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# Transient behavior in the Lorenz model

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## Abstract

Dynamical systems like the one described by the three-variable Lorenz model may serve as metaphors for complexity in nature. When natural systems are perturbed by external forcing factors, they tend to relax back to their equilibrium conditions after the forcing has shut off. Here we investigate the behavior of such transients in the Lorenz model by studying its trajectories initialized far away from the asymptotic attractor. Perhaps somewhat surprisingly, these transient trajectories exhibit complex routes and, among other things, sensitivity to initial conditions akin to that of the asymptotic behavior on the attractor. Thus, similar extreme events may lead to widely different variations before the perturbed system returns back to its statistical equilibrium.

## 1 Introduction

The by now famous Lorenz system (Lorenz, 1963), which arises via a truncation of Saltzman's equations (Saltzman, 1962) for convective motion – a paramount feature in climate – is described by the following system of ordinary differential equations:

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= -xz + rx - hy,\\ \dot{z} &= xy - bz.\end{aligned} \tag{1}$$

Here the dot denotes a time derivative, while the parameters $\sigma$ and $r$ correspond to Rayleigh and Prandtl numbers, respectively. The choice of parameters $\sigma = 10$, $r = 28$, $h = 1$, and $b = 8/3$ results in asymptotic (statistically equilibrated) aperiodic behavior on a strange attractor, with smooth trajectories alternating irregularly between loops around one of the two nontrivial unstable equilibrium points. The topological structure and properties of this attractor have been investigated and reported in a plethora of papers and books since the mid 1970s (see, for example, Sparrow, 1982).

Because of the great interest in the structural details of this – and other – chaotic attractors, their numerical simulations are usually initialized near the attractor itself. In this case the transients, defined as phase-space trajectories connecting the initial condition and the attractor, are short and uninteresting (Tsonis, 1992). Little attention thus far was, however, paid to the transient behavior in situations when the system is numerically integrated from the states located far from its asymptotic attractor. Investigating such transients is important because extreme far-from-equilibrium events do occur in nature due to either external forcing factors or due to self-amplifying interactions between various subcomponents of complex natural systems. Examples of the two types of phenomena in climate include the response of the climate system to forcing associated with volcanic aerosols and climate adjustment to particularly strong ENSO events, respectively; see (Ghil and Childress, 1987) for a topical account of other important climatic interactions. The purpose of this note is to point out some interesting properties of post-extreme-event transients in the Lorenz model.

## 2 Duration of transients and its relationship to trajectory-averaged local Lyapunov exponents

Asymptotically, as time $t \to \infty$, trajectories of the model Eq. (1) are confined within a bounded region $B$ of the model's $(x, y, z)$ phase space (Ghil and Childress, 1987). Here we objectively defined region $B$ numerically, as a rectangular cuboid with $x$-, $y$-, and $z$-ranges based on maximum and minimum values of the corresponding variables from a long model simulation initialized on the attractor; these ranges are (−19, 19), (−25, 25) and (4, 46), respectively. We also defined the approximate center of the attractor as the long-term time mean of $(x, y, z)$ from the same simulation: the point (0, 0, 24). We then performed numerical simulations of transient behavior in the Lorenz system (Eq. 1) using a set of extreme initial conditions equidistant from the attractor center so computed and thus located on a sphere $S$ with the radius $a = 150$; these initial conditions are all well beyond the attractor region $B$. The transients were defined





NPGD
1, 1905–1917, 2014

as trajectories emanating from the sphere $S$ and followed until their first entry into the region $B$.

The first characteristic of the Lorenz-system transients we looked at was their duration (Fig. 1). A typical time scale associated with a single revolution of the long-term trajectory around either lobe of the butterfly-shaped asymptotic attractor for our choice of model parameters is on the order of unity (not shown). This also happens to be the duration of the longest transients for initial conditions on the sphere $S$; the fastest transients take as short as 0.2 time units to reach the attractor region $B$, while the mean duration of transients is around 0.6 time units. The most striking property of the transient times distribution is, however, its non-uniformity and, in particular, the presence of two "blue" regions of initial conditions leading to extremely short-duration transient trajectories, as well as the presence of a relatively narrow "red" spiral belt of the initial conditions corresponding to the trajectories with the longest transient-period durations.

