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Horton laws for Hydraulic-Geometric variables and their scaling exponents in self-similar river networks

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Abstract

An analytical theory is presented to predict Horton laws for five Hydraulic-Geometric (H-G) variables (stream discharge Q, width W, depth D, velocity U, slope S, and friction n'). The theory builds on the concept of dimensional analysis, and identifies six independent *dimensionless River-Basin numbers*. We consider self-similar Tokunaga networks and derive a mass conservation equation in the limit of large network order in terms of Horton bifurcation and discharge ratios. It is applied to obtain self-similar solutions of type-1 (SS-1), and predict Horton laws for width, depth and velocity as asymptotic relationships. Exponents of width and the Reynold's number are predicted. Assuming that SS-1 is valid for slope, depth and velocity, corresponding Horton laws and the H-G exponents are derived. The exponent values agree with that for the Optimal Channel Network (OCN) model, but do not agree with values from three field experiments. The deviations are substantial, suggesting that H-G in network does not obey optimality or SS-1. It fails because slope, a dimensionless River-Basin number, goes to 0 as network order increases, but, it cannot be eliminated from the asymptotic limit. Therefore, a generalization of SS-1, based in self-similar solutions of Type-2 (SS-2) is considered. It introduces two anomalous scaling exponents as free parameters, which enables us to show the existence of Horton laws for channel depth, velocity, slope and Manning's friction. The Manning's friction exponent, y, is predicted and tested against observed exponents from three field studies. We briefly sketch how the two anomalous scaling exponents could be estimated from the transport of suspended sediment load and the bed load. Statistical variability in the Horton laws for the H-G variables is also discussed. Both are important open problems for future research.

1 Introduction

Horton (1945) first discovered Horton laws in quantitative geomorphology with the aid of maps. The original motivation was to define stream size based on a hierarchy of tributaries. The most common method for defining a spatial scale in a hierarchical branched network is the method of Horton–Strahler ordering, or Strahler ordering for short, because Strahler (1952, 1957) modified the ordering system that Horton had introduced. Strahler ordering assigns, $\omega = 1$ to all the unbranched streams. They contain the highest level of spatial resolution for a network and thereby define a spatial scale. Continuing downstream through the network, where two streams of identical order ω meet, they form a stream of order $\omega + 1$. Where two streams of different orders meet, the downstream channel is assigned the higher of the two orders. This continues throughout the network, labeling each stream, and ending with the stream of order Ω. By definition, any network contains only one stream of order Ω called the network order. Strahler ordering defines a one-to-one map under pruning, i.e., if the streams of order 1 are pruned and the entire tree is renumbered, the order 2 streams identically become the new order 1 streams, the order 3 streams become order 2, and so on throughout the network. The order of the entire network decreases by one. This is a necessary condition for defining self-similarity for a hierarchical branched network.

Strahler ordering led to the discovery of the "Horton laws of drainage composition". Often referred to simply as the Horton laws, the most common of these laws include a relationship between stream orders and stream numbers. Similar relationships are observed for lengths, slopes, and areas. These are not formal laws as they have not been proved from first principles, however, they are widely observed in real river networks. The most famous of the Horton's laws is the law of stream numbers for N_ω, denoting the number of streams of order ω in a network of order Ω. It is traditionally written as

$$\frac{N_\omega}{N_{\omega+1}} = R_\mathrm{B}, \quad 1 \le \omega \le \Omega. \tag{1}$$

The number R_B is called the bifurcation ratio. Observations from real river networks show a limited range of R_B values between three and five. The Strahler ordering and the Horton laws concepts had a big impact to model growth of plants, and other hierarchical biological structures such as animal respiratory and circulatory systems, in the order

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of register allocation for compilation of high level programming languages and in the analysis of social networks (Jarvis and Woldenberg, 1984; Pries and Secomb, 2011; Viennot and Vauchaussade de Chaumont, 1985; Park, 1985; Horsfield, 1980; Borchert and Slade, 1981; Berry and Bradley, 1976). The widespread appearance of Horton laws suggests that perhaps a "fundamental principle" underlies them. Indeed, recent research has shown that Horton laws arise in "self-similar networks", which is a form of scale invariance. Theoretical river network models based in self-similarity have been developed that prove Horton laws are asymptotic relations, as cited in Sect. 2. Horton laws as asymptotic relations in a self-similar network serve as the foundation for the theory presented in this paper.

In a classic paper, Leopold and Miller (1956), discovered that channel discharge varies as a function of drainage area as a power law, $Q = kA^c$. At the time, the Horton law for drainage area was known (Jarvis and Woldenberg, 1984). They tested the Horton law for discharge, and asserted that the Horton laws hold for the entire suite of hydraulic-geometric (H-G) variables as functions of discharge, e.g., width, depth, velocity, slope, channel roughness, and sediment transport. Until this paper was published, the Horton laws had been discovered for only the topologic and geometric variables (Jarvis and Woldenberg, 1984). The H-G relations in river networks followed the earlier pioneering work on "at-a-station" (temporal) and downstream (spatial) H-G relations (Leopold and Maddock, 1953). By extending the Horton laws to H-G variables, the Leopold and Miller (1956) paper showed how river basin geomorphology, hydrology and channel hydraulics are linked. Consequently, it opened a new door to understanding how the geometry, statistics and dynamics in river networks are mutually coupled on many spatial scales, which has far-reaching implications for understanding and modeling river flows and sediment transport in river networks. This major objective has not been realized because the theoretical underpinning of the Horton laws and the H-G exponents in channel networks has remained elusive. It is a fundamental, long-standing open problem in Hydro-geomorphology. We develop an analytical theory to predict Horton relationships for five H-G variables (stream discharge Q_ω, width W_ω,

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depth D_ω, velocity U_ω, slope S_ω, and Manning's friction n'_ω) in self-similar Tokunaga river networks (Tokunaga, 1978; Peckham, 1995a). The H-G exponents for W_ω and n'_ω are predicted and tested against observed exponents from three field studies.

Section 2 gives a brief review of the literature. In Sect. 3, our theoretical framework builds on the concept of similarity or similitude that is based in dimensionless numbers (Barenblatt, 1996). We identify a total of six independent *dimensionless River-Basin numbers*. The self-similarity concept has been successfully applied to derive the Horton laws for channel network topology and geometry as asymptotic relations. Mcconnell and Gupta (2008) derived these results for the self-similar Tokunaga networks that are considered in this paper for developing the H-G theory. In Sect. 4, we formulate a mass conservation equation for a river network indexed by Strahler order. A solution of this equation in the limit of large network order Ω is obtained in terms of Horton bifurcation, area and discharge ratios. It applies to small order streams, $\omega = 1, 2, 3 \ldots$

In Sect. 5, we consider three H-G variables, W_ω, D_ω and U_ω, and show that they are power law functions of discharge. By definition, $Q_\omega = U_\omega W_\omega D_\omega$. Horton laws are obtained as asymptotic relations for these three H-G variables. We show that self-similar solutions of type-1 (SS-1) hold asymptotically for the width and the Reynold's number, and values for their H-G exponents are predicted.

In Sect. 6, it is assumed that the SS-1 framework from Sect. 5 is valid for S_ω, D_ω and U_ω. Horton laws for these three H-G variables are derived asymptotically, and their exponents are predicted, which agree with that of the Optimal Channel Network (OCN) model (Rodríguez-Iturbe et al., 1992). But the OCN model uses optimality assumptions and does not consider Horton Laws for these H-G variables. In this sense, foundations of our theory based in self-similarity are very different from that of the OCN model. Ibbitt et al. (1998); McKerchar et al. (1998) conducted two comprehensive field experiments in New Zealand to test the OCN predictions. We show using H-G data from one of these field studies that they obey Horton laws. Moreover, the field values of the exponents don't agree with the OCN predictions. The deviations are substantial suggesting that H-G in network does not obey either optimality or SS-1.

In Sect. 7, we explain that the reason for the failure of SS-1 is that slope, as one of the dimensionless numbers, goes to 0 as network order increases. But, slope cannot be eliminated from the asymptotic limit. Therefore, a generalization of SS-1 theoretical framework is necessary. It requires the concept of “Asymptotic Self-Similarity of Type-2 (SS-2)” discussed in Barenblatt (1996) with many physical examples. SS-2 introduces two anomalous scaling exponents in the theory, which enables us to show the existence of Horton laws for channel depth, velocity, slope and Manning’s friction. The two scaling exponents are free parameters, which cannot be predicted from dimensional considerations. We estimate them from the field-observed values of velocity and slope exponents, which creates a problem with field-testing our theory. To make progress on this front, we consider Manning’s friction coefficient that could be estimated from values of depth and slope, as well as predicted from our theory without any free parameters. Predictions are tested against three field data sets.

Two fundamental physical processes that shape the H-G of channels are transport of suspended sediment load and the bed load that are not considered above. In Sect. 8, we sketch how these two physical processes could be used to determine the two anomalous scaling exponents. Inclusion of statistical variability in the Horton laws for the H-G variables is also discussed. Both are important open problems for future research. The paper is concluded in Sect. 9.

2 Background

Leopold and Maddock (1953) first introduced the H-G of rivers at-a-station and in the downstream direction. Langbein (1964) used energy considerations in developing a theoretical framework to predict downstream H-G exponents (Leopold et al., 1964, p. 266–271). A dedicated body of literature came out in the 1970s devoted to predicting at-a-station H-G relations that were based in the idea of “optimality”. Griffiths (1984) severely criticized these efforts, and called them “an illusion of progress”. Recently, Griffiths (2003) published an article on downstream H-G using ideas of similarity or similitude. An extensive literature has developed on these topics; see Singh (2003) for a recent review of the literature. This body of literature, though important, is not directly relevant to the objectives of our paper. Singh concluded, "The work on hydraulic geometry of channels serves as an excellent starting point to move on to the development of a theory of drainage basin geometry and channel network evolution. This will permit integration of channel hydraulics and drainage basin hydrology and geomorphology." We take an innovative step towards developing a theory of H-G in river networks in this paper.

