Nonlinear Processes in Geophysics (2000) 7: 117–126

Nonlinear Processes in Geophysics

Deterministic description of a phase transition in a medium of interacting waves

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Received: 25 May 1999 – Accepted: 31 August 1999

Abstract. We describe an effect of phase-locking catastrophe arising in an ensemble of a great number of oscillators interacting by means of their emitting waves. These waves can be either pulsatile, that is, soliton-like, or continuous stationary waves generated by the oscillators considered as resonators. Each one of these waves will introduce certain perturbations among the phases of the oscillators of the ensemble in such a way that it is possible to follow in time the distribution of these phases. In fact, we deduce the p.d.e's governing the evolution in time of this distribution, which displays a tendency of accumulating around certain of its values (phase-locking), and also of sudden increasing of the intensity of the physical effect (a "phase transition").

1 Introduction

The sudden emergence of a coherent wave of high intensity in a medium which up to then was the siege of a great number of (apparently) random incoherent identical waves with very small intensity in weak interaction is a phenomenon arising in many different fields, from earthquakes in geophysics to brain waves in biophysics. Generally speaking they can be gathered under the general denomination of phase transitions, and considered as such. In fact, such "big" waves are due to some particular, non random distributions of the phase differences of the small waves, creating a superposition with great values for the resulting amplitude. It is the aim of this paper to describe, without any "heuristic" hypothesis, and in a deterministic (that is, non probabilistic) way some physical models that may account for such phenomena, and deduce the corresponding evolution equations leading to such abrupt bifurcations.

Obviously the whole theory rests on the physical nature of the elementary components (small waves, oscillators emitting these waves, etc.) of the ensemble, on the different kinds of interaction that may take place between them, and also on the number of such components of the ensemble. We shall then start by some previous considerations about the same phenomenon with only two interacting identical oscillators. Not only the mathematical treatment can be carried much further, but also this case will provide some useful information about the corresponding phenomenon in an ensemble with many oscillators.

2 The Huyghens effect (two inter- acting clocks)

Though this effect has always been well known in the theory of nonlinear and nonconservative oscillations yet it remained unexplained up to very recently, and lies at the basis of many subsequent developments, namely those presented in this paper. Huyghens himself described very precisely that effect in a letter to his father (Huyghens, 1893): Whenever two identical clocks were put near one another, the phase difference between their pendula tend to a certain asymptotical constant value. This was usually the opposition of phases, though other values could also occur depending on the initial conditions, the structure of the clocks, etc. (It should be reminded that Huyghens was, so to speak, the inventor of modern clocks, provided with the so-called escapement system that was going to change so deeply the measure of time.). The physical idea underlying this phenomenon may be resumed as follows :

In its evolution the pendulum of each (isolated) clock undergoes a shock with its own escapment, by means of which small equal amounts of energy are periodically supplied to the pendulum as a compensation for its losses by dissipation, friction, etc. In clocks with simple internal structure, these shocks arise once in a period and whenever the pendulum goes through a certain fixed value of the phase, usually $\phi = \pi/2$. These shocks are simply the tick-tock we could hear in all clocks up to the appearance of our modern quartz clocks. (Must it be reminded that, precise and reliable as these later clocks are, they still call for a good theory accounting for their marvelous properties?). It seems natural to assume that in each of these "tocks" (due to the collision, in each clock, of the arm of the pendulum with the trigger of the escapment) a wave-like soliton is created, with very small amplitude (energy), that propagates in space and will perturb the other clock (if it is not too far), then giving rise in its pendulum to a small modification of the phase. A straightforward calculation gives for this perturbation the value $\delta = \pm \aleph \sin \phi_0$ where ϕ_0 is the phase of the clock immediately before the collision, and $\aleph$ is a small positive constant that depends on the physical model of the clock. Sign + (resp -) must be taken if the soliton comes from the right (resp. left). Now this <u>second</u> clock, when undergoing its own next "tock", generates its own solitonic wave, identical to the above one with the only difference of the slight modification of the phase. It then follows that the <u>first</u> clock, when receiving the soliton coming from the second one, actually receives a perturbation in which he has, so to speak, introduced a modification of its own, that is, an information about its own state. And so on. This is the way by which the phase difference between the two clocks changes in time. Happily enough, we have at our disposal a beautiful theory of the clock given by Andronov in the early forties (Andronov et al., 1963) which is the exactly fitted tool we needed in order to describe the Huyghens effect. The Andronov clock is simply a damped harmonic oscillator (the dissipation being due to "fluid" friction, i.e., proportional to the velocity) which is periodically supplied in energy from the outside. The amount of energy in each supply is always the same and takes place instantaneously once in a period whenever the oscillator takes a certain fixed value of its phase. It is then easily seen (Andronov et al., 1963) that such non linear, dissipative system is a limit cycle. With it, it is possible to deduce an o.d.e. governing the evolution of the phase difference $\tau(t)$ between two clocks according to the reasoning above. The deduction can be seen in (Vassalo Pereira, 1981) and leads to

$$\tau(t) \quad : \qquad \frac{d\tau}{dt} = -\Gamma \cos \tau$$

where Γ is a positive constant depending on the physical characteristics of the clock. From its exact analytical solution

$$tg\,\frac{\tau(t)}{2} \quad = \quad \frac{\left(1 + tg\,\frac{\tau_0}{2}\right)\exp\left(-\Gamma(t - t_0)\right) - \left(1 - tg\,\frac{\tau_0}{2}\right)}{\left(1 + tg\,\frac{\tau_0}{2}\right)\exp\left(-\Gamma(t - t_0)\right) + \left(1 - tg\,\frac{\tau_0}{2}\right)}$$

$$\tau_0 \equiv \tau(t = t_0)$$

we may derive the essential properties of the system, namely the asymptotical values $\tau_\infty \equiv \tau(t = \infty) = \frac{3\pi}{2}, \frac{7\pi}{2}$ of the phase difference, which are precisely those observed by Huyghens three centuries ago.

Let us point out that the Andronov limit cycle fulfills some of the physical requirements expected for a sound description of the effect, namely, periodicity, dissipation of energy, and stability. This last requirement implies the existence of a strong attractive basin for the limit cycle such that its representative point (RP) in coordinates r, ϕ never leaves the neighbourhood of the unperturbed orbit. More precisely, the perturbation arising from the small solitons originating in the other clocks introduces a change in the phase ϕ but does not modify the unperturbed value of the radius vector r. This "separation" of the variables is an essential ingredient throughout the theory in as much as the state of a clock can thus be given merely by the value of its phase regardless the unperturbed value of the radius vector. This fact has a far reaching importance when we go over to the description of a great number of such clocks in interaction. One could only object that the Andronov limit cycle model for the clock is perhaps not a very suitable one, according to some reasons of structural stability. Yet the essential features of the above mentioned results are broadly independent of that particular model, as it was shown by Abraham in a paper (Abraham, 1990) where the original theory was generalized both from the mathematical and the physical point of view.