We will see later that longer-duration transient trajectories are also the ones that exhibit the most interesting evolution. A useful diagnostic for a potentially complex behavior is the trajectory-averaged maximum local Lyapunov exponents. The local Lyapunov exponents are defined as the eigenvalues of the dynamical operator $L$ for the tangent-linear model that describes the local spread of trajectories of our original model (Eq. 1) in the close neighborhood of an arbitrary point $(x_0, y_0, z_0)$ in the system's phase space:

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix} = L \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}; \qquad L = \begin{pmatrix} -\sigma & \sigma & 0 \\ -z_0 + r & -1 & -x_0 \\ -y_0 & x_0 & -b \end{pmatrix}. \tag{2}$$

We computed the maximum (in its real part) of the three local Lyapunov exponents for all points along each transient trajectory between the sphere $S$ and the asymptotic attractor region $B$ and took an average of these values to characterize a given trajectory (Fig. 2). Interestingly, the absolute majority of initial conditions on the sphere $S$ are characterized by the positive average Lyapunov exponents so obtained, thereby indicating potential sensitivity to initial conditions, which we will indeed confirm in Sect. 3





NPGD
1, 1905–1917, 2014

below. Furthermore, there is a clear correspondence between the pattern of trajectory-averaged maximum local Lyapunov exponents in Fig. 2 and that of the transient duration in Fig. 1. In particular, the "blue" regions on the sphere $S$ that initialize the transients with fastest decay toward the asymptotic attractor are also the regions with negative average Lyapunov exponents, while the "red" ribbon of initial states corresponding to the longest transients is also the region of the maximum positive trajectory-averaged Lyapunov exponents. Thus, the longest transients are naturally associated with trajectories that tend to "travel sideways" along the dynamical slopes of the Lorenz-system landscape, rather than go straight downhill toward the asymptotic attractor.

## 3 Types of transient behavior

### 3.1 Sensitivity of transients to initial conditions

A typical example of transient behavior for initial states chosen near the ribbon of longest transient times (Fig. 1) or, equivalently, that of largest trajectory-averaged Lyapunov exponents (Fig. 2), is shown in Fig. 3. Here a bunch of trajectories that emanate from close-by initial conditions splits in two diverging sets of trajectories, which follow very different routes prior to reuniting near the asymptotic attractor location; the latter attractor region is evident as a small butterfly-shaped cluster of trajectories close to the origin. The immediate consequence of such transient behavior in the Lorenz system is that it can apparently be as unpredictable as the asymptotic behavior in the sense that similar extreme perturbations may result in completely unrelated transients as the system relaxes back to the state of its statistical equilibrium.

### 3.2 Geometric complexity of transients

Another interesting observation is that transient approach to the asymptotic attractor may be characterized by fairly complex trajectories, which appear, in some cases, to emulate the attractor itself (Fig. 4a and b). In particular, the trajectories here exhibit





NPGD

1, 1905–1917, 2014

larger-scale butterfly-shaped excursions prior to ending up on the similarly shaped asymptotic attractor near the origin. We will refer to this phenomenon as to the "ghost" transient attractor.

Qualitatively, this behavior can be understood in the following way. We saw previously that some transients are able to stay away from the attractor for longer time periods than others (Fig. 1). For such longer transient trajectories, the variables in the model (Eq. 1) can be locally rescaled in space and time to focus on their relatively persistent (large) local phase-space distances from the origin and fast phase speeds. Effectively, this rescaling will produce the system of equations completely analogous to the Eq. (1), but with different set of parameters ($\sigma$, $r$, $h$, $b$). If so, it may not seem improbable that for some regions of the phase space, this "transient" Lorenz model will exhibit dynamical structures and trajectory shapes akin to those known to arise for other parameter sets in the asymptotic limit of statistically equilibrated behavior.

To present a concrete example of the rescaling mentioned above, we introduce the following change of variables

$$(x', y', z', t) = \varepsilon(x, y, z, t'). \tag{3}$$

Note that for $\varepsilon < 1$, Eq. (3) corresponds to squishing the spatial coordinates and stretching the time so that the large values of the non-transformed variables on the order of $\varepsilon^{-1}$ will correspond to the transformed variable values on the order of one, while the order-of-one changes in the stretched time $t'$ will span the short interval on the order of $\varepsilon$ when measured in original time units $t$. This rescaling is thus appropriate, in principle, for the trajectories in the region situated far from the origin and during relatively short transient period before returning to the asymptotic attractor behavior.