In a classic paper Leopold and Miller (1956) extended the H-G relations to channel networks (Jarvis and Woldenberg, 1984, chap. 19). It included Horton laws for the H-G variables, channel discharge, width, depth and slope as functions of stream order. The literature is minimal on understanding the Horton laws for the H-G variables in channel networks. An attempt to predict the H-G exponents in river networks without the Horton laws is the theory of optimal channel networks (OCN) (Rodríguez-Iturbe et al., 1992). OCNs have been analytically shown to produce three universality classes in terms of scaling exponents, but none of these predictions agree with data (Maritan et al., 1996). Two comprehensive field programs were carried out in New Zealand to test the OCN predictions (Ibbitt et al., 1998; McKerchar et al., 1998). However, the observed values of the H-G exponents substantially deviated from the OCN predictions. Other attempts building on optimality ideas have used data from these two New Zealand basins (Molnar and Ramirez, 2002). But, a foundational understanding of the geophysical origins of Horton laws for the H-G variables and their exponents has remained elusive.

West et al. (1997) recently tackled a somewhat similar problem in the allometric theory of biological networks. Our treatment of the H-G problem has some similarities but major differences. For example, West et al. (1997) appeal to an "optimality assumption" by maximizing or minimizing a function. By contrast the present theory uses no optimality assumption, but uses "self-similarity" as its basic building block.

The complexity resulting from space-time variability in climate and lithology can be contrasted with the empirical observations like the Horton laws that suggest regularities

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related to similarity across scales, or self-similarity. How the network adjusts its geometry and hydraulics to increasing discharge in a channel network is related to conservation laws governing network development and maintenance that cannot be understood only by focusing attention on climate and physiographic complexities. The regularity across scale suggests the existence of more simple physical principles like the Horton laws that have been known for nearly 70 years (Jarvis and Woldenberg, 1984). Self-similarity concept has been applied successfully to channel network topology and geometry. A partial list of references is Peckham (1995b); Peckham and Gupta (1999); Dodds and Rothman (1999); Veitzer and Gupta (2000); Troutman (2005); Veitzer et al. (2003); Mcconnell and Gupta (2008). This paper extends the self-similarity framework to include H-G variables in river networks. Although we don't consider fluctuations within the present theory, concrete ideas regarding this important issue are discussed in Sect. 8. Other points of comparison with the existing literature are given throughout this paper when required.

3 Definitions and assumptions

3.1 A brief review of self-similar Tokunaga River networks

Assume that a self-similar Tokunaga river network is given (Mcconnell and Gupta, 2008). The topological fractal dimension D_T for Tokunaga networks is given by, $D_\mathrm{T} = \log R_\mathrm{B}/\log R_\mathrm{C}$, where R_B is Horton bifurcation ratio and R_C is the link ratio (Peckham, 1995a). Since $R_\mathrm{B} = R_\mathrm{A}$, and assuming constant link lengths, $R_\mathrm{C} = R_\mathrm{L}$, it follows that, $R_\mathrm{L} = R_\mathrm{A}^{1/D_\mathrm{T}}$, where R_A is Horton area ratio, and R_L is the length ratio. For OCNs, $D_\mathrm{T} = 2$ (Maritan et al., 1996). For natural river networks data sets show that typically, $1.7 < D_\mathrm{T} < 1.8$. The class of Tokunaga networks predicts values of D_T less than or equal to 2. The Hack exponent for Tokunaga networks is, $\beta = 1/D_\mathrm{T} \geq 1/2$, and the area exponent $\alpha \leq 1/2$ as observed empirically (Peckham, 1995b). Moreover, for Tokunaga networks, $\alpha + \beta = 1$ (Peckham and Gupta, 1999). A new theory of random self-similar

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networks (RSN) has been developed to include statistical variability, and the Tokunaga is shown as a special case for a subclass of RSN that obey mean self-similarity (Veitzer and Gupta, 2000). The RSN theory provides the topologic and geometric foundations on which a H-G theory incorporating statistical fluctuations can be developed in the future.

3.2 Defining stream discharge in complete Strahler streams

Let Q_ω denote the discharge at the bottom of a complete Strahler stream of order ω. The discharge is computed below using a mass conservation equation for a network indexed by the Strahler order. It is similar to a mass conservation equation for a channel network indexed by link magnitudes rather than by Strahler order (Gupta and Waymire, 1998). We assume that a channel network obeys Tokunaga self-similarity that is used in solving this equation.

In the current literature, spatial variability of bank-full flows is treated using an empirical-statistical method based in quantiles. However, this approach is unsuitable for the present purposes because it need not conserve mass on a channel network for individual rainfall–runoff events. For example, Furey and Gupta (2000) noted that disaggregation of flows for unnested subbasins using quantiles can give negative discharges. Though it is recognized that the most meaningful discharge to consider in morphology studies is the one that forms and maintains the channel, these considerations plus the availability of data lead us to the consideration of a mean discharge for this theory, at least as a first approximation (Leopold et al., 1964, p. 241). A further argument is that the exponents in scaling relations among variables related to bank-full variables are preserved in relations among mean variables provided that there is a fixed relation between the respective mean and bank-full variables. For instance if bank-full depth, D_B is related to bank-full discharge Q_B as, $D_\mathrm{B} \propto Q_\mathrm{B}^f$, and both $Q_\mathrm{B} = c_1\overline{Q}$, and $D_\mathrm{B} = c_2\overline{D}$, it follows that $\overline{D} \propto \overline{Q}^f$.

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Our treatment considers only spatially variable stream flows in networks rather than space-time variable flows. For this purpose, we assume a steady state condition w.r.t to storages in channels. Our formulation shows rigorously that $Q_\omega = \mathrm{R}A_\omega, \forall \omega \geq 1$ holds asymptotically in the limit of large Ω for Tokunaga self-similar networks. Here A_ω is the cumulative drainage area for a complete Strahler stream, and R is a constant of proportionality. Based on dimensional considerations, it can be interpreted as a spatially uniform discharge rate per unit area, and it has the dimension $[\mathrm{LT}^{-1}]$. Furey and Gupta (2000) explicitly computed such a rate using subsurface discharge from hills in the context of low flows for a network. Their computations show that $Q_\omega = \mathrm{R}A_\omega, \forall \omega \geq 1$ holds in a statistical-mean sense. This result can be extended to a RSN using the results obtained by Veitzer and Gupta (2000), but we will not get into considering fluctuations in the network topology and geometry in this paper as mentioned earlier. Moreover, the two drainage basins from New Zealand that are used to test our theory in Sect. 7.2 measured H-G exponents for low stream flows that can be assumed to obey our steady state condition.

3.3 Dimensionless River-Basin numbers

The fundamental physical variables governing hydraulic-geometry in drainage networks are Q_ω, D_ω, U_ω, W_ω, H_ω, L_ω, S_ω, A_ω, ν, ρ, R, g. Here, Q_ω is river discharge rate, and $D_\omega, U_\omega, W_\omega$ are the corresponding channel depth, velocity, and width, respectively at the bottom of a complete Strahler stream of order $\omega \geq 1$. The elevation drop, or the elevation difference between the two end junctions of a complete Strahler stream is denoted by H_ω. L_ω denotes the corresponding stream length, and slope, $S_\omega = H_\omega/L_\omega$. Kinematic viscosity is ν, water density is ρ, runoff rate is R, and the acceleration due to gravity is g. We have identified a total of nine independent variables, because, $Q_\omega = U_\omega W_\omega D_\omega$, $S_\omega = H_\omega/L_\omega$, and $L_\omega = A_\omega^{1/D_\mathrm{T}}$ reduce three independent variables from the set of twelve listed above. These variables include three basic repeat dimensions, Length (L), time (T) and mass (M). The Buckingham–Pi theorem gives

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us six independent dimensionless numbers, which are specified using physical arguments rather than formal dimensional analysis. Some of the dimensionless numbers were considered in Peckham (1995b).

The first dimensionless number is given by,

$$\Pi_1(\omega) = \frac{Q_\omega}{\mathrm{R}A_\omega}. \tag{2}$$

Here, R is the mean runoff rate per unit area, and it has the dimension $[\mathrm{L}\,\mathrm{T}^{-1}]$. The spatial uniformity of R implies that river basin is homogeneous with respect to runoff generation. This assumption can be easily relaxed but we want to keep this presentation simple. Discharge rate Q_ω is taken to be a linear function of drainage area. This is observed in many humid climates for low and even mean flows. Mean flow has been widely used in H-G investigations (Leopold et al., 1964).

The second dimensionless number is,

$$\Pi_2(\omega) = \frac{R\sqrt{A_\omega}}{D_\omega U_\omega}. \tag{3}$$

It is suggested by mass conservation involving the ratio of runoff per unit width of drainage basin in the numerator, and discharge per unit channel width in the denominator.

The relation between gravitational and inertia forces in river networks suggest the third dimensionless number $\Pi_3(\omega)$. Specifically, we define the "Basin Froude Number" as,

$$\Pi_3(\omega) = \frac{U_\omega}{\sqrt{gH_\omega}} = \frac{U_\omega}{\sqrt{gS_\omega L_\omega}}, \tag{4}$$

where the channel slope,

$$\Pi_4(\omega) = S_\omega = H_\omega / L_\omega \tag{5}$$

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is the fourth dimensionless number. The drop H_ω defines the length scale governing gravitational force. It should be differentiated from a channel Froude number in open channel hydraulics where flow depth defines the length scale.