3 An ensemble of oscillators (limit cycles) with pulsatile interaction

The next step in the theory is naturally the description of an ensemble of N>>1 identical Andronov oscillators, that is, limit cycle clocks in pulsatile interaction as it was described above. Each particular clock of the ensemble is now enduring small phase perturbations under the form of collisions due to small identical solitons proceeding from the other N-1 clocks. That same particular clock, when emitting its own solitons, will in turn perturb the phase values of the other N-1 clocks. Now since the state of any clock is unambiguously given by the value of its phase (that is ϕ, instead of r and ϕ - see the hypothesis above - it follows that the state of the whole ensemble can be given by means of some distribution function $\omega(t,\phi)$ such that $N.\omega(t,\phi)\, d\phi$ is the number of oscillators of the ensemble with values of the phase within the interval $(\phi, \phi + d\phi)$ at instant t.

Before we go on by briefly exposing the deduction of the evolution equation for $\omega(t,\phi)$, we must previously recall a useful result in probability theory concerning the composition (convolution) of a great number of gaussian distributions with slowly varying dispersion. It can be viewed as a consequence of the central limit theorem and it states that the final distribution is still of gaussian-type with amplitude and dispersion given by certain functionals depending on the cor- responding quantities of the distribution components. To be more precise:

Let x be some parameter starting at instant $t = t_0$ with value $x(t = t_0) \equiv x_0$, and enduring random variations of magnitude $\Delta x = \pm\delta(t)$ (with equal probability for having sign + or -). Then the probability density for x to be $x(t = t') \equiv x'$ at instant $t = t'$ is given by

$$P\{x_0, t_0; x', t'\} = \left(4\pi\int_{t_0}^{t'} D(s)ds\right)^{\frac{1}{2}} . \exp\left(\frac{-(x' - x_0)^2}{4\int_{t_0}^{t} D(s)ds}\right),$$

with $D(t) \equiv \frac{1}{2}\delta^2(t)\rho(t)$, and $\rho(t)$ denoting the number of variations/perturbations endured by the parameter x in unit time at instant t. (This result is a straightforward generalisation of the well known particular case with δ, ρ constants, available in any textbook)

Let us then follow along time interval $(t, t + dt)$ some particular oscillator in the ensemble, whose phase is incessantly perturbed by the collisions with the small identical solitons proceeding from the other N-1 oscillators. If we assume that each solitonic wave is created whenever the phase of some clock goes through the value $\phi = \pi/2$ of its phase, then the number of perturbations endured by a particular oscillator during $(t, t + dt)$ is given by

$\rho(t)\,dt = N\,\omega(t,\phi = \pi/2)\,dt$. As it was said above, the magnitude of each perturbation is given by $\delta = \pm\aleph\sin\phi$. (Here we may of course assume that we have equal a priori probability for any soliton to come either from the left or from the right). In the limit $\aleph = 0$ we have the unperturbed case: the state of each clock is totally independent from the others and ω is then given by some "ne varietur" function of argument $\phi - t$.

Let then be one among the N clocks having a certain value $\phi(t_0) = \phi_0$ of its phase at a given instant t_0. At instant $t_0 + dt$ its phase will then be $\phi_0 - dt - \theta$, where θ is the deviation (with regard to the unperturbed value) due to the collisions with the solitonic waves coming from the other N-1 clocks. According to the above mentioned result on gaussian distributions, the probability density for the clock to have phase $\phi_0 - dt - \theta$ at instant $t_0 + dt$, starting from the initial condition $\phi(t_0) = \phi_0$, is equa lto

$$P\{\phi_0, t_0; \phi - dt - \theta, t_0 + dt\} = \left(4\pi D(t_0)dt\right)^{-\frac{1}{2}} \exp\left(\frac{-\theta^2}{4D(t_0)dt}\right)$$

with $\delta^2(t) = \aleph^2 \sin^2\phi(t) \cong \aleph^2 \sin^2\phi(t_0 - dt)$. It follows that the number of clocks of the ensemble whose phases at instant $t_0 + dt$ are found in $(\phi, \phi + d\phi)$ is given by the usual convolution, that is,

$$\omega(t,\phi) = \int \omega(t_0,\phi).\,P\{s,t_0; s - dt - \theta, t_0 + dt\}ds =$$

$$= \int \frac{\omega(t_0,\phi)}{\sqrt{4\pi D(t_0)dt}} \exp\left(-\frac{(s - dt - \phi)^2}{4D(t_0)dt}\right) ds$$

By making use of an approximative algorithm for the sharp gaussian function in the integrand (and without assuming any further <u>physical</u> hypothesis!) it is possible to write this integral equation for $\omega(t,\phi)$ under the equivalent form of a partial differential equation. This (strictly mathematical) deduction can be seen in detail in (Vassalo Pereira, 1989) and leads to the following nonlinear parabolic p.d.e.,

$$\frac{\partial}{\partial t}\omega(t,\phi) = \frac{\partial}{\partial\phi}\omega(t,\phi) + \frac{N\aleph^2}{6}\omega(t,\phi = \tfrac{\pi}{2})\frac{\partial^2}{\partial\phi^2}\left(\omega(t,\phi).\sin^2\phi\right)$$

Clearly if the interaction between the oscillators (that is, the amplitude of the solitons) is neglected, then $\aleph = 0$. Our equation thus takes the simplified form

$$\frac{\partial\omega}{\partial t} = \frac{\partial\omega}{\partial\phi}$$

and we have $\omega(t,\phi) = f(\phi - t)$, that is, a "ne varietur" distribution of the phases in time. The parabolic nature of this final equation is not surprising for it is well known that, in broad terms, such equations (Fourier, Fokker-Planck, etc.) always emerge in close connection with the physical description of diffusion fields (such as heat, brownian motion, etc) where random microscopical collisions play an essential role. Now this is precisely what happens in our model, with the perturbations due to the collisions of the solitons with the clocks. This remark will prove useful later, when dealing with perturbations of an utterly different kind (continuous stationary waves), then leading to hyperbolic instead of parabolic state equations - see below equations (1) and (2) - Some non linear parabolic equations of this kind are studied in mathematical physics under the denomination of "reaction-diffusion equations", and their behaviour often includes the existence of phase-locking-type solutions, cascade frequencies, etc. In the present case, such study can be achieved by writing $\omega(t,\phi)$ as a Fourier series and analysing the infinite system of ordinary differential equations obtained for the Fourier coefficients. We can also write $\omega(t,\phi)$ as a power series of the small parameter $\frac{1}{6}N\aleph^2$, and integrate the sequence of first order linear p.d.e providing the sequential coefficients of the series (each p.d.e requiring the solution of the preceding one). An application of this last method was given in (Karatchentzeff et al., 1994), where we could find the multiplication of frequencies and secular terms associated to different orders of the approximation.

Let us point out that the phase-locking strongly depends on the value of $\frac{1}{6}N\aleph^2$, a "small" parameter which carries inform-ation concerning the dimension of the ensemble as well as the interaction intensity among the elements of the ensemble, since it is the product of the "big" integer N (the number of oscillators of the ensemble) by the "very small" constant $\aleph$ (the amplitude of the solitons).