Substituting the transformation (Eq. 3) into the system (Eq. 1) and introducing the new set of parameters

$$(\sigma', r', h', b') = \varepsilon(\sigma, r, h, b) \tag{4}$$

results, for this case, in the *same* system of equations as the original system (Eq. 1), but for the primed variables. This implies that topological behavior of the trajectories that





NPGD
1, 1905–1917, 2014

are somehow *able to persist* in a far-away-from-attractor region of the phase space in which Eq. (3) is valid can be qualitatively described by the asymptotic behavior of the Lorenz system with a different set of parameters rescaled by $\varepsilon^{-1}$, where $\varepsilon$ corresponds to the ratio of the radius-vector of defining the far-away location of the persistence region to the typical value of the trajectories' radius-vector in the original Lorenz system. Indeed, the topology of the Lorenz system (Eq. 1) with $(\sigma, r, h, b)$ parameters rescaled in this way using the value of $\varepsilon = 1/3$ (Fig. 4c and d) looks qualitatively similar to the transient trajectories in Fig. 4a and b.

Note that in the rescaling example Eqs. (3) and (4), the notion of the single parameter $\varepsilon$ controlling the stretching of all three phase-space variables, as well as time, is completely arbitrary. Furthermore, and more importantly, the local stretching (Eq. 3) tells nothing about where in the phase space the transient trajectories must be for the stretched regime to be persistent; the latter persistence is essential for these trajectories to have sufficient time to reveal the structure of the stretched-system attractor during transient evolution. For these reasons, a qualitative demonstration above should be regarded as nothing more as an empirical one-parameter fit to illustrate the concept of the "ghost" transient attractor.

## 4 Summary and discussion

We studied transient behavior in numerical simulations of the three-variable Lorenz model (Eq. 1) initialized far away from the region of its asymptotic chaotic attractor. These transients were shown to have a range of durations, with the longest transients corresponding to the trajectories having largest average Lyapunov exponents and complex routes emulating sensitivity to initial conditions, as well as exhibiting the "ghost" attractors akin to their asymptotic siblings.

Persistent chaotic transients in the Lorenz system have been studied before in the particular case when the Rayleigh number was chosen to be just below the critical value required for chaotic behavior (Shimizu and Morioka, 1978; Yorke and Yorke,





NPGD
1, 1905–1917, 2014

1979). With this choice of parameters, the model trajectories initially evolved along the attractor that was close to the asymptotic chaotic attractor of the system with a slightly higher Rayleigh number, but slowly decayed from chaos to the final state of a steady flow. This situation is very different from the one considered in the present paper, where the parameters of the Lorenz model were set to correspond to the chaotic regime; the non-trivial transients arise here due to the dynamical properties of the system considered in the phase space regions situated far from the asymptotic attractor.

Transient behavior of dynamical systems recently drew a lot of attention in the ecological literature (see Hastings, 2004, and references therein). The discussion in (Hastings, 2004) evolved around recognizing the fact that the transient behavior is closely associated with the inherently multi-scale character of natural systems, including the timescale asymmetries stemming from the presence of the stable and unstable manifolds in these systems' dynamics; incidentally, this presence is the root of the strange chaotic attractor in the Lorenz model. Cushing et al. (1998) described laboratory experiments and numerical simulations of the transient behavior in an underlying population model, which depended on the choice of the initial conditions near the stable or, alternatively, unstable manifold of an equilibrium point of this model. This sensitivity of the transient evolution to initial conditions is qualitatively similar to the behavior we report here, but involves completely different dynamics, which lack, for example, the "ghost-attractor" behavior.

The properties of the transient behavior in the Lorenz model discussed here – which are likely to be typical for arbitrary equations exhibiting chaos – are not just beautiful, but may also have important implications in understanding the evolution of complex nonlinear systems such as climate, economy, ecosystems, sociological networks and so on, if these systems are somehow taken far from their equilibrium states. In particular, similar extreme perturbations in such systems may exhibit widely different variations before relaxing back to the statistical equilibrium.