The fifth dimensionless number is given by the Reynolds number. Leopold et al. (1964, p. 158) have discussed its significance in the context of laminar and turbulent flows. In natural streams, the flow is largely turbulent.

$$\Pi_5(\omega) = \frac{U_\omega D_\omega}{\nu}. \tag{6}$$

The sixth dimensionless number incorporates the factors controlling flow velocity. Total frictional force along the channel boundary is, $\tau_\omega(2D_\omega + W_\omega)L_\omega \approx \tau_\omega W_\omega L_\omega$, where τ_ω is the shear stress per unit area. It is proportional to the square of the mean velocity for turbulent flows if the boundary does not change with variations in flow (Leopold et al., 1964, p. 157). Gravitational force due to the mass of water along the channel length L_ω is given by $\rho g W_\omega D_\omega L_\omega S_\omega$. Dimensionless ratio of these two forces gives,

$$\Pi_6(\omega) = \frac{U_\omega^2}{g D_\omega S_\omega}. \tag{7}$$

The term $\sqrt{g D_\omega S_\omega}$ is known as the shear velocity. Π_6 is proportional to the Darcy–Weisbach resistance coefficient. Leopold et al. (1964, Fig. 6.5) illustrated for the Bryandywine Creek, PA that $1/\sqrt{\Pi_6}$ is linearly related to the logarithm of relative roughness defined by the ratio of flow depth to the height of roughness elements.

4 Mass conservation in Tokunaga self similar networks

We will formulate a mass conservation equation for a river network indexed by Strahler order. Recent developments show that Strahler order rather than link magnitude is most natural in understanding self-similarity in channel networks (Tokunaga, 1978; Peckham, 1995b; Veitzer and Gupta, 2000; Mcconnell and Gupta, 2008).

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Let $\overline{S}_{\omega}(t)$ denote the storage in a Strahler stream of order $\omega \geq 1$ defined by,

$$\overline{S}_{\omega}(t) = W_{\omega}(t) D_{\omega}(t) L_{\omega}(t). \tag{8}$$

The dependence of storage on time t comes from temporal variations of streamflows in the downstream direction, which results in width and depth to vary with time.

Let $T_{\omega,\omega-k}, k = 1,2,\ldots,\omega-1$ denote the number of side tributaries of order $\omega - k$ joining a stream of order ω. We assume that a network has a self-similar topology defined as,

$$T_{\omega,\omega-k} = T_k, \quad k = 1,2,\ldots,\omega-1; \quad \omega \geq 1. \tag{9}$$

We further assume that the network obeys Tokunaga self-similarity defined by,

$$T_k = ac^{k-1}, \quad k = 1,2,\ldots,\omega-1, \tag{10}$$

where the parameters a, c are positive constants. Mcconnell and Gupta (2008) proved the Horton laws of stream numbers and magnitudes (equivalent to stream areas) in Tokunaga networks.

Total number of junctions denoted by C_{ω} is the same as the total number of links in a complete Strahler stream of order ω. Let t_i, $i = 1,2,3,\ldots,C_{\omega}$ be a sequence of travel times for water to reach the bottom of a complete Strahler stream from successive junctions enumerated from the bottom. This means that t_1 represents the travel time from the first junction from the bottom, t_2 from the second junction and so on. For example, since all the links are assumed to have the same length l in Sect. 3.1, and if water flows with a uniform velocity u, then $t_i = il/u$.

Let $R_i(t)$, $i = 1,2,3,\ldots,C_{\omega}$ denote the volumetric runoff rate from the ith hill along a complete Strahler stream of order ω. Let Q_{k_i}, $i = 1,2,\ldots,C_{\omega}$ denote the discharge from the side tributary at the ith junction from the bottom. Here the subscripts $k_1, k_2, \ldots$ denote the Strahler orders of the side tributaries coming into the junctions counted from the bottom of a stream. Let $Q_{\omega}(t)$ denote the discharge at the bottom of a stream of

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order ϖ, and $Q^1_{\varpi-1}(t)$ and $Q^2_{\varpi-1}(t)$ denote the discharges in the two tributaries at the top of the stream. Each of them is of order $\varpi - 1$ by the definition of Strahler order.

Considering a Strahler stream as a finite control volume, the mass conservation equation can be written as,

$$\frac{d\overline{S}_\varpi(t)}{dt} + Q_\varpi(t) = Q^1_{\varpi-1}(t - t_{\varpi-1}) + Q^2_{\varpi-1}(t - t_{\varpi-1}) + \sum_{i=1}^{C_\varpi} Q_{k_i}(t - t_i) + 2\sum_{i=1}^{C_\varpi} R_i(t). \tag{11}$$

For $\varpi = 1$, Eq. (11) reduces to the link magnitude-based mass conservation equation in Gupta and Waymire (1998) that the reader may easily check.

As a first step, we have chosen to focus solely on the spatial analysis in the context of H-G. In particular, we will seek a spatial solution of Eq. (11) by ignoring the time dependence of $Q_\varpi(t)$, and denoting it as $Q_\varpi(t) = Q_\varpi$. This is tantamount to assuming that $d\overline{S}_\varpi(t)/dt = 0$, $R_i(t) = 0$, $\forall i, t > 0$, and the travel times $t_i = 0, \forall i$. Physically, these sets of assumptions can be interpreted to mean that a fixed quantity of runoff rate per unit area, say R, is applied uniformly throughout the network at time $t = 0$. Moreover, water is assumed to travel in a very short time throughout the network so that travel times are ignored.

In a recent paper on a space-time theory of low flows for river networks, travel times were ignored throughout the basin compared to the subsurface response time for hillslopes, and R was computed from hillslope processes under idealized conditions. The theoretical results so obtained compared well with observations (Furey and Gupta, 2000). Similarly, in the present context, these idealized assumptions are necessary to make progress on this complex problem. Further developments of the H-G theory for networks will require that one or more of these assumptions be relaxed in order to develop a broad understanding of space-time variability of physically relevant streamflows on channel networks. A general space-time solution, either analytical or numerical, of Eq. (11) is beyond the scope of this paper because numerous issues remain unsolved. For some complementary on-going work on the spatial scaling structure of peakflows

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governed by the link-based mass conservation in deterministic networks and RSNs, see Gupta et al. (2007).

In view of above assumptions, Eq. (11) simplifies to,

$$Q_\omega = Q^1_{\omega-1} + Q^2_{\omega-1} + \sum_{i=1}^{C_\omega} Q_{k_i}. \tag{12}$$

The key problem is to compute a solution for Q_ω. In view of the definition of self-similarity given by Eq. (9), Eq. (12) reduces to,

$$Q_\omega = 2Q_{\omega-1} + \sum_{k=1}^{\omega-1} T_k Q_{\omega-k}, \tag{13}$$

where $T_k = T_{\omega,\omega-k}$, $k = 1, 2, \ldots, \omega - 1$ denotes the number of side tributaries of order $\omega - k$ joining a stream of order ω. Equation (13) has been solved rigorously under the assumption that T_k's obey Tokunaga self-similarity given by Eq. (10). The solution can be written as (Mcconnell and Gupta, 2008),

$$\lim_{\Omega-\omega\to\infty} \frac{Q_{\omega+1}}{Q_\omega} = R_Q. \tag{14}$$

Because the recursion equation (Eq. 13) for Q_ω is the same as the ones for N_w and A_w, we assert from Mcconnell and Gupta (2008) that

$$R_Q = \lim_{\Omega-\omega\to\infty} \frac{Q_{\omega+1}}{Q_\omega} = R_\mathrm{B} = \lim_{\Omega-\omega\to\infty} \frac{N_\omega}{N_{\omega+1}} = R_\mathrm{A} = \lim_{\Omega-\omega\to\infty} \frac{A_{\omega+1}}{A_\omega} \tag{15}$$

and

$$R_Q = R_\mathrm{B} = R_\mathrm{A} = \frac{(2+a+c) + \sqrt{(2+a+c)^2 - 8c}}{2}.$$

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Equation (15) implies that $Q_\varpi = \mathrm{RA}_\varpi$ that is used in defining the first dimensionless number in Sect. 3.3. It follows from the definitions of Horton ratios given in Eq. (14), and from the equality, $R_Q = R_\mathrm{B}$ that,

$$Q_{\varpi+1}N_{\varpi+1} = Q_\varpi N_\varpi, \quad \text{as } \Omega - \varpi \to \infty. \tag{16}$$

This is a foundational result governing mass conservation in Tokunaga networks indexed by Strahler order. Even though, Eq. (16) is valid in the limit of large network order, the result holds for small values of ϖ. West et al. (1997) used a similar equation for perfect branching biological networks in which no side tributaries are present and each parent branch bifurcates into two branches. In that case, it is simple to write down Eq. (16) as a special case without involving any limit. West et al. (1997) used it to obtain some remarkable results governing allometry in biological networks. We apply Eq. (16) to extend the geometric and topological Horton laws in Tokunaga networks to include the H-G variables. Figure 1 shows a Horton law for channel widths in a drainage network that was mentioned along with other H-G variables in Sect. 1. The key equation providing this link was a power-law relation between discharge and drainage area, and a Horton law for drainage areas (Leopold and Miller, 1956, p. 19-20). This important issue is discussed in greater detail in Sect. 7.2.

5 Asymptotic Self-Similarity of Type-1

5.1 Horton laws for channel width, depth and velocity

It follows from the definition of Π_1 (Eq. 2) and the fact that $R_Q = R_\mathrm{A}$ (Eq. 14) in Tokunaga networks that

$$\lim_{\Omega-\varpi\to\infty} \frac{\Pi_1(\varpi+1)}{\Pi_1(\varpi)} = R_{\Pi_1} = 1, \quad \varpi = 1, 2, \ldots \ll \Omega. \tag{17}$$

This is a very important result that comes from the self-similarity of Tokunaga networks and the assumption of spatial homogeneity of runoff. It probably is a valid assumption

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for widely varying climatic regions and a broad range of spatial scales. For example, the three river basins, one from the United States (US) and two from New Zealand (NZ) that we use to test the predictions of our theory in Sect. 7.2 have different climates, and different sized drainage areas. We also test if $R_Q = R_A$ holds for one of the NZ basins in Sect. 7.2.