4 An ensemble of N>1 resonators with continuous wave-like interaction

In the preceding model each clock produces more or less instantaneously a small soliton once in each period and whenever the emitting clock goes over some particular state of its own (In the Andronov clock such state is simply specified by the value $\phi = \pi/2$ of the phase). It is by means of these solitons that the oscillators of the ensemble interact. Now a more realistic model would obviously require a continuous instead of a pulsatile interaction, that is, with each oscillator perturbing the other N-1 at any instant of time (and not only at the instants of recurrence upon some particular state) and carrying information of the instantaneous state of the perturbing clock at any moment.

It then seems natural to assume, in an ameliorated model, that each oscillator perturb the remainder N-1 by means of a stationary wave generated and maintained by the oscilator itself in its vibrating motion (a model quite similar to Planck resonators supporting the black body field of radiation), that is, with very small amplitude and with phase coinciding at any instant with the phase of the oscillator itself. As we are going to see, this new and also mathematicaly more demanding model displays some very interesting properties that are absent in the preceding ones (namely, the possibility of phase transitions). Let us then make a few preliminary comments about the essential "ingredients" of this new model:

a) <u>Oscillators with strong "radial stability"</u>.

For the same reasons that were invoked above, we also need here an oscillator endowed with some kind of stability and dissipation of energy. Such was the case of the Andronov clock which proved so useful in the above model, and if we are now compelled to give it up, that is "merely" due to mathematical, not physical reasons. In other words, if we retained the Andronov clock in our new model it would prove technically impossible to carry on the analytical calculations after a certain step of the theory. We thus have to look for another, more mathematically "workable" oscillator replacing the former one and, of course, fulfilling the above required physical properties. Now this oscillator will be simply an harmonic perturbed oscillator coupled to some self-acting device (that one may think of as an "ersatz" of the trigger of a clock) acting once in a period and bringing instantaneously the (perturbed) value of the amplitude to the corresponding unperturbed value. By so doing, the essential property is maintained, that is : the amplitude (but, of course, not the phase) of the perturbed oscillator always lies in the neighbourhood of the unperturbed (constant) value. As it happened with our first model, the state of any oscillator is thus entirely defined by the sole value of its phase, since the amplitude may be regarded as keeping the same, unperturbed value in course of time. Roughly speaking, this oscillator may be regarded as a "coarse limit cycle" whose interest lies mainly on the fact that we may follow the values of the perturbed phase while being sure that the amplitude will not diverge. It then follows that when we consider a great number of such oscillators, the state of this ensemble will then be completely defined by means of some distribution on the phases, that is, by the density function $\omega(t,\phi)$ introduced above and whose physical meaning is kept in this new model.

b) The theorem of Rayleigh

Since each resonator produces a stationary wave of the same kind (same period and same small amplitude), any resonator will then be perturbed by a small stationary wave which is the superposition of $N-1 \cong N$ similar waves, and whose phase depends on the instantaneous distribution of the phases of the ensemble, that is, on the density function $\omega(t,\phi)$. We now make use of a theorem (Rayleigh, 1965) which is a fundamental tool in order to deduce the probability distribution governing the values of the phase of that superpositon. We now briefly expose its content: Let us consider a great number N of harmonic vibrations, all with same period and same amplitude κ. The resulting superposition of such waves will then be a (non-harmonic) vibration with same period, and whose amplitude R and phase Φ depend on the phase differences between the component vibrations. Clearly, a probability distribution for the phases of the N components will induce some certain probability distribution $f = f(R,\Phi;N)$ for the amplitude and phase of the resulting wave. Now what Rayleigh did was to assume equal "a priori" probability for the phase components and deduce from it a p.d.e. for $f = f(R,\Phi;N)$ the which, strangely enough, was but the Fourier equation for heat diffusion! More precisely, and written in cartesian coordinates $x = R\cos\Phi$, $y = R\sin\Phi$, he found

$$f(R,\Phi;N) = f\left(R = \sqrt{x^2+y^2}, \Phi = tg^{-1}\left(\frac{y}{x}\right)\right) \quad :$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{4}{\kappa^2}\frac{\partial f}{\partial N}$$

The normalised solution is the gaussian distribution (see figure 1)

$$f(N,x,y) = \frac{1}{\pi\kappa^2 N}\exp\left(-\frac{x^2+y^2}{\kappa^2 N}\right),$$

with its well-known properties: The probability is the same for a fixed value of the resulting amplitude independently of the resulting phase; it takes its maximum value for null amplitude of the resulting vibration; it tends quickly to zero for increasing values of

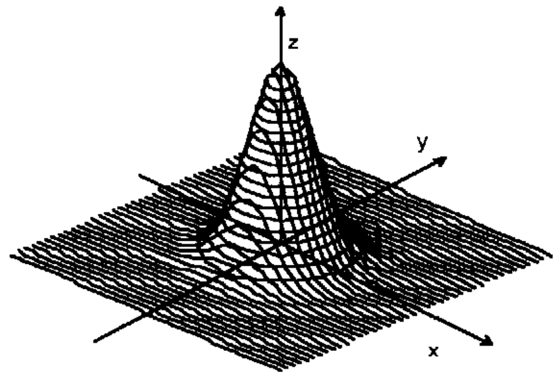

Figure 1. Gaussian distribution.

the resulting amplitude, etc. Let us recall that, strictly speaking, the resulting amplitude of N identical harmonic vibrations with same amplitude κ can never be greater than $N\kappa$. This apparently trivial remark will appear under a different light when giving up the hypothesis of equal "a priori" probability of the phases, as we shall see.

In fact, in order to account for the physical properties of our new model we must look for a generalisation of the theorem of Rayleigh for the case when the distribution of the phases of the elementary vibrations, that is, the density function $\omega(t,\phi)$, is not a constant, which means that we no longer have equal probability regardless the values of the phases). As it is shown in Appendix, where we give the deduction of the generalized theorem of Rayleigh, we now find a new linear p.d.e. giving the probability distribution $f(R,\Phi;N) = f(x,y;N)$ for the amplitude and phase of the resulting wave, whose coefficients are functionals of the density phase function $\omega(t,\phi)$, depending in a simple way upon the first Fourier coefficients of $\omega(t,\phi)$. To be more precise, we have

$$A\frac{\partial^2 f}{\partial x^2} + 2B\frac{\partial^2 f}{\partial x\,\partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} = \frac{\partial f}{\partial N}$$

where

$$A = \frac{\kappa^2}{4}\left(1+\pi\alpha^2\right), \qquad B = \frac{\kappa^2}{4}\pi\beta^2$$

Actually, and for mathematical simplicity, the Fourier coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ do not refer to the function $\omega(t,\phi)$ itself but to $\Omega(t,\psi) \equiv \omega(t,\phi = -t+\psi)$, which is obviously the same function ω now described in a rotating frame turning with the angular velocity of the unperturbed phase $\phi(t) = -t$. In other words, $\Omega(\psi)d\psi$ is the probability for the representative point of any oscillator of the ensemble to have a phase difference with regard to the unperturbed phase $\phi(t) = -t$ within $(\psi, \psi + d\psi)$. The physical difference is irrelevant and the mathematical treatment becomes much simpler.