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NPGD
1, 1905–1917, 2014

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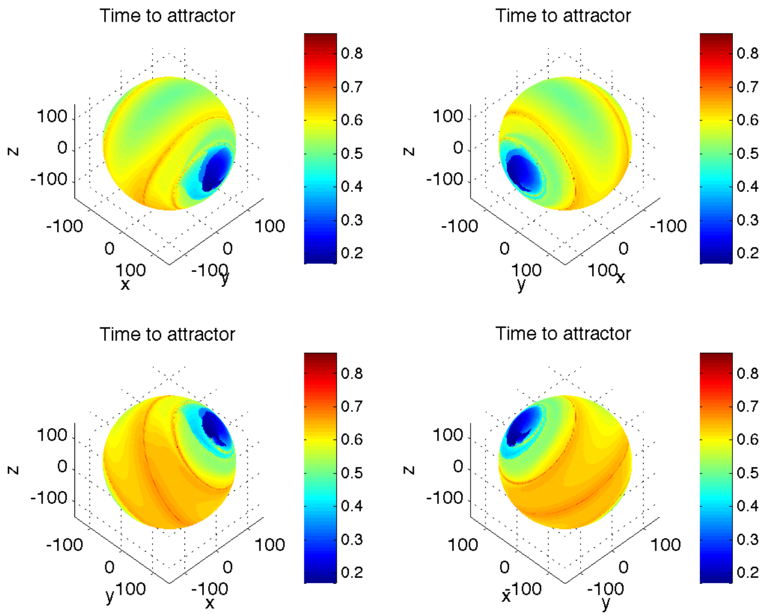


**Figure 1.** Distribution of transient times (color shading) to the Lorenz attractor for initial conditions on the sphere $S$ with the radius $a = 150$ centered at the point (0, 0, 24); the center of this sphere was chosen to be close to the asymptotic time mean of trajectories simulated by the Lorenz model (Eq. 1) with $\sigma = 10$, $r = 28$, $h = 1$, and $b = 8/3$. The trajectory initialized on $S$ was considered transient until its first entry into the rectangular cuboid region $B$ bounded by $x$-, $y$- and $z$-ranges of (–19, 19), (–25, 25) and (4, 46), respectively. The Lorenz attractor for the model parameters considered is located within this region. The four figure panels display the same quantity, but from different view angles. *Comment*: note non-uniformity of the transient-time distribution, with a spiraling belt of relatively long durations.





NPGD

1, 1905–1917, 2014

Transient behavior in the Lorenz model

S. Kravtsov et al.

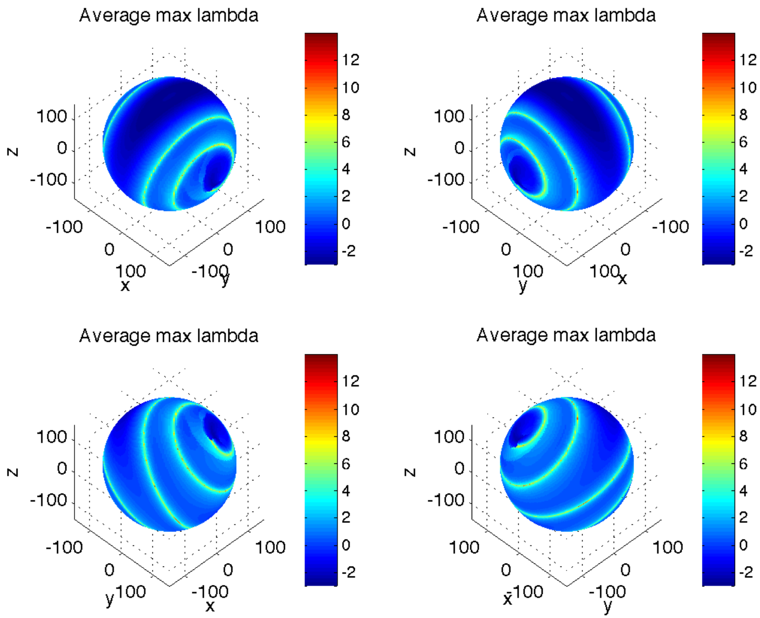


**Figure 2.** Distribution of the averaged maximum local Lyapunov exponents computed over the transient portion of trajectories (before first entry into the attractor region *B*) initialized on the same sphere *S* as in Fig. 1. The same four orientations as in Fig. 1 are shown. *Comments*: most of the trajectories exhibit positive Lyapunov-exponent averages. There is a clear correspondence between the spiraling belt of largest averaged Lyapunov exponents and that of longest transient duration times in Fig. 1.