All the five H-G variables, U, W, D, S and the Manning's friction coefficient n' considered in this paper vary as discharge Q varies. Therefore, we assume that all the H-G variables are homogeneous functions of Q. This means that the functions do not depend on any other parameter except Q. We can write it as $U = f_1(Q)$, $W = f_2(Q)$ etc. To determine the form of these functions, assume that they are self-similar in a Tokunaga network. Consider $U = f_1(Q)$. Self-similarity can be represented by the functional equation $f_1(Q_1+Q_2) = f_1(Q_1)f_2(Q_2)$ (Gupta and Waymire, 1998, p. 102–103), whose solution is a power law.

$$U = f_1(Q) \propto Q^m.$$

The above argument applies to all the functions. Therefore, the H-G variables can be written as power law functions of discharge,

$$U_\omega \propto Q_\omega^m, \quad W_\omega \propto Q_\omega^b, \quad D_\omega \propto Q_\omega^f, \quad S_\omega \propto Q_\omega^z \quad \text{and } n'_\omega \propto Q_\omega^y. \tag{18}$$

Our notations for the H-G exponents are the same as in Leopold et al. (1964, p. 244). If we consider the ratio, $Q_{\omega+1}/Q_\omega$, and take the limit as $\Omega - \omega \to \infty$, it follows from (Eq. 14) that it converges to a constant, R_Q, and thereby obeys a Horton law. To extend Horton laws to the H-G variables, let us consider velocity

$$\lim_{\Omega-\omega\to\infty} \frac{U_{\omega+1}}{U_\omega} = \frac{Q_{\omega+1}^m}{Q_\omega^m} = R_Q^m = R_U, \tag{19}$$

which follows from the fact that the ratios are positive and monotonic in ω as shown in (Eq. 18) (Rudin, 1976, p. 44).

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Similarly, $R_W = R_Q^b$ and $R_D = R_Q^f$. By definition, $Q_\omega = U_\omega W_\omega D_\omega$. Therefore, the Horton ratios for velocity, width and depth can be written as,

$$R_U = R_Q^m, \quad R_W = R_Q^b, \quad R_D = R_Q^f, \quad m + b + f = 1. \tag{20}$$

Equation (20) required that, (i) Horton laws for channel widths, depths and velocities hold in Tokunaga self-similar networks, (ii) runoff generation is spatially homogeneous, and (iii) channel width, depth and velocity depend monotonically on channel order. For further reference we will call a network satisfying these three conditions as self-similar homogeneous networks (SSHN).

5.2 Prediction of the width exponent and Reynolds number exponent

We will now use the above results to show that in SSHN, topologic and geometric self-similarity extends to "asymptotic self-similarity of Type-I", or SS-1 for short, in the limit of large order that is based in dimensional analysis (Barenblatt, 1996, p. 148). Let us first consider the dimensionless number $\Pi_2(\omega)$ defined by Eq. (3), and the ratio given by,

$$R_{\Pi_2}(\omega) = \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \times \frac{D_\omega U_\omega}{U_{\omega+1} D_{\omega+1}}. \tag{21}$$

Substituting, $D_\omega U_\omega = Q_\omega / W_\omega$, in the above expression gives,

$$R_{\Pi_2}(\omega) = \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \times \frac{Q_\omega W_{\omega+1}}{Q_{\omega+1} W_\omega}. \tag{22}$$

We have already shown that the right hand side converges to a constant in Sect. 5.1. It follows that the left side of Eq. (22) also converges to a constant. Stated mathematically,

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$$\lim_{\Omega-\omega\to\infty} \frac{\sqrt{A_{\omega+1}}}{\sqrt{A_\omega}} \frac{Q_\omega W_{\omega+1}}{Q_{\omega+1} W_\omega} = \frac{R_A^{1/2} R_W}{R_Q} = \lim_{\Omega-\omega\to\infty} R_{\Pi_2}(\omega) = R_{\Pi_2}. \quad (23)$$

The asymptotic constancy of the ratio $R_{\Pi_2}(\omega)$ of the dimensionless number Π_2 across different Strahler orders holds in SSHN. Since, $R_Q = R_A$, Eqs. (23) and (20) can be combined to obtain,

$$R_W = R_{\Pi_2} R_Q^{1/2} = R_Q^b. \quad (24)$$

Therefore, $R_{\Pi_2} = 1$, and the width H-G exponent is,

$$R_W = R_Q^{1/2}, \quad b = 1/2. \quad (25)$$

It directly follows from Eqs. (6), (20) and (25) that a Horton law for Reynolds number in SSHN can be written as,

$$R_{\Pi_5} = R_U R_D = R_Q^{m+f} = R_Q^{1/2}. \quad (26)$$

6 OCN model exponents and SS-Type-I

As an application of the above results, we test our predictions of the H-G exponents against the optimal channel network (OCN) model of Rodríguez-Iturbe et al. (1992), and show that the two are the same. However, our theory differs from the OCN model in a fundamental manner as will be explained below. In the following developments, we assume that SS-1 applies to slope, S_ω, and that the Horton ratio for slope converges to R_S following a similar reasoning as given in Eq. (19). However, this assumption is invalid as explained at the end of this section. It is being made only to compare the H-G exponents from our theory with the OCN model.

Define the Horton ratio for the Basin Froude number from Eq. (4). Following similar arguments as given in Eq. (19), and given about the length ratio in Sect. 3.1. We assert the convergence of the Basin Froude number because the Horton ratio of each term in it converges.

$$\lim_{\Omega-\varpi\to\infty}\frac{\Pi_3(\varpi+1)}{\Pi_3(\varpi)}=R_{\Pi_3}=\frac{R_U}{\sqrt{R_\mathrm{L}R_\mathrm{S}}}. \tag{27}$$

From Eq. (18) $R_\mathrm{S}=R_\mathrm{A}^z=R_Q^z$. Invoking, $R_\mathrm{L}=R_\mathrm{A}^{1/D_\mathrm{T}}=R_Q^{1/D_\mathrm{T}}$ from Sect. 3.1 and assuming that the Tokunaga network is space filling as discussed there for the OCN model, it follows that $D_\mathrm{T}=2$. Substituting $R_U=R_Q^m$ from Eq. (19) into Eq. (27) gives,

$$R_U=R_Q^m=R_{\Pi_3}R_Q^{1/4}R_Q^{z/2}. \tag{28}$$

Equation (28) predicts that $R_{\Pi_3}=1$, and

$$m=\frac{1}{2}(z+1/2). \tag{29}$$

Similarly, consider the Horton ratio for the dimensionless number proportional to the Darcy–Weisbach resistance coefficient given by Eq. (7), and take limit. We have demonstrated the convergence of each term in it. Therefore,

$$\lim_{\Omega-\varpi\to\infty}\frac{\Pi_6(\varpi+1)}{\Pi_6(\varpi)}=R_{\Pi_6}=\frac{R_U^2}{R_DR_\mathrm{S}}. \tag{30}$$

We get an expression for the depth exponent by rewriting Eq. (30) as

$$R_D=R_Q^f=\frac{R_U^2}{R_{\Pi_6}R_\mathrm{S}}=\frac{R_Q^{2m}}{R_{\Pi_6}R_Q^z}. \tag{31}$$

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It predicts, $R_{\Pi_6} = 1$, and

$$f = 2m - z. \tag{32}$$

Solving Eqs. (29) and (32) gives, $f = 1/2$, $m = 0$, $z = -1/2$, which also satisfy the constraint that $m + f = 1/2$. In summary, our predictions for the OCN model H-G exponents $b = 1/2$, $m = 0$, $f = 1/2$, $z = -1/2$ correspond to those of Rodríguez-Iturbe et al. (1992). However, no optimality ideas are used in making these predictions. We take the self-similar Tokunaga network model, derive Horton laws for the H-G variables, and predict the H-G exponents. By contrast, the OCN model directly predicts the H-G exponents using optimality and without Horton laws for the H-G variables. Consequently, the connections between our theory and the OCN model remain unclear to us. But in our context, SS-1 does not hold in river networks as explained below.

Two comprehensive field experiments in NZ were conducted to test the OCN predictions. Figure 2 illustrates the measurement sites in network of the Taieri River Basin. These two basins are described in Sect. 7, where the field-measured values of the H-G exponents are also given. These values don't agree with the OCN predictions except for $b = 1/2$. The deviations are substantial suggesting that H-G in network does not obey SS-1 or the optimality. We also show the existence of Horton laws for some of the H-G variables for the Ashley River Basin in Sect. 7.

To understand the reason within a dimensional analysis framework, note that one of the six dimensionless numbers, slope, $S_\omega \to 0$ as $\Omega - \omega \to \infty$. Therefore, SS-1 framework is not applicable to asymptotic relations that include slope. The general theoretical framework of renormalization group approach is required to include slope into our theory as given in the next section.

7 Asymptotic Self-Similarity of Type-2 in Tokunaga networks

7.1 Theoretical expressions for the H-G exponents

Slope appears in dimensionless numbers given by Eqs. (4), (5) and (7). The stream drop in Eq. (5) is bounded but stream length increases with order. Therefore, slope $S_\omega \to 0$ as $\Omega - \omega \to \infty$. Moreover, slope cannot be eliminated from the asymptotic limit. Therefore, a generalization of the dimensional analysis is necessary. It requires the concept of "Asymptotic Self-Similarity of Type-2", or SS-2 for short, discussed in Barenblatt (1996, chap. 5). He has given a recipe for the applications of similarity analysis and renormalization group approach with many physical examples including turbulent shear flows, fractals, biological allometry, and groundwater hydrology. We apply it to show the existence of Horton laws for channel depth, velocity, slope and Manning's friction. This enables us to make progress in predicting the corresponding H-G scaling exponents.

