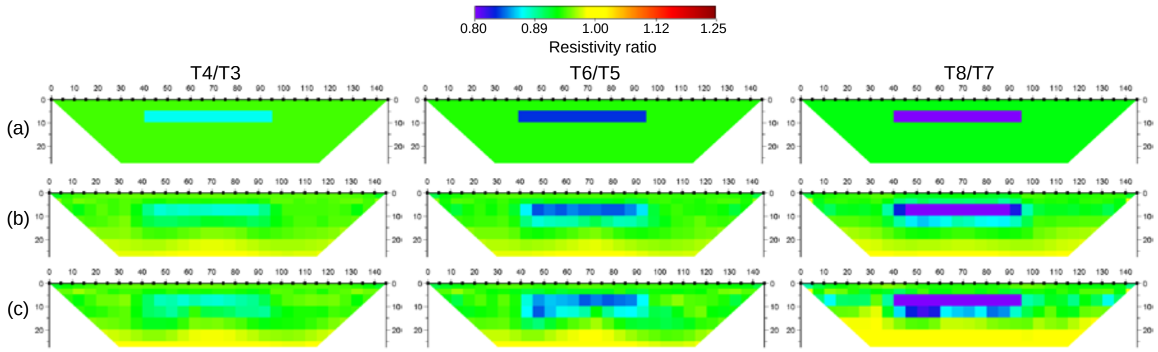

Fig. 3: Inversion experiments based on the test model of Figure 2. (a) The true model changes between sequential time steps. (b) L2D_L2S_L1T. (c) L2D_L2S_L1T. Regularization parameters were automatically determined.

As illustrated in the first model experiments, the problems of too smoothly varying model with time can be solved by minimizing the L1 norm of the time-domain roughness. However, numerous numerical experiments, for instance, the above two synthetic examples, led us not to conclude that a particular norm (L1 or L2) inversion would be the best choice for inverting monitoring data in a 4D inversion manner. Either L1 or L2 norm criterion should be selected through the careful consideration of the behavior of data and inverse model parameters in the 4D space. Particularly for the minimization of the time-domain model roughness, the L1 norm would be better when the subsurface changes are locally confined, while the L2 norm approach would be more reasonable when the changes are expected widespread.

Field application: landslide monitoring

Austrian Geological Survey has been operating six test sites for the monitoring of landslides for the TEMPEL project. DC resistivity data observed at the Bagnaschino site in the north-western Italy were chosen for the field application test of the developed algorithms. At the test site, significant displacements up to about 100 mm were recorded during the period of 15 to 18 March 2011, and we selected the 9 time-lapse data sets recorded from 14 March to 16 March (Figure 3).

The reconstructed difference images illustrated in Figure 4 implies that the ground condition changes have mainly happened horizontally at the depth interval of 4-8 meters, which is expressed as conductivity increase. Movements of the changes with time are also recognizable; at the monitoring step T3 the change started in the zone of about 90-110 meters and propagated leftwards along the slope until T5. At the T7 phase, the anomalous zone became more resistive; the original ground condition somehow started to be recovered. According to the displacements recorded by a high-precision borehole inclinometer, ground movements were dominant from the surface down to about 8 meter depth; the ground mass above the 8 meter depth slid. These observations well match the understanding of ground changes from the difference images.

Comparing the L1 norm minimization of the time-domain roughness with the L2 norm one, the anomalies in the L1 norm results look are more focussed, while the L2 ones are more horizontally elongated. Nevertheless, both results well agree with the observations of ground displacement. It is hardly concluded which particular approach would be superior to the other one, since the true changes are not precisely known and both reasonably match the known information.

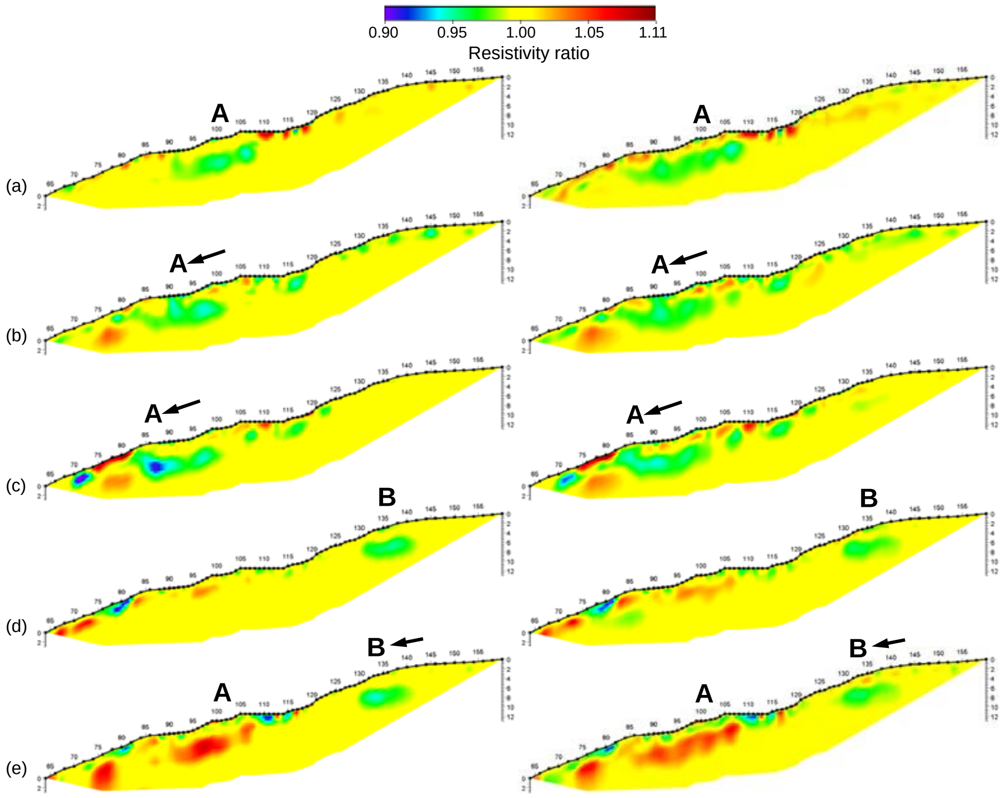

Fig. 4: Reconstructed difference images between two sequential time steps. The left column is the results by the L2D_L2S_L1T approach while the right is L2D_L2S_L2T. (a) T3/T2. (b) T4/T3. (c) T5/T4. (d) T6/T5. (e) T7/T6.

Conclusions

A new 4D inversion algorithm of resistivity monitoring data is presented for precisely estimating the ground condition changes. Through the developed method, we intend to provide a way to adopt either L1 norm or L2 norm minimization of any penalty values in the 4D inversion based on the characteristics of expected subsurface model as well as measured data. The most innovative aspect of the developed algorithm is that the optimal values of the regularization parameters controlling the two smoothness constraints can automatically be updated at each iteration step as the data misfits and the model roughness varies.

Comparisons with the inversion results adopting many different pre-determined values of the regularization parameters confirmed the effectiveness of the newly devised automatic methods.

The automatic determination method always resulted in the difference images much closer to the ground truth than the approaches of using pre-determined parameter values. The problems of too smoothly varying model with time can be solved simply by minimizing the L1 norm of the time-domain roughness. However, either L1 or L2 norm criterion should be selected through the careful consideration of the behavior of data and inverse model parameters in the 4D space. Particularly referring to the time-domain model roughness, the L1 norm minimization would be better when the subsurface changes are locally confined, while the L2 norm approach would be more reasonable when the changes are expected widespread.

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