NPGD
1, 1905–1917, 2014
Transient behavior in the Lorenz model
S. Kravtsov et al.

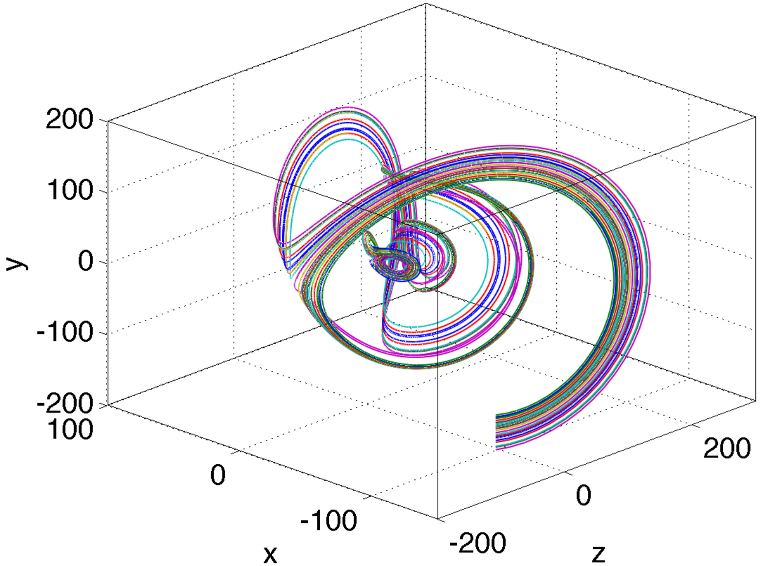


**Figure 3.** An example of transient trajectories sensitive to initial conditions. *Comment*: this is a typical situation for initial conditions taken in and around the spiraling belts of longest transient times (Fig. 1) and highest averaged Lyapunov exponents (Fig. 2).





NPGD
1, 1905–1917, 2014

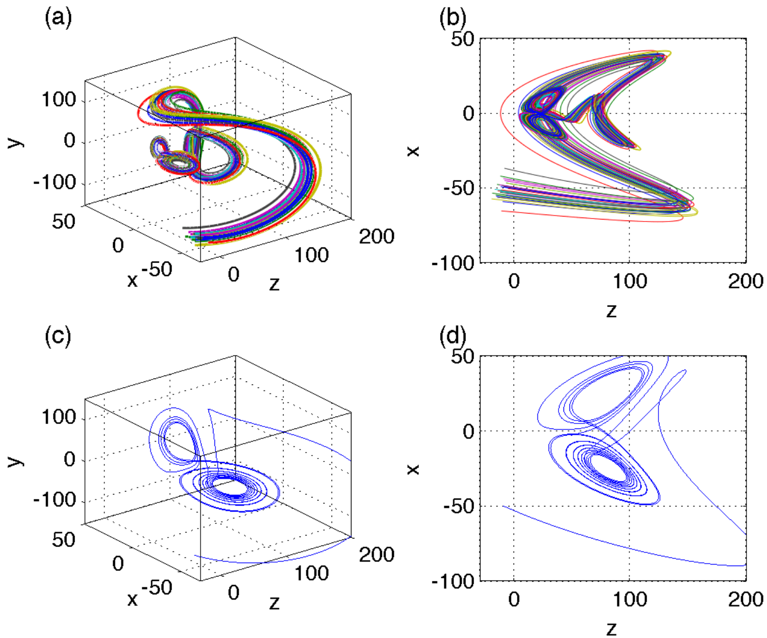


**Figure 4.** (**a** and **b**) An example of a "ghost" transient attractor in the simulation of the Lorenz model (Eq. 1) with $\sigma = 10$, $r = 28$, $h = 1$, and $b = 8/3$; (**c** and **d**) a trajectory of the Lorenz system with parameters $(\sigma, r, h, b)$ stretched by $\varepsilon^{-1} = 3$. See text for details. *Comments*: the transient trajectory in (**a** and **b**) is not a mere spiraling toward the attractor, but exhibits a complex path reminiscent of that on asymptotic attractor. The path of rescaled Lorenz system in (**c** and **d**) shares geometrical similarity with the transient path in (**a** and **b**).





NPGD
1, 1905–1917, 2014