SS-2 is a consequence of a dimensionless number that, despite being too small (or large if you consider its reciprocal), cannot be ignored in the limit. For the problems in which dimensional analysis has proved successful, there is a clear way of separating the important variables from the ones that do not play a significant role because they are too small. Mathematically this corresponds to the case of a function converging to a finite limit different from zero when the variable in question goes to zero. But if the limit does not exist, or is zero or infinity, one cannot discard the variable. The simplicity of SS-1, that consists of the possibility of discarding small variables and obtaining the scaling exponents from dimensional analysis is lost in this case. In SS-2, small variables continue to play a role in the problem, which requires an introduction of anomalous exponents that cannot be derived from dimensional analysis. It requires different physical arguments for their determination.

Renormalization group theory has developed along two separate lines, the first one in statistical physics and the second one in fluid mechanics. Barenblatt (1996) has explained that these two approaches are equivalent. For our purposes, the fluid-mechanical approach is more natural than the statistical physical approach, because it is based on a generalization of the dimensional analysis framework. We will follow the fluid-mechanical approach in the subsequent developments. The reader is referred to Barenblatt (1996, chap. 5) for an expository overview of the renormalization group approach based in a generalization of dimensional analysis.

Following Barenblatt, we define two "renormalized dimensionless numbers" in which slope appears. Equations (4) and (7) modify to,

$$\Pi_3^*(\omega) = \frac{U_\omega}{\sqrt{gL_\omega S_\omega^\alpha}}, \tag{33}$$

$$\Pi_6^*(\omega) = \frac{U_\omega^2}{gD_\omega S_\omega^\beta}. \tag{34}$$

Here α and β are "anomalous scaling exponents" that cannot be predicted from dimensional analysis. In principle they can be predicted from physical arguments involving sediment transport. This is a task for future research as explained in Sect. 8.

Following similar arguments as given in Sect. 5 to justify Eq. (19), we assert the convergence of the normalized dimensionless numbers,

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_3^*(\omega+1)}{\Pi_3^*(\omega)} = R_{\Pi_3^*} = \frac{R_U}{\sqrt{R_L R_S^\alpha}}. \tag{35}$$

and,

$$\lim_{\Omega-\omega\to\infty} \frac{\Pi_6^*(\omega+1)}{\Pi_6^*(\omega)} = R_{\Pi_6^*} = \frac{R_U^2}{R_D R_S^\beta}. \tag{36}$$

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Recall from Sect. 3.1 that $R_\mathrm{L} = R_\mathrm{A}^{1/D_\mathrm{T}} = R_Q^{1/D_\mathrm{T}}$ and from Sect. 5 that, $R_\mathrm{S} = R_\mathrm{A}^z = R_Q^z$, and $R_U = R_Q^m$. Therefore, Eq. (35) gives,

$$R_U = R_Q^m = R^*_{\Pi_3} R_Q^{1/2D_\mathrm{T}} R_Q^{z\alpha/2}. \quad (37)$$

Equation (37) predicts that $R^*_{\Pi_3} = 1$, and

$$m = \frac{1}{2}(z\alpha + 1/D_\mathrm{T}). \quad (38)$$

Since $m + f = 1/2$, an expression for the depth exponent follows directly from Eq. (38),

$$f = \frac{1}{2}(1 - z\alpha - 1/D_\mathrm{T}). \quad (39)$$

We get a second expression for the depth exponent by rewriting Eq. (36) as

$$R_D = R_Q^f = \frac{R_U^2}{R_{\Pi_6^*} R_\mathrm{S}^\beta} = \frac{R_Q^{2m}}{R^*_{\Pi_6} R_Q^{z\beta}}. \quad (40)$$

It predicts, $R_{\Pi_6^*} = 1$, and, in view of Eq. (38),

$$f = 2m - z\beta = z(\alpha - \beta) + 1/D_\mathrm{T}. \quad (41)$$

Equating the expressions for f from Eqs. (41) and (39), we obtain an expression for the slope scaling exponent as,

$$z(3\alpha - 2\beta) = 1 - 3/D_\mathrm{T}. \quad (42)$$

Equations (38) and (42) together generalize the H-G theory for a channel network based on an application of the renormalization group theory, and SS-2.

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To summarize, given the topological fractal dimension D_T, and prediction of the width exponent, $b = 1/2$ by Eq. (26), we have two Eqs. (38), (42) that give theoretical expressions for H-G exponents m, z in terms of two unknown parameters, α, β. Our theoretical expressions for the H-G exponents can be written as,

$$
\begin{aligned}
b &= 1/2 \\
z &= (1 - 3/D_T)/(3\alpha - 2\beta) \\
m &= (z\alpha + 1/D_T)/2 \qquad (43) \\
f &= 1/2 - m.
\end{aligned}
$$

Throughout, we will fix, $D_T = 7/4$, a realistic value for river networks (La Barbera and Rosso, 1989; Maritan et al., 1996). The scaling exponents α and β are free parameters, which are not predicted by our theory in this paper. We estimate them from the observed values of m and z. Therefore, the depth exponent, $f = 1/2 - m$ is also estimated from data, which creates a problem in testing our theory. To make progress on this front, we consider Manning's friction coefficient that can be estimated from values of depth and slope as explained below. But first, we derive a theoretical expression for the Manning's friction exponent.

Rewrite $\Pi_6^*(\omega)$ given by Eq. (34) as,

$$\Pi_6^*(\omega) = 1 = \frac{U_\omega^2}{gD_\omega S_\omega^\beta} = \frac{U_\omega^2}{gD_\omega S_\omega S_\omega^{-1+\beta}}, \qquad (44)$$

so it may be expressed in the form of the well-known Chezy's equation,

$$U_\omega = (gD_\omega S_\omega)^{1/2} S_\omega^{(-1+\beta)/2}. \qquad (45)$$

Therefore an expression for the Chezy's friction parameter is given as,

$$C_\omega^* = (g)^{1/2} S_\omega^{(-1+\beta)/2}, \qquad (46)$$

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provided, $\beta < 1$. This constraint is mentioned because the slope exponent must be negative to be consistent with data. Manning's friction coefficient n'_ϖ is related to Chezy's by means of (Leopold et al., 1964, p. 158),

$$U_\varpi = C^*_\varpi \sqrt{D_\varpi S_\varpi} = \frac{1.49}{n'_\varpi} D_\varpi^{2/3} S_\varpi^{1/2} = \frac{1.49}{n'_\varpi} D_\varpi^{1/6} \sqrt{D_\varpi S_\varpi}.$$

Therefore,

$$n'_\varpi = 1.49 D_\varpi^{1/6} / C^*_\varpi. \tag{47}$$

It can be expressed as the ratio using Eqs. (46) and (47) as,

$$\frac{C^*_{\varpi+1}}{C^*_\varpi} = \left[\frac{D_{\varpi+1}}{D_\varpi}\right]^{1/6} \left[\frac{n'_\varpi}{n'_{\varpi+1}}\right]. \tag{48}$$

Taking the limit as, $\Omega - \varpi \to \infty$, gives

$$R_{n'} = R_D^{1/6} R_\mathrm{S}^{-(-1+\beta)/2}. \tag{49}$$

Equation (18) defined $R_{n'} = R_Q^y$. Using $R_\mathrm{S} = R_Q^z$ as before gives an expression for the H-G scaling exponent related to the Manning's equation,

$$R_{n'} = R_Q^y = R_Q^{f/6} R_Q^{-z(-1+\beta)/2}, \tag{50}$$

which gives a theoretical prediction for the Manning friction exponent as,

$$y = f/6 - z(-1+\beta)/2 \tag{51}$$

provided, $\beta < 1$. There are no adjustable parameters in this expression. It is tested against the field value of the Manning friction exponent in the next section. We will use the same sets of field data from channel networks to support that estimated $\beta < 1$, and $\beta < 3\alpha/2$ from Eq. (42).

Mantilla et al. (2006) have described the H-G form of Chezy's friction coefficient that they deduced from empirical observations. They showed that the expression for Chezy's friction coefficient played a key role in testing the presence of statistical self-similarity involving Hortonian relationship in peak flows in the Walnut Gulch basin, Arizona.

7.2 Test of the theory: prediction of Manning Scaling exponent for Three Field Studies

As mentioned in Sect. 7.1, we estimate the free parameters, α, β using the empirical values of f and z from three field experiments. This exercise allows us to compute the numerical values of the scaling exponents, α, β and check that Eqs. (44) and (42) hold as required by the theory. We use the published data for H-G exponents from these three basins, and test the prediction of the scaling exponents b and y corresponding to the width and the Manning friction as a test of our theory.

Our first two basins are from NZ. The largest part of NZ has a pleasant sea climate with mild winters and warm summers. Köppen climate classification lists it as type Cf. The first field experiment was conducted in the Taieri River Basin (Ibbitt et al., 1998) that was introduced earlier. It has an estimated mean annual precipitation of 1400 mm. Basin Area is 158 km^2. Mean discharge, as measured at the streamflow recorder over the discontinuous 14 year period 1983–1996, is 4.90 m^3 s^{-1}, representing an average runoff rate of 980 mm yr^{-1} from the basin.