In the particular case considered by Rayleigh of equal probability in the phases, we have

$$\Omega(\psi) = \text{const} = (2\pi)^{-1},$$

and we obtain the simplified form of the original equation.

Now the geometrical structure of the solution of this generalised equation is very different from the above, non-generalised one. In fact, for some values of the coefficients, i.e., for some values of the Fourier coefficients of $\Omega(t,\psi)$ - or, equivalently, of $\omega(t,\phi)$ - it shows a gaussian-type structure, not very different from the usual, non generalised form studied by Rayleigh. To be more precise (see Appendix for the details), as long as we have $\Delta \equiv 1 - \pi^2(\alpha_2^2 + \beta_2^2) > 0$, where α_k, β_k denote the second Fourier coefficients of $\Omega(t,\psi)$, we have for f the analytical expression (3) of the Appendix, corresponding to the geometrical form shown in figure 2. In particular, such must be the case at the initial stage of the evolution of the ensemble, for it seems natural to

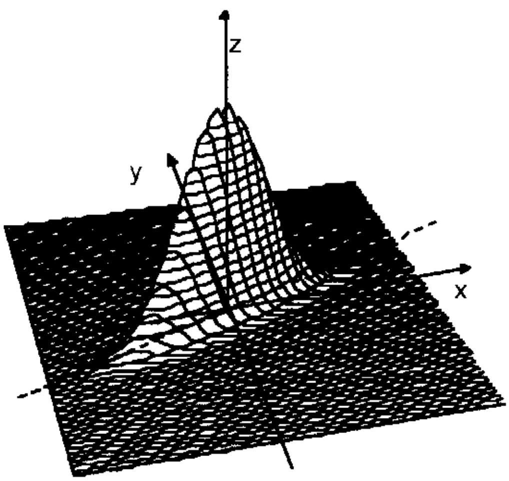

Figure 2. Gaussian-like distribution.

assume an equal lacking of information about the state of the oscillators, that is, $\alpha_k = \beta_k = 0 \quad (k \geq 1)$, which implies $\Delta \cong 1 > 0$. Yet if at some instant of time α_2, β_2 are such that $\Delta \equiv 1 - \pi^2(\alpha_2^2 + \beta_2^2) < 0$, (and nothing "a priori" prevents such possibility), then $f = f(R, \Phi; N)$ takes the anaytical expression (4) of the Appendix, corresponding to a very different, hyperboloid-like structure, which no longer tends to zero for increasing values of R. This means that the above trivial remark concerning the rigorous null probability for a resulting amplitude of N identical harmonic vibrations with same amplitude κ to be greater than $N\kappa$, now appears of utmost importance, implying for f the discontinuous geometrical forms shown in figures 3 and 4 with $f = 0$ for $R^2 \equiv x^2 + y^2 \geq (N\kappa)^2$. As we are going to show, it is this sudden and radical change of the geom -

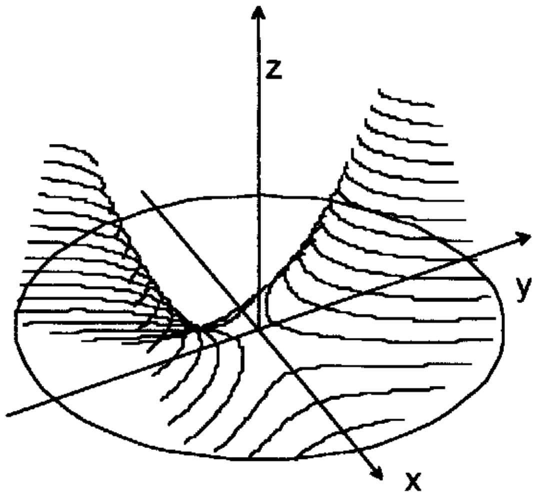

Figure 3. Hyperboloid-like distribution.

etrical properties of the Rayleigh distribution $f = f(R, \Phi; N)$ that is responsible for a bifurcation phenomenon in the analytical structure of the distribution of the phases $\omega(t,\phi)$.

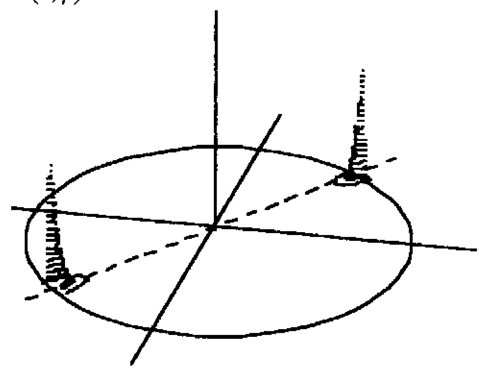

Figure 4. Hyperboloid-like distribution (near the limit).

Let us then expose the underlying physical idea of this model:

We thus start with $N >> 1$ identical oscillators of the above described type, whose phases are distributed according to $\omega(t,\phi)$, i.e., with $N.\omega(t,\phi)\,d\phi$ oscillators with phase in the interval $(\phi, \phi + d\phi)$ at instant t. Each oscillator acts like a resonator, i.e., it generates a stationary wave with very small amplitude κ, and phase coinciding at any instant with that of the oscillator itself. Any oscillator then comes perturbed by the superposition of the waves originated by the other $N - 1$, that is, if we denote by ϕ_j the phase of oscillator n° j, then oscillator n° i is perturbed by a small wave that can be written under the form

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \kappa \sin \phi_j \cong \sum_{j=1}^{N} \kappa \sin \phi_j = \sum_{j=1}^{N} \kappa \sin\left(-t + (\phi_j + t)\right)$$

$$= R_0 \sin\left(-t + \Phi_0\right)$$

Now since $\omega(t,\phi)$ is not constant, we make use of the generalised Rayleigh equation in order to calculate the probability for the superposition of the $N \cong N-1$ waves to have certain values R_0, Φ_0 for the resulting amplitude and phase, the which is precisely provided by the Rayleigh function $f(N,x,y)$. In other words

$$f(N, x = R_0 \cos\Phi_0, y = R_0 \sin\Phi_0)\, dxdy,$$

with

$$dx = dR_0 \cos\Phi_0 - d\Phi_0 R_0 \sin\Phi_0$$
$$dy = dR_0 \sin\Phi_0 + d\Phi_0 R_0 \cos\Phi_0 \;,$$

is the probability for that superposition to have, at instant t, its amplitude and phase within the intervals $(R_0, R_0 + dR_0)$ and $(\Phi_0, \Phi_0 + d\Phi_0)$. The deduction of the equation governing the evolution of $\omega(t,\phi)$ in time can now be described in the following way:

Any oscillator, starting with phase ϕ at instant t, would have, if unperturbed, a certain value $\bar{\phi}$ of its phase at instant t+dt. But due to the small waves generated by the other $N \cong N-1$ oscillators, that particular oscillator is perturbed by a small wave which is the sum of these $N \cong N-1$ waves, and which has precisely the probability $f(N, x = R_0\cos\Phi_0, y = R_0 \sin\Phi_0)\, dxdy$ of taking certain values R_0, Φ_0 for its resulting amplitude and phase. From here it is possible to calculate the probability for the oscillator to have, at instant t+dt a certain value $\tilde{\phi}$ of its phase, that is, a certain value $\theta \equiv \tilde{\phi} - \bar{\phi}$ for the deviation of the phase with regard to its unperturbed value. This probability will be denoted below by $\Pr ob(\theta)$. The expression of ω at instant t+dt then follows by the usual reasoning in probabilities, by means of an integral equation which is merely the "convolution" of $\Pr ob(\theta)$ with the expression of ω at instant t. This integral equation, after some analytical work, can finally be put under the equivalent form of a p.d.e. which is the equation for the evolution of of ω we were looking for. Let us now turn this reasoning into a more mathematically detailed way:

The phase difference θ between two oscillators starting with the same initial condition $\phi(t = t_0) \equiv \phi_0$, where one is unperturbed and the other is perturbed by a small stationary wave generated by the other $N-1$ oscillators of the ensemble, is given by an elementary calculation and we obtain

$$\tilde{\phi} - \bar{\phi} = \Delta t\, R_0 \sin(-t_0 + \Phi_0)\cos\Phi_0$$
$$\equiv \theta(\phi_0, t_0, t = t_0 + \Delta t;\, R_0, \Phi_0) \;,$$

with $R_0 \equiv R(t_0)$, $\Phi_0 \equiv \Phi(t_0)$. As it was said above, at the beginning of the evolution (we then have $1 - \pi^2(a_2{}^2 + b_2{}^2) > 0$) the values of R_0, Φ_0 are distributed according to a gaussian-like probability density f(N,x,y) as the one shown in figure 2 whose analytical expression can be found by the usual methods in second order linear p.d.e. (see Appendix). From this, it follows that the probability density for for θ, which we denote by $\Pr ob(\theta)$, is given by

$$d\theta \Pr ob(\theta)(t_0, \Phi_0; t = t_0 + \Delta t) = \iint_D f(N,x,y)\, dxdy$$

$$D \equiv \left\{ x,y:\;\; \theta \le \theta(t_0, \Phi_0; t = t_0 + \Delta t; R_0, \Phi_0) \le \theta + d\theta \right\} =$$
$$= \left\{ x,y:\;\; \frac{\theta}{\Delta t \cos\Phi_0} \le y\cos t_0 - x \sin t_0 \le \frac{\theta + d\theta}{\Delta t\cos\Phi_0} \right\}$$

where the formula given above for θ appears in the definition of the domain of integration. We find then

$$d\theta \Pr ob(\theta)(t_0, \Phi_0; t = t_0 + \Delta t) =$$
$$= \frac{d\theta}{|\Delta t \cos\Phi_0 \cos t_0|} \int_{-\infty}^{+\infty} f(N, x, y = x\, tg t_0 + \frac{\theta}{\Delta t \cos\Phi_0 \cos t_0})\, dx$$

We only have to introduce in the integrand the expression obtained for f(N,x,y) (see (3) of Appendix) by solving the generalized Rayleigh equation (always assuming the initial situation in which it is $1 - \pi^2(a_2{}^2 + b_2{}^2) > 0$), and carry on the analytical calculations. We omit that tedious but elementary procedure, and simply present the final form of $\Pr ob(\theta)$. We find

$$\Pr ob(\theta)(t_0, \Phi_0; t = t_0 + \Delta t) =$$
$$= \left|N\pi\kappa^2 (\Delta t)^2 \wp_1(t_0) \cos^2\phi_0\right|^{\frac{1}{2}} . \exp\left(\frac{-\theta^2}{\left|N\kappa^2(\Delta t)^2 \wp_1(t_0)\cos^2\phi_0\right|}\right)$$

with $\wp_1(t) \equiv 1 - \pi\, a_2(t)$, where a_k, b_k are the Fourier coefficients of ω:

$$\omega(t,\phi) = \frac{1}{2\pi} + \sum_{k=1}^{\infty}\left(a_k(t)\cos k\phi + b_k(t)\sin k\phi\right) .$$

Following the reasoning exposed above, it is now obvious that $\omega(t_0 + \Delta t, \phi)$ is given by the "convolution" of $\omega(t_0, \phi)$ and $\Pr ob(\theta)$:

$$\omega(t_0 + \Delta t, \phi) =$$
$$= \int_{(\phi_0)} \omega(t_0, \phi_0) \Pr ob(\theta = \phi_0 - \phi - \Delta t)(t_0, \phi_0; t = t_0 + \Delta t)\, d\phi_0$$

and by making use of the expression obtained above for $\Pr ob(\theta)$,

$$\omega(t_0 + \Delta t, \phi - \Delta t)$$
$$= \int_{(s)} \frac{\omega(t_0, s)}{\left|N\pi\kappa^2 \wp_1(t_0)(\Delta t)^2 \cos^2 s\right|^{\frac{1}{2}}} \exp\left(\frac{-(\phi - s)^2}{\left|N\kappa^2 \wp_1(t_0)(\Delta t)^2\cos^2 s\right|}\right) ds$$

From this integral equation it is possible to deduce an equivalent p.d.e. for $\omega(t,\phi)$ in a strictly mathematical way, that is, without assuming any further physical hypothesis than those employed so far in our model. The analytical "modus faciendi" can be seen in (Vassalo Pereira, 1996) and merely makes use of a certain approximate expression for the very sharp gaussian function in the integrand. We again omit the calculations and simply write the final form of the evolution equation for the phase state function of the ensemble $\omega(t,\phi)$. At this stage of the theory, and for the sake of simplicity, it is better to consider instead of $\omega(t,\phi)$, the function $\Omega(t,\psi) \equiv \omega(t,\phi=-t+\psi)$, which is simply the expression of $\omega(t,\phi)$ taken in the rotating frame turning with the same angular velocity of the unperturbed phase. Not only the physical meaning is more obvious, but the mathematical structure of the evolution equation for this new function is much simpler than with the former one. To be more precise, we find (as long as we have $\Delta > 0$)

$$\frac{N\kappa^2}{6}\wp_1(t)\frac{\partial^2}{\partial\psi^2}\left(\Omega(t,\psi)\cos^2(\psi-t)\right)=\frac{\partial^2}{\partial t^2}\Omega(t,\psi) \qquad (1)$$

