Nonlinear Processes in Geophysics (1999) 6: 145–148

Nonlinear Processes in Geophysics

Localized Alfvénic solutions of nondissipative and compressible MHD

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Received: 16 June 1999 – Revised: 5 November 1999 – Accepted: 8 November 1999

Abstract. Alfvénic solutions of nondissipative MHD are entirely determined by their magnetic configuration. With the supplementary assumption of incompressibility any solenoidal field can be used to construct an Alfvénic solution. It is demonstrated that for nondissipative and compressible MHD the energy equation constrains the magnetic field of Alfvénic solutions to have a constant strength along field lines. Some topological solitons known in nondissipative and incompressible MHD do not have this property. New localized axisymmetric Alfvénic solutions of nondissipative and compressible MHD are explicitly constructed.

1 Introduction

Exact solutions of equations modeling the evolution of nonlinear physical models have an intrinsic value for theory but also for applications, sometimes more as accuracy tests of numerical simulation methods than as reference solutions for comparison with observations. Among these solutions, localized solutions having topological invariants are perhaps the most difficult to construct. In nondissipative and incompressible MHD Alfvénic solutions, for which the velocity of the fluid is everywhere equal or opposite to the local Alfvén velocity, can be built on any solenoidal field. Using a Hopf mapping and a stereographic projection between the sphere S^3 of the Euclidean space R^4 and the three-dimensional Euclidean space R^3 Kamchatnov (1982) constructed a very interesting localized magnetic configuration made of closed field lines, each of which being linked to all the other ones. Thus Kamchatnov's field has non null magnetic helicity (see for example Biskamp (1993) for the definition of the concept and its properties):

$$H = \int_V \vec{A} \cdot \vec{B}\, d^3 r \qquad (1)$$

Hence Kamchatnov built a topological soliton of incompressible MHD. Unfortunately this solution is not valid when compressibility is taken into account because any solenoidal field is not necessarily the field of an Alfvénic solution in compressible MHD. Section 2 gives a detailed derivation of general Alfvénic solutions which relaxes the usually made assumption of constant density. In this section we derive explicitly the conditions to be satisfied by the magnetic field of such solutions. Section 3 presents Kamchatnov's magnetic field, some of its properties and an heuristic argument to derive it. It is also demonstrated in this section that Kamchatnov's field does not satisfy the necessary constraint on $\vec{B}$ to support an Alfvénic solution in compressible MHD. At last section 4 presents a pedestrian construction of exact localized Alfvénic solutions unfortunately lacking the mathematical elegancy of Kamchatnov's work. Unfortunately also this elementary construction does not allow any conclusion about the magnetic helicity of the structure, although it is very likely non null. This last point remains to be checked.

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2 Exact Alfvénic solutions of nondissipative MHD

Making use of the International System of units and respectively substituting $\vec{B}$ and $\vec{J}$ for $\vec{B}/\sqrt{\mu_0}$ and $\sqrt{\mu_0}\vec{J}$ the equations of nondissipative MHD can be written in conservative form as follows:

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (2)$$

$$\partial_t \vec{B} + \vec{\nabla} \cdot \left(\vec{B} \otimes \vec{v} - \vec{v} \otimes \vec{B} \right) = 0 \qquad (3)$$

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad (4)$$

$$\partial_t (\rho \vec{v}) + \vec{\nabla} \cdot \left(\rho \vec{v} \otimes \vec{v} - \vec{B} \otimes \vec{B} + P^* \overleftrightarrow{I} \right) = 0 \qquad (5)$$

$$\partial_t U + \vec{\nabla} \cdot \left((U + P^*)\vec{v} - (\vec{v} \cdot \vec{B})\vec{B} \right) = 0 \qquad (6)$$

where $U = e + \rho u^2/2 + B^2/2$ and $P^* = P + B^2/2$ are respectively the total energy density and the total pressure, e being the internal energy density of the fluid. The solenoidality condition (2) is written first, then Faraday's equation

(3) followed by the conservation equations for mass (4), momentum (5), and energy (6). The displacement current has been neglected and use has been made of Ampere's equation $\vec{\nabla} \times \vec{B} = \vec{J}$. The transport of a fluid element is supposed to be adiabatic. In incompressible fluids the energy equation only determines the temperature field of the fluid given the velocity field; in this sense the energy equation decouples from the other equations and the pressure, or the total pressure in MHD, is determined by a Poisson equation obtained by taking the divergence of the momentum equation. Investigating the behaviour of a plasma in the MHD approximation does not allow the use of the incompressibility approximation even if it happens that some solutions, very commonly observed in space plasmas, do not compress the medium.

The Alfvén velocity $\vec{v}_A = \vec{B}/\sqrt{\rho}$ plays a key role in MHD as evidenced by peculiar solutions for which $\vec{v} = \pm\vec{v}_A$. Chandrasekhar (1961) called these solutions *equipartition solutions* accordingly to their property $\rho v^2/2 = B^2/2$; we will also quote them as Alfvénic solutions. Alfvénic solutions satisfy the following set of equations (let us emphasize that, for the moment being, we do not make any assumption concerning the mass density ρ):

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{7}$$

$$\partial_t \vec{B} = 0 \tag{8}$$

$$\partial_t \rho \pm \vec{\nabla} \cdot \left(\sqrt{\rho}\vec{B}\right) = 0 \tag{9}$$

$$\pm\partial_t \left(\sqrt{\rho}\vec{B}\right) + \vec{\nabla} P^* = 0 \tag{10}$$

$$\partial_t e \pm \vec{\nabla} \cdot \left((e + P^*)\frac{\vec{B}}{\sqrt{\rho}}\right) = 0 \tag{11}$$

When $\vec{\nabla} \cdot \left(\sqrt{\rho}\vec{B}\right) = 0$, or equivalently $\vec{B} \cdot \vec{\nabla}\rho = 0$, equation (9) leads to $\partial_t \rho = 0$, then equation (10) combined with (8) gives $\vec{\nabla} P^* = 0$. If moreover $\vec{B} \cdot \vec{\nabla} e = 0$ then equation (11) reduces to $\partial_t e = 0$. The internal energy per unit volume e being a function of P and ρ this results in $\partial_t P = 0$ and $P^* = P_0^*$, a constant. The condition $\vec{B} \cdot \vec{\nabla} e = 0$ is equivalent to $\vec{B} \cdot \vec{\nabla} B^2 = 0$, due to:

$$\vec{\nabla} e = \frac{\partial e}{\partial \rho}\vec{\nabla}\rho + \frac{\partial e}{\partial P}\vec{\nabla}\left(P_0^* - \frac{B^2}{2}\right)$$

Thus, given fields $\vec{B}(\vec{r}), \rho(\vec{r})$ and $\vec{v} = \pm\vec{B}/\sqrt{\rho}$ satisfying the following conditions:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{12}$$

$$\vec{B} \cdot \vec{\nabla}\rho = 0 \tag{13}$$

$$\vec{B} \cdot \vec{\nabla} B^2 = 0 \tag{14}$$

$$P + \frac{B^2}{2} = P_0^* \tag{15}$$

are exact and stationnary solutions of the nondissipative MHD equations (2-6). For planar solutions defined by fields $\vec{B}, \rho$ which are functions of $\zeta = \vec{k} \cdot \vec{r}$, where $\vec{k}$ is a constant vector, equations (12-14) become:

$$\vec{k} \cdot \vec{B} = \vec{k} \cdot \vec{B}_0 \tag{16}$$

$$\frac{d\rho}{d\zeta}\,\vec{k} \cdot \vec{B} = 0 \tag{17}$$

$$\frac{d(B^2)}{d\zeta