The field values of the H-G exponents are, $b = 0.517$, $z = -0.315$, $m = 0.238$ and $f = 0.247$. The empirical width exponent is close to the predicted value, $b = 1/2$. Ibbitt et al. (1998) do not give an empirical value of Manning's friction exponent, but it can be computed from other exponents given above, and the empirical Manning equation,

$$U_\omega = 1.49 D_\omega^{2/3} S_\omega^{1/2} / n'_\omega,$$

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as,

$$y = (2/3)0.247 - (1/2)0.315 - 0.238 = -0.231. \tag{52}$$

To make a theoretical prediction of y, we take, $D_\mathrm{T} = 7/4$, since river networks show a value between 1.7 and 1.8 (La Barbera and Rosso, 1989; Maritan et al., 1996). Then, using the empirically computed values of the scaling exponents f and z in Eq. (41) we get

$$\alpha - \beta = (f - 1/D_\mathrm{T})/z = 1.030, \tag{53}$$

and from Eq. (42)

$$3\alpha - 2\beta = (1 - 3/D_\mathrm{T})/z = 2.268. \tag{54}$$

Solving Eqs. (53) and (54) gives the values of the scaling exponents as, $\alpha = 0.208$, and $\beta = -0.822$, which satisfy the constraints on α, β described above. The predicted value of the Manning-scaling exponent from Eq. (51) is,

$$y = f/6 - z(-1 + \beta)/2 = -0.246, \tag{55}$$

which is very close to the observed Manning exponent given in Eq. (52), which supports our theory.

The second field experiment was conducted in the 121 km^2 Ashley River Basin (McKerchar et al., 1998). Annual precipitation increases in a northwesterly direction across the basin from 1200 to about 2000 $\mathrm{mm\,yr}^{-1}$. Mean discharge, as measured at the stream gauge over the 20-year period 1977–1996 is 3.99 $\mathrm{m}^3\,\mathrm{s}^{-1}$, representing an average runoff rate of 1040 $\mathrm{mm\,yr}^{-1}$ from the basin.

Tradionally Horton laws have been known in terms of statistical means. Peckham and Gupta (1999) reformulated the Horton laws in terms of probability distributions as explained in Sect. 8. We name it as statistical self-similarity in ordered networks (SS-SON) for an easy reference here and in future research articles on this topic. Mantilla

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(2014) is testing the presence of SSSON for the H-G variables in the two NZ basins considered here and a few others basins for which he has data. He has kindly shared some of his analysis with us for the Ashley basin that has $\Omega = 6$. Mantilla (2014) extracted the Ashley basin geomorphology from the Digital Elevation Model (DEM) data using the software CUENCAS (Mantilla and Gupta, 2005). His first set of results pertains to the Horton laws for drainage area and stream numbers as shown in Figs. 3 and 4. The Horton laws hold quite well, and the observed $R_A = 4.47$ agrees well with $R_B = 4.5$ as predicted for the Tokunaga network in Sect. 4.

Next, the Horton law for mean stream flow is considered. In making this plot, the theoretical condition $\Omega - \omega \to \infty$ is incorporated by omitting order 6 and 5 streams from the analysis. Mantilla (2014) found that the basin has a large number of the 1-st order streams that are mostly missed in the map that McKerchar et al. (1998) presented. Therefore, the Horton plot is made for streams of order $\omega = 2, 3, 4$, shown in Fig. 5. $R_Q = 3.05$ is observed. It shows that $R_Q = R_A^{\theta}$, where $\theta = 0.74$. The reason is that all the streams in a network need not contribute to stream flows. Many physical processes play a role, like space-time variable rainfall, state of dryness or wetness of soil in a basin at the time rainfall begins, which governs infiltration into soil and evaporation from it and so on. The physical parameter, θ, represents the aggregate behavior of the physical processes governing runoff generation, and can take a values less than or equal to 1. It is written as,

$$R_Q = R_A^{\theta}. \tag{56}$$

Galster (2007) analyzed several basins to test the relationship $Q = kA^c$. His results show that the studied watersheds could be grouped into two broad categories based on their respective c values: (1) those where $c = 1$ or nearly 1, and (2) those where c is significantly < 1 like 0.8 or 0.5. Other research efforts have been made on understanding the nature of the scaling exponent θ from physical processes (Poveda et al., 2007; Gupta et al., 2010; Furey et al., 2013). Clearly, the Ashley basin shows that $\theta < 1$. Our derivation in Eq. (15) that $R_Q = R_A$ applies to category (1) basins in Galster (2007), but

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not to category (2) like the Ashley. Therefore, it needs to be generalized to incorporate such basins for which $\theta < 1$.

The observed H-G scaling exponents are, $b = 0.44$, $z = -0.317$, $m = 0.318$ and $f = 0.242$. The empirical width exponent, $b = 0.44$ shows some deviation from the predicted value, $b = 1/2$. The Horton law for the width, taken from Mantilla (2014) is shown in Fig. 6. The observed value of $R_W = 1.61$, which shows some deviation form the predicted Horton ratio for the width exponent $R_Q^{1/2} = 1.74$. Since $R_Q^{0.44} = 1.63$, the width exponent that McKerchar et al. (1998) presented is consistent with the observed value of the Horton width ratio of 1.61 that Mantilla (2014) obtained. Other H-G variables not shown here support that the Horton laws hold as predicted in our work, and $R_W R_D R_U = 2.96$ that is close to the value of $R_Q = 3.05$ (see Eq. 20).

McKerchar et al. (1998) do not give an empirical value of the Manning's friction exponent, but it can be computed from above exponents and the empirical Manning equation. The value is,

$$y = (2/3)0.242 - 0.318 - (1/2)0.317 = -0.315. \quad (57)$$

We take, $D_\mathrm{T} = 7/4$. Using the observed values of f and z, the empirically computed values of the scaling exponents are (using Eq. 53):

$$\alpha - \beta = \frac{0.242 - 4/7}{-0.317} = \frac{0.329}{0.317} = 1.038. \quad (58)$$

Similarly, using Eq. (54) and the empirical exponents we obtain

$$3\alpha - 2\beta = \frac{1 - 12/7}{-0.317} = \frac{0.714}{0.317} = 2.253. \quad (59)$$

Solving Eqs. (58) and (59) gives, $\alpha = 0.175$, $\beta = -0.864$, which satisfy the constraints on α and β described above.

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The predicted value of the Manning scaling exponent using Eq. (51) is,

$$y = \frac{f}{6} - \frac{z(\beta - 1)}{2} = \frac{0.242}{6} + \frac{0.317(-1 - 0.864)}{2} = -0.255. \tag{60}$$

There is some discrepancy between the observed and the predicted values. It seems to come from the observed exponents of width and velocity. The discrepancy in the width exponent affects the velocity exponent, which in turn affects the Manning friction exponent. The reader may compare the empirical H-G exponents in the Taieri basin with those in the Ashley basin to get a comparative idea of the field measured values of the H-G exponents in these two basins that have comparable scales and climates. The measured values of f and z are comparable as one expects, but not of b and m. Reasons for this potential discrepancy may lie in $Q = kA^c$ relationship if $c < 1$. This is a topic for future research as stated in Sect. 9.

The third example is for the classic Brandywine creek, PA in the US as given in Leopold et al. (1964, Table 7.5, p. 244), where the H-G exponents are also given. It has humid subtropical climate with cool to cold winters, hot, humid summers, and generous precipitation throughout the year, approx 1100 mm yr^{-1}. Köppen climate classification lists it as type Cfa. It has a drainage Area of 777 km^2 at the mouth. Average discharge is 12 m^3 s^{-1}.

The observed values of the H-G exponents are $b = 0.42$, $f = 0.45$, $m = 0.05$, $z = -1.07$ and $y = -0.28$. We fix $D_T = 7/4$. The empirically computed values of the scaling exponents, which correspond to these H-G exponents, are: $\alpha = 0.441$, and $\beta = 0.327$. They satisfy the theoretical constraints, $\beta < 1$ and $\beta < 3\alpha/2$. Moreover, Eq. (51) correctly predicts the empirical value of the Manning's exponent, $y = -0.285$. However, the empirical values related to the width, depth and velocity do not satisfy $b + f + m = 1$, instead they add to 0.92. Assuming that depth and velocity exponents are correct, because they correctly predict the Manning's exponent, the value of $b = 1 - f - m = 1/2$ agrees with our theoretical prediction.

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Table 1 presents a summary of the observed and predicted H-G scaling exponents for the three basins considered above. Predicted values for the exponent y that we presented above using Eq. (51), and for the exponent $b = 1/2$ from Eq. (43).

8 Future research problems: two examples

The above theory can be generalized along several lines. We illustrate two important problems. The first is that the anomalous scaling exponents α and β need to be predicted using physical arguments. Two fundamental physical processes that shape the H-G of channels are transport of suspended sediment load and the bed load that we have not considered so far. There is a huge literature on this subject (Leopold et al., 1964; Singh, 2003). Our ideas on how these two physical processes can be used to determine α and β are rudimentary and are only meant for illustration.

The suspended load increases in proportion to discharge. Therefore, suspended sediment concentration, defined as the ratio of the two, does not change. Leopold et al. (1964, p. 269) gave an expression for sediment concentration, $C \propto (UD)^{0.5} S^{1.5}/n^4$, constancy of C implies that $0.5m + 0.5f + 1.5z - 4y = 0$, or, $0.25 + 1.5z - 4y = 0$ since, $m + f = 1/2$. It gives the first equation in terms of α and β.

The second equation can be developed from considering stream power per unit of bed area, $\varpi = \rho g QS/W$, which plays a basic role in the bed load transport (Molnar, 2001). Essentially all the theories of bed load transport assume that there is a threshold shear stress, stream power, or mean flow speed, and no erosion occurs below it. During floods, these variables exceed the threshold, and bed load is transported that creates erosion. We expect that a second equation can be obtained from these considerations in terms of α and β. The two equations can be solved to compute α and β.