NB: By making use of the Fourier series of $\Omega(t,\psi)\equiv\omega(t,\phi=-t+\psi)$,

$$\Omega(t,\psi)\equiv\frac{1}{2\pi}+\sum_{k=1}^{\infty}\left(\alpha_k(t)\cos k\psi+\beta_k(t)\sin k\psi\right)$$

we obtain

$$\alpha_k = a_k\cos kt - b_k\sin kt$$
$$\beta_k = a_k\sin kt + b_k\cos kt$$

We thus find for the functional $\wp_1$ the equivalent expression

$$\wp_1(t)\equiv 1-\pi\, a_2(t) = \\ = 1-\pi\left(\alpha_2(t)\cos 2t+\beta_2(t)\sin 2t\right)$$

and for the discriminant Δ,

$$\Delta\equiv 1-\pi\left(a_2^{\,2}+b_2^{\,2}\right)=1-\pi\left(\alpha_2^{\,2}+\beta_2^{\,2}\right).$$

Let us present some comments about this equation:

It is obvious that starting at some instant of time in a situation of equal "a priori" knowledge (or lacking of it) about the distribution of the values of the phases (i.e, $\omega=\Omega=const.=(2\pi)^{-1}$), such distribution imediately changes in time. Another almost obvious property is that ω and Ω are periodic in the phase arguments with period π - and not only 2π . This implies, namely, that the phases of the oscillators will never be found "concentrated" in a neighbourhood of only one value of the phase. In fact, whenever such concentration occurs, then a similar concentration will also exist with the same number of oscillators and in opposition of phase.

Furthermore, the study of the evolution of the Fourier coefficients of Ω (or of ω) allows us to conclude that there is a "transfer" in time of the intensities of these coefficients from lower to higher order, i.e., from smaller to higher frequencies. In broad terms, this denotes a tendency for the phases to "accumulate" in the neighbourhood of some precise values of these phases (as we pointed above, there are at least two of them, and in opposition of phase.)

The presence of the constant $N\kappa^2$ in (1) also deserves some comments. This constant is the product of two factors : the parameter N (the number of the existing resonators in the global system, which can be taken as giving a measure of the "dimension" of the ensemble) and κ^2 (related to the intensity of the mutual interaction among the resonators of the ensemble). The value of this product is obviously of primary importance in the whole theory, and the fact that both parameters appear in the equation solely through its product shows that there exists some sort of reciprocal "compensation" between the dimension of the global system and the smallness of the interaction among its components. In other words: An ensemble with relatively few oscillators and a (relatively) strong interaction will display the same behaviour than an ensemble with a higher number of oscillators and a weaker interaction - as long as the value of the product $N\kappa^2$ remains the same in both cases.

Let us now assume that after some instant of time the discriminant takes negative values:

$$1-\pi\left(a_2^{\,2}+b_2^{\,2}\right)<0\ .$$

According to Appendix, the probability density f has now a very different analytical expression, with very different geometrical properties (see figure 4), with negligeable values everywhere unless in the neighbourhood of two points in the intersection of the circle $x^2+y^2=R^2=N^2\kappa^2$ with a straight line whose expression is exactly provided by the theory . We thus have to resume the same mathematical procedure followed above with another expression for f, now given by (4) of Appendix. Since there is no other difference in the reasoning, we omit again the intermediate calculations and merely state the final result, that is, the new form of the state equation for the evolution of $\Omega(t,\psi)$ (Let us recall that Ω is simply the description of ω in the rotating frame of the unperturbed phase) :

$$N^2\kappa^2\wp_2^{\,2}(t)\cos^2 t\ \frac{\partial^2}{\partial\psi^2}\left(\Omega(t,\psi)\cos^2(\psi-t)\right)=\frac{\partial^2}{\partial t^2}\Omega(t,\psi) \qquad (2)$$

By $\wp_2$ we denote another functional of ω, whose expression is different from $\wp_1$, present in (1):

$$\wp_2(t)=\frac{\pi\beta_2}{\sqrt{(\pi\beta_2)^2+(1-\pi\alpha_2)^2}}\left(\frac{1-\pi\alpha_2}{\pi\beta_2}-tg\ t\right)$$

Comparing equations (1) and (2), we see that they both share the same analytical structure, with the only difference of the non linear factors in their first members. For a vanishing interaction among the resonators both equations reduce to the same simplified form

$$\frac{\partial^2}{\partial t^2}\Omega(t,\psi)=0\,,$$

in which the phase distribution of the ensemble is conserved and turns as a whole with the unperturbed angular velocity.

Let us recall that with the above model of the cloks, in which the interation was due to discontinuous, shock-like perturbations, we obtained a parabolic equation for the state equation of the ensemble. In the present model, with the interaction due to continuous, wave-like perturbation it is not surprising that at the end we find a hyperbolic state equation, as it is generally the case in the description of vibrating fields (d'Alembert equation, vibrating string, etc).

Since the analytical structure of both equations is the same, we may transpose for (2) what was said above about the properties of equation (1). Yet we must point out an essential difference between the two, namely, the presence in equation (2) of the coefficient $N^2\kappa^2$, instead of $N\kappa^2$ as it was the case in (1). This means that when (and if) the discriminant $1-\pi\left(a_2{}^2+b_2{}^2\right)$ goes from positive to negative values, the "weight" N of the ensemble ($N >> 1$ is the number of identical oscillators of the whole ensemble) then starts playing a fundamental role in the physical behaviour of the global ensemble. In fact, even for $N\kappa^2$ small (let us recall that κ is the very small amplitude of the stationary elementary waves generated by the oscillators and by which they interact among them in the ensemble), the parameter $N^2\kappa^2$ may assume significant, non negligeable values. In other words, the physical effect then comes "multiplied by N.

5 Concluding remarks

We have thus described in a rigorous, non heuristic way, a bifurcation phenomenon which consists in an abrupt change (from (1) to (2)) of the state equation of a statistical ensemble of interacting oscillators . Such transition occurs whenever the Fourier coefficients a_2, b_2 of the state function ω take values for which the discriminant $1-\pi\left(a_2{}^2+b_2{}^2\right)$ changes its signal. But adding to this "catastroph", there is still a sudden increase of the intensity of the physical effect, of the order of the "dimension" of the system, i.e., of the number of elementary systems of the global ensemble.

Appendix The generalized Rayleigh equation

Let us denote by R and Φ the amplitude and phase shift of the superposition of N identical harmonic vibrations all with same period 2π and same amplitude κ:

$$R\sin(-t+\Phi) \equiv \sum_{j=1}^{N}\sin(-t+\psi_j).$$

We assume that the values ψ_j of the phases of the component vibrations are distributed according to some probability density $\Omega(\psi)$, not necessarily a constant. It follows that any superposition of these N vibrations will fix a pair of values R,Φ, represented by a point in a plane (x,y) by means of polar coordinates: $x = R\cos\Phi,\quad y = R\sin\Phi$. Inversely, any point (x,y) of the plane represents a wave due to the superposition of N N elementary harmonic waves with amplitude $R=+\sqrt{x^2+y^2}$ and phase shift $\Phi = tg^{-1}(y/x)$.

If we now consider a great number n of different combinations of the N elementary waves, the corresponding n representative points in the plane (x,y) will then be distributed according some density function f(N,x,y), such that n.f(N,x,y) dx dy is the number of superpositions whose amplitude R and phase shift Φ are found within the neighbourhood dx.dy of the point x,y. Obviously, the probability density f(N,x,y) for the resulting wave must be zero for $+\sqrt{x^2+y^2} > N\kappa$, since $N\kappa$ is the biggest possible value for the resulting amplitude, corresponding to the situation of the N oscillators in phase. Yet there is no need to emphasize such restriction since the analytical expression found for f(N,x,y) is gaussian-like (at least as far as case $1-\pi^2\left(\alpha_2{}^2+\beta_2{}^2\right)>0$ is concerned - see below), and thus quickly tends to zero with increasing r.

Let us assume that to each of these n superpositions we add one elementary wave of the above form $\kappa\sin(-t+\psi)$, whose phase shift ψ has the probability density $\Omega(\psi)$ for its values. Among the n superpositions (of N elementary waves), let us find those that can lead to a final superposition (of N+1 elementary waves) with R,Φ belonging to the neighbourhood dx.dy of the point x,y. It is clear that only those superpositions whose representative points x',y' are on the circle centered in (x,y) and radius κ (that is, $x' = x+\kappa\cos\psi'$, $y' = y+\kappa\sin\psi'$,) can produce a final superposition of the desired kind, and that by simply adding an elementary wave with amplitude κ and a suitable value of the phase, which is precisely $\psi'+\pi$. Now the number of superpositions (of N elementary waves) whose representative points are found within dx'dy' is given by n.f(N,x',y')dx'dy', and among these only $\Omega(\psi')d\psi'$ may produce a superposition (of N+1 elementary waves) with representative point in dx.dy.

It follows that in order to obtain the number of superpositions (of N components) providing a superposition (of N+1 components) with representative point in dxdy, we only have to consider all the points x'y' situated on the circle centered in x,y and radius κ, and integrate over the phase angle $\psi' \in (0,2\pi)$:

$$n\,dx'dy'\int_0^{2\pi} f\left(N,\ x' = x+\kappa\cos\psi',\ y' = y+\kappa\sin\psi'\right)\Omega(\psi')\,d\tau' \ .$$

Since N>>1 we may take this argument as continuous and write

$$n\,dxdy\,f(N+1,x,y) \cong$$

$$\cong n\,dxdy\left[f(N,x,y)+1.\frac{\partial f}{\partial N}(N,x,y)\right] .$$

If we then develop f(N,x',y') in the powers of $\kappa >> 1$,

$$f(N,x+\kappa\cos\psi, y+\kappa\sin\psi) =$$
$$= f(N,x,y)+\kappa\left(\frac{\partial f}{\partial x}\cos\psi+\frac{\partial f}{\partial y}\sin\psi\right)+$$
$$+\frac{\kappa^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\cos^2\psi+2\frac{\partial^2 f}{\partial x\partial y}\cos\psi\sin\psi+\frac{\partial^2 f}{\partial y^2}\sin^2\psi\right)+\ldots$$

we find the equation

$$\frac{\partial f}{\partial N}= \kappa^2\frac{\partial^2 f}{\partial x^2}\cdot\tfrac{1}{2}\int\Omega(\psi)\cos^2\psi\, d\psi+\kappa^2\frac{\partial^2 f}{\partial y^2}\cdot\tfrac{1}{2}\int\Omega(\psi)\sin^2\psi\, d\psi$$
$$+\kappa^2\frac{\partial^2 f}{\partial x\partial y}\cdot\int\Omega(\psi)\cos\psi\sin\psi\, d\psi$$
$$+\kappa\frac{\partial f}{\partial x}\cdot\int\Omega(\psi)\cos\psi\, d\psi+\kappa\frac{\partial f}{\partial y}\cdot\int\Omega(\psi)\sin\psi\, d\psi+\ldots$$

The functionals of $\Omega(\psi)$ in the second member can be expressed by means of the Fourier coefficients of $\Omega(\psi)$. Let us recall that $\Omega(\psi)\,d\psi$ is the probability density for the representative point of any oscillator of the ensemble to have a phase difference ψ with regard to the unperturbed phase $\phi(t)=-t$. If we then write

$$\Omega(\psi)\equiv\frac{1}{2\pi}+\sum_{k=1}^{N}\left(\alpha_k\cos k\psi+\beta_k\sin k\psi\right),$$

with

$$\alpha_k\equiv\frac{1}{\pi}\int_0^{2\pi}\Omega(s)\cos ks\, ds,\qquad \beta_k\equiv\frac{1}{\pi}\int_0^{2\pi}\Omega(s)\sin ks\, ds,$$

we obtain the values of these functionals:

$$\tfrac{1}{2}\int_0^{2\pi}\Omega(\psi)\cos^2\psi\, d\psi=\tfrac{1}{4}\left(1+\pi\alpha_2\right)$$
$$\tfrac{1}{2}\int_0^{2\pi}\Omega(\psi)\sin^2\psi\, d\psi=\tfrac{1}{4}\left(1-\pi\alpha_2\right).$$
$$\int_0^{2\pi}\Omega(\psi)\sin\psi\cos\psi\, d\psi=\frac{\pi}{2}\beta_2$$
$$\int_0^{2\pi}\Omega(\psi)\sin\psi\, d\psi=\pi\beta_1$$
$$\int_0^{2\pi}\Omega(\psi)\cos\psi\, d\psi=\pi\alpha_1$$

Hence the final form for the p.d.e. defining f(N,x,y):

$$\frac{\partial f}{\partial N}=\pi\kappa\alpha_1\frac{\partial f}{\partial x}+\pi\kappa\beta_1\frac{\partial f}{\partial y}+$$
$$+\frac{\kappa^2}{4}\left(1+\pi\alpha_2\right)\frac{\partial^2 f}{\partial x^2}+\frac{\kappa^2\pi}{2}\beta_2\frac{\partial^2 f}{\partial x\partial y}+\frac{\kappa^2}{4}\left(1-\pi\alpha_2\right)\frac{\partial^2 f}{\partial y^2}$$

To this linear homogeneous p.d.e. with constant coefficients we impose the probabilistic normalization of f(N,x,y) and also the additional natural condition

$$\lim_{N\to 0} f(N,x,y)=\delta(x,y).$$

Since this equation can be easily solved by making use of the standard methods in linear p.d.e., we refer to (Vassalo Pereira, J.,1996) where that integration was first performed, and merely state the final results in this Appendix. That integration follows two distinct paths according to the sign of the discriminant $\Delta\equiv 1-\pi^2\left(\alpha_2^{\,2}+\beta_2^{\,2}\right)$. For $\Delta>0$ it can be shown that f(N,x,y) takes the form

$$f(N,x,y)=$$
$$=\frac{1}{N\pi\kappa^2\sqrt{\Delta}}\exp\left\{\frac{-(1-\pi\alpha_2)x^2+2\pi\beta_2 xy-(1+\pi\alpha_2)y^2}{N\kappa^2\Delta}\right\}\qquad(3)$$
$$=\frac{1}{N\pi\kappa^2\sqrt{\Delta}}\exp\left\{-\frac{1}{2}\left(a_{11}x^2+2a_{12}xy+a_{22}y^2\right)\right\}$$