The second problem is to generalize the Horton laws for the H-G variables that include SSSON. Peckham and Gupta (1999) presented such a framework for drainage areas and channel lengths. Specifically, they gave observational and some theoretical arguments to show that probability distributions of all drainage areas rescaled by their means, $A_\omega/\overline{A}_\omega$ collapse into a common probability distribution. Let us consider drainage areas, $A_\omega/\overline{A}_\omega$ to illustrate SSSON. There are two components to this argument.

i. A Horton law for the mean drainage areas, $\overline{A}_\omega$ of order ω, holds that can be written as,

$$\overline{A}_\omega = R_{\mathrm{A}}^{\omega-1}\overline{A}_1, \quad \omega = 1, 2, \ldots, \tag{61}$$

where R_{A}, is the Horton's area ratio. It is illustrated in the Whitewater basin, Kansas, USA in Fig. 7.

ii. SSSON is defined as

$$A_{\omega+1}/\overline{A}_{\omega+1} \stackrel{d}{=} A_\omega/\overline{A}_\omega, \tag{62}$$

or,

$$A_{\omega+1} \stackrel{d}{=} \left(\overline{A}_{\omega+1}/\overline{A}_\omega\right) A_\omega, \quad \omega = 1, 2, \ldots \tag{63}$$

where $\stackrel{d}{=}$ means that the probability distributions of the rescaled areas on both sides of Eq. (63) are the same. Since the Horton law holds for the mean areas given in Eq. (61), it follows from Eq. (63) that,

$$A_{\omega+1} \stackrel{d}{=} R_{\mathrm{A}} A_\omega, \quad \omega = 1, 2, \ldots \tag{64}$$

This feature is illustrated for drainage areas in the Whitewater basin, Kansas, USA in Fig. 8.

Let us consider the dependence of channel widths on discharge. Both are treated as random variables. Therefore the results obtained in Sect. 5.2 can be interpreted as for the means and written as, $\overline{W}(Q_\varpi) = c\overline{Q}_\varpi^{\,b}$. We conjecture that SSSON holds for the rescaled channel widths. Then,

$$W(Q_{\varpi+1}) = \left(\frac{\overline{Q}_{\varpi+1}}{\overline{Q}_\varpi}\right)^b W(Q_\varpi), \quad \varpi = 1,2,\ldots \tag{65}$$

This is an equality among random variables as shown for drainage areas in Eq. (63). But in the context of channel widths, a H-G variable, it also means that the probability distribution of $W(Q_{\varpi+1})$ can be computed from the probability distribution of $W(Q_\varpi)$ provided a Horton law of mean widths and the value of b are known. Both these features, as predicted in Sect. 5 for SS Tokunaga networks, can be interpreted as those for the means. This conjecture is made in the light of the result described in Sect. 3 that the Tokunaga networks are a special case for a subclass of RSN that obey mean self-similarity (Veitzer and Gupta, 2000). In view of these arguments, we can write,

$$\overline{W}_\varpi = R_W^{\varpi-1}\overline{W_1}, \quad \varpi = 1,2,\ldots, \tag{66}$$

where, $R_W = R_Q^b$ is the Horton ratio for the mean widths. We conjecture based on these arguments that Horton laws hold for all the H-G variables measured in the two New Zealand basins that were analyzed in the Sect. 7.2. We support this conjecture for the validity of Horton laws for widths and stream flows in the Ashley basin in Sect. 7.2. Both the basins have the necessary data sets to test our conjecture regarding applicability of SSSON. Mantilla (2014) is conducting this research as mentioned in Sect. 7.2.

9 Conclusions

There has been important progress in topological and geometric theories to explain the related Horton's law for stream bifurcation, drainage areas and stream lengths as asymptotic relations. But progress on Horton laws for the H-G variables has been long

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overdue. We made a contribution to this important problem, and laid the theoretical foundations of a H-G theory in the SS Tokunaga networks. Our main findings are summarized below:

1. We used the Buckingham–Pi theorem and identified six dimensionless basin numbers in Sect. 3, which served as a basis to develop the theory in the subsequent sections.

2. A mass conservation equation was specified in Strahler ordered networks. A link-based equation was shown to be a special case of it. We solved it in Tokunaga SS networks using the results from Mcconnell and Gupta (2008) and derived a mass conservation equation in the limit as, $\Omega - \omega$ goes to infinity in terms of Horton bifurcation and discharge ratios in Sect. 4.

3. We gave an analytical derivation of the H-G relations as power-law functions of discharge. The derivation is based on the assumptions that the H-G variables are homogeneous and self-similar functions of discharge. The Horton laws are extended to width, depth and velocity in Tokunaga SS networks using the results from Sect. 4. Within the dimensional analysis framework, the SS-1 given in Barenblatt (1996) is used to predict the width exponent, $b = 1/2$. These results are given in Sect. 5.

4. Assuming that SS-1 holds for slope, we predicted the Horton's laws for S, U and D, and their exponents. Our predictions agree with the exponents given in the optimal channel network model (OCN) (Rodríguez-Iturbe et al., 1992), but they don't predict Horton laws. Our theoretical framework is based in self-similarity, and does not use any optimality assumptions. Published previous field studies cited here have shown that the OCN predictions do not agree with observations. These results are given in Sect. 6. We assert following Barenblatt (1996) that the problem lies in the assumption that SS-1 holds for slopes, because the slope goes to zero in the limit of large basin order.

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5. SS-2 is required to deal with the case when one or more dimensionless numbers go to zero in the limit (Barenblatt, 1996). Therefore, we consider that SS-2 holds for slopes, which gives rise to two anomalous scaling exponents, α and β that come from two dimensionless numbers in Sect. 3. We derived Horton's law for S, D, and U in Sect. 7, but the H-G exponents become functions of α and β. To predict these two anomalous scaling exponents from geophysics, we suggest that it would require consideration of sediment transport, as briefly discussed in Sect. 8.

6. We tested the predictions of our theory against observations using published H-G data from three river basins in Sect. 7.2. Since we do not give a physical prediction of α and β, we back calculate them using observed exponents for D and S. In this process we lose testability. To make progress with testing our theory, we consider a fifth H-G variable, namely Manning's friction, that could be estimated from data on slope, velocity, width and depth, and predicted from our theory. The predictions are good as given in Sect. 7.2. Even though there are few network field studies of H-G, the theory passes these tests reasonably well. The estimation of the anomalous exponents from physical principles and the consideration of sediment transport are needed for a definite test of the theory.

7. The Two NZ basins analyzed here show statistical variability in the H-G variables. We showed some results from Mantilla (2014) for the Ashley basin for the existence of Horton laws for widths and stream flows. He is testing for the presence of SSSON in all the H-G variables for a further development of this theory. Last but not least, more H-G data on river networks is needed in different climates to test theoretical predictions as they become available. This is an expensive proposal, which would require international cooperation to make progress.

8. The empirical observation that $R_Q \neq R_\mathrm{A}$, illustrated here for the Ashley basin, is true more generally as Galster (2007) discussed. This is very important and needs further considerations. In particular, some of the assumptions leading to

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Eq. (13) do not hold, because it was shown rigorously that under those assumptions $R_Q = R_A = R_B$. In the text some possible physical explanations were suggested. However, to incorporate this hydrologic feature in Tokunaga networks, the generator expression given in Eq. (10) needs to be modified so that all the streams that don't contribute to stream flows are removed in the derivation of Eq. (13).

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Table 1. Summary of observed and predicted H-G scaling exponents. The sources of the data are Ibbitt et al. (1998) for the Taieri River Basin in New Zealand; McKerchar et al. (1998) for the Ashley River Basin in New Zealand; Leopold et al. (1964, Table 7.5, p. 244) for the Brandywine creek, PA in the United States.

		Basin		
Variable	Exponent	Taieri	Asheley	Brandywine
		Observed		
$U \propto Q^m$	m	0.238	0.318	0.050
$W \propto Q^b$	b	0.517	0.440	0.420
$D \propto Q^f$	f	0.247	0.242	0.450
$S \propto Q^z$	z	−0.315	−0.317	−1.070
$n' = Q^y$	y	−0.231	−0.315	−0.280
Estimated using $D_T = 7/4, f$ and z				
	α	0.208	0.175	0.441
	β	−0.822	−0.864	0.327
Predicted using Eqs. (43) and (51)				
$W \propto Q^b$	b	0.500	0.500	0.500
$n' \propto Q^y$	y	−0.246	−0.255	−0.285

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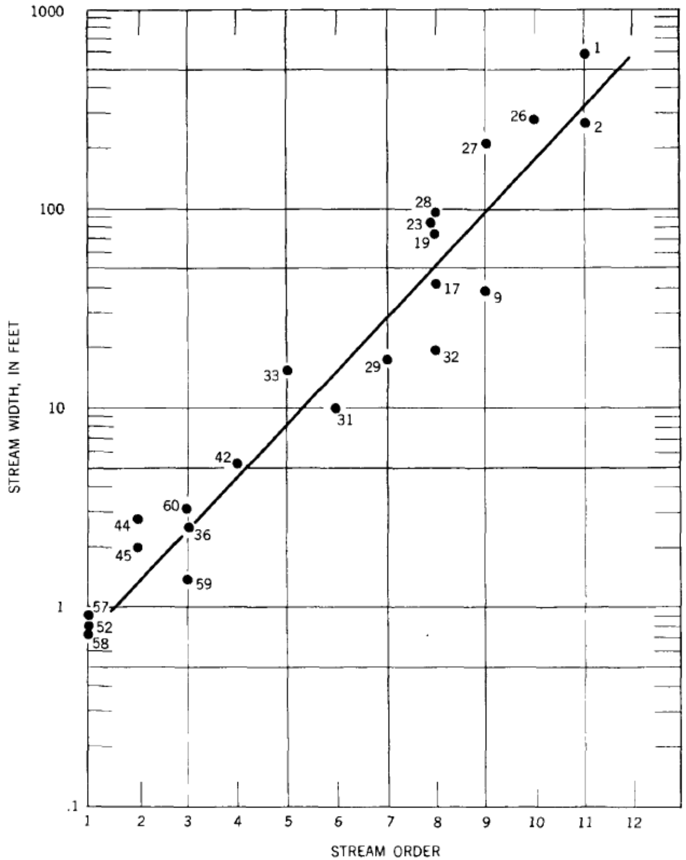

Fig. 1. Reproduction of the original figure of Leopold and Miller (1956, Fig. 19, p. 24) showing the relation of stream width to stream order in arroyos. Numbers beside points correspond to different points in the network.

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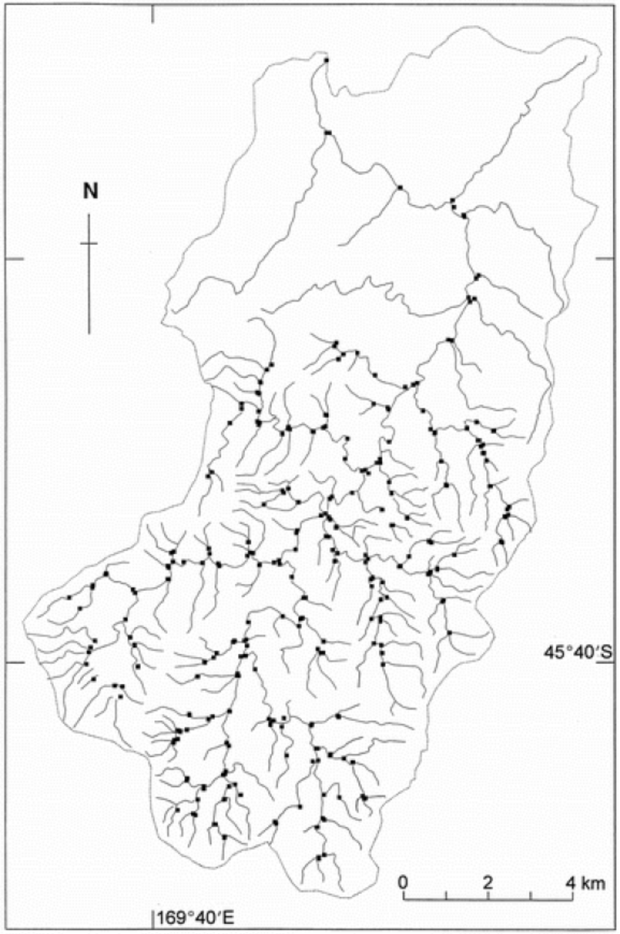

Fig. 2. Reproduction of the original figure of Ibbitt et al. (1998, Fig. 1) showing the river network of the Taieri basin in New Zealand along with measurements sites in the network.

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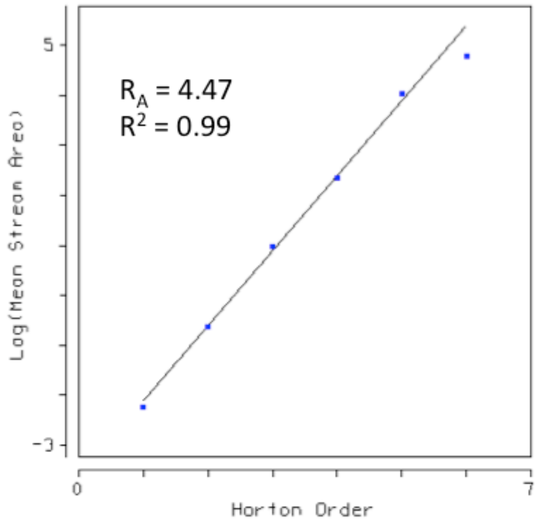

Fig. 3. Horton analysis of upstream areas (including orders 2, 3, and 4) for Ashley River Basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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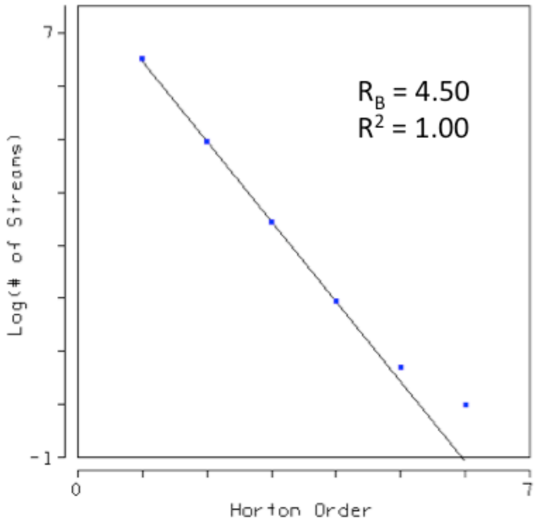

Fig. 4. Horton analysis of stream numbers (including orders 2, 3, and 4) for Ashley River Basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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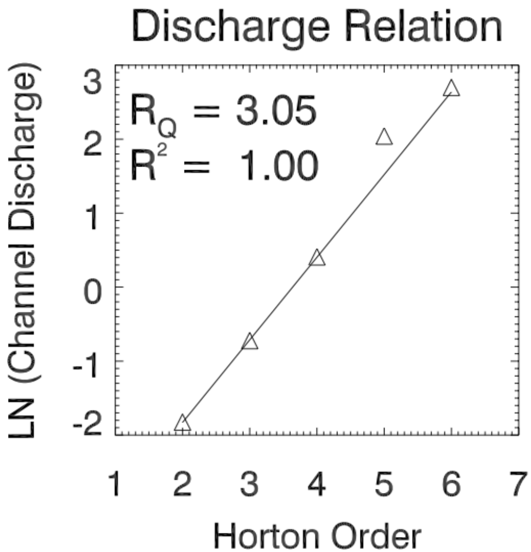

Fig. 5. Horton plots for discharge (including order 2, 3, and 4) for Ashley River Basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

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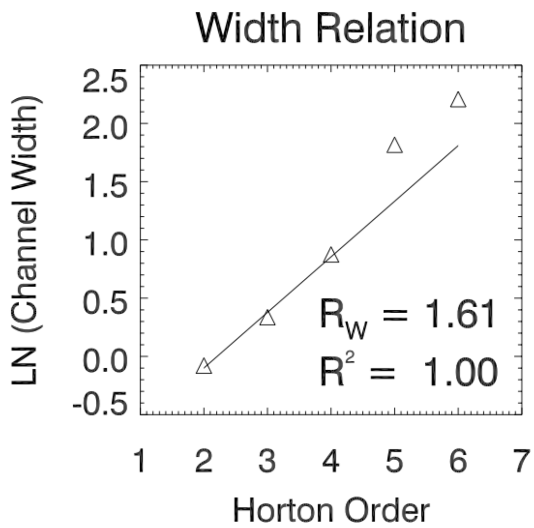

Fig. 6. Horton plots for Hydraulic Geometric variables (including order 2, 3, and 4) for Ashley river basin (McKerchar et al., 1998), results kindly provided by Mantilla (2014).

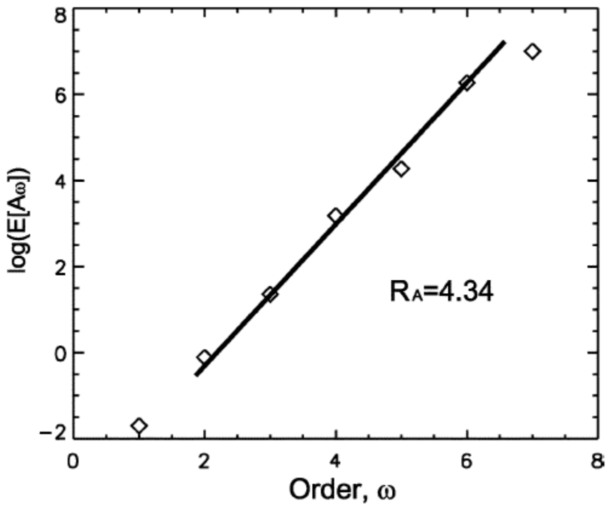

Fig. 7. Reproduction of the original figure of Mantilla and Gupta (2005, Fig. 2) showing the scaling of mean drainage area with order (Horton law) of the river network of the Whitewater Basin, Kansas, US.

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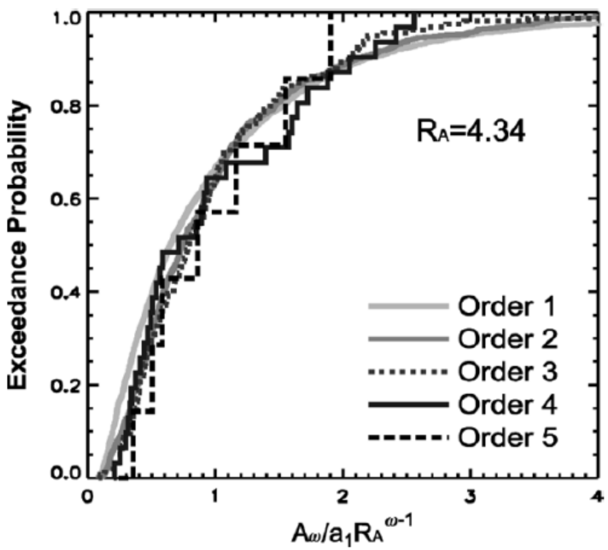

Fig. 8. Reproduction of the original figure of Mantilla and Gupta (2005, Fig. 2) showing the statistical scaling of the probability distribution of drainage area with order (generalized Horton law) of the river network of the Whitewater basin, Kansas, US.

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