Since $\Delta>0$, it is obvious that a_{11} and a_{22} cannot be both negative. Assuming then that we have $a_{11}>0$, we may write

$$f(N,x,y)=$$
$$=\frac{1}{N\pi\kappa^2\sqrt{\Delta}}\exp\left\{-\frac{1}{2}\left[\frac{\left(a_{11}x+a_{12}y\right)^2}{a_{11}}+\frac{a_{11}a_{22}-a_{12}^{\,2}}{a_{11}}\cdot y^2\right]\right\}$$

If we change from x,y to new variables X,Y such that

$$X^2\equiv\frac{\left(a_{11}x+a_{12}y\right)^2}{2a_{11}}\qquad Y^2\equiv\frac{a_{11}a_{22}-a_{12}^{\,2}}{2a_{11}}\cdot y^2,$$

then f takes the gaussian-like form

$$f(N,x,y)=\frac{1}{N\pi\kappa^2\sqrt{\Delta}}\exp\left\{-\left[X^2+Y^2\right]\right\}.\qquad(3')$$

Figure 2 shows some of the essential properties of this form of f(N,x,y)): In the x,y plane, OX is coincident with Ox, and OY is the straight line

$$y=-\frac{a_{11}}{a_{12}}x=\frac{1-\pi\alpha_2}{\pi\beta_2}x.$$

The probability density f(N,x,y) has an absolute maximum at the origin x=y=0. Besides, on each one of the ellipses

$$X^2(x,y)+Y^2(x,y)\equiv\tfrac{1}{2}a_{11}x^2+a_{12}xy+\tfrac{1}{2}a_{22}y^2=C=const$$

we have the same value of f, decreasing exponentially as C increases. For very small values of the discriminant Δ all these ellipses approach the straight line OY. In the limit,

$$\Delta\equiv 1-\pi^2\left(\alpha_2^{\,2}+\beta^2\right)\to 0\quad\Rightarrow\begin{cases}\left|\vec{e}_x\right|\to 0\\ \left|\vec{e}_y\right|\to\neq 0,\infty\end{cases},$$

and f takes non negligeable values only in the neighbourhood of OY, with a maximum at x=y=0, and decreasing exponentially out of the origin.

Finally, if $a_{11}>0$, then we must have $a_{22}>0$, and the preceding considerations are easily transposed for this case, without any significant changes in the properties of f(N,x,y).

Turning now to the case $\Delta\equiv 1-\pi^2\left(\alpha_2^{\,2}+\beta_2^{\,2}\right)<0$, we are going to meet an unexpected property of the distribution probability f(N,x,y). In fact, the analytical expression of f(N,x,y) in the whole plane x,y, is now

$$f(N,x,y) = \frac{Const}{N}\exp\left\{\frac{-(1-\pi\alpha_2)x^2+2\pi\beta_2 xy-(1+\pi\alpha_2)y^2}{N\kappa^2\Delta}\right\} \quad (4)$$

$$\equiv \frac{Const}{N}\exp\left\{-\frac{1}{2}\left(a_{11}x^2+2a_{12}xy+a_{22}y^2\right)\right\}$$

that is, the same expression found above in the case $\Delta > 0$, with the only difference of the changed sign in the exponent. Clearly a_{11} and a_{22} cannot be both positive: if we then assume $a_{11} > 0$ ($a_{22} > 0$ would lead to similar conclusions) the transformation

$$X^2 \equiv \frac{(a_{11}x+a_{12}y)^2}{2a_{11}} \qquad Y^2 \equiv \frac{\left|a_{11}a_{22}-a_{12}^2\right|}{2a_{11}}.y^2$$

gives for f the form

$$f(N,x,y) = \frac{Const}{N}\exp\left\{-X^2+Y^2\right\} \quad (4')$$

In these new coordinates OX is coincident with Ox, and OY is the straight line

$$y = -\frac{a_{11}}{a_{12}}x = \frac{1-\pi\alpha_2}{\pi\beta_2}x$$

In the limit, for values of the discriminant approaching zero we have

$$\Delta \equiv 1-\pi^2\left(\alpha_2^2+\beta^2\right)\to 0 \quad \Rightarrow \begin{cases} \left|\vec{e}_X\right| \to 0 \\ \left|\vec{e}_Y\right| \to \neq 0,\infty \end{cases}$$

The geometrical form of f(N,x,y) can be seen in figure 4 : it is an hyperboloid with saddle point at the origin x=y=0, and asymptotes

$$X = \pm Y \quad \Longleftrightarrow$$

$$y = \frac{a_{11}}{-a_{12}\pm\sqrt{\left|a_{11}a_{22}-a_{12}^2\right|}} = \frac{1-\pi\alpha_2}{\pi\beta_2\pm\sqrt{|\Delta|}}.$$

The lines of equal probability are now the hyperbolae

$$-X^2(x,y)+Y^2(x,y) \equiv \tfrac{1}{2}a_{11}x^2+a_{12}xy+\tfrac{1}{2}a_{22}y^2 = C = const$$

For C=0 we have the two asymptotes, and for C>0 (resp.<0) f takes increasing (resp.decreasing) values as C increases (resp.decreases).

For $\Delta \equiv 1-\pi^2\left(\alpha_2^2+\beta_2^2\right)$ very close to zero f(N,x,y) will only take significant values (that is, the exponent is positive) in the narrow region lying between the asymptotes, in which all the hyperbolae approach OY (see figure 3). It follows that f takes two equal absolute maxima at the intersections of OY with the circle of radius $N\kappa$. This is simply due, as we have pointed out in the text, to the strict impossibility for a superposition of N waves of amplitude κ to have a resulting amplitude greater than $N\kappa$:

$$x^2+y^2 > (N\kappa)^2 \quad \Rightarrow \quad f(N,x,y) \equiv 0$$

Such remark, which was trivial when the gaussian-like character of f(N,x,y) implied negligeable values for x,y far from the origin, is now of utmost importance. Furthermore, since the probability distribution f is normalized in the domain $x^2+y^2 \leq (N\kappa)^2$ (and N>>1) these maxima are very sharp and f is almost negligeable out of their neighbourhood. In fact, these conclusions are a consequence of both the normalization of the probability and the increasing exponential values taken by f on the straight line OY (and far from the origin). In other words, a transition from positive to negative values of Δ denotes a "bifurcation" behaviour of the probability distribution for the superposition of $N >> 1$ identical harmonic vibrations. It is this fact that is responsible for the existence of two different equations - namely, (1) and (2) - governing the evolution of the ensemble of oscillators, and thus describing deterministically a phase transition of the statistical ensemble as it is shown in the text.

We must still point out that even for Δ negative but not necessarily very close to zero, the same geometrical conclusions concerning the two (then not so sharp) maxima of f are true, namely the existence of high values in the neighbourhood of the above mentioned intersections of OY with the circle of radius $N\kappa$ (see figure 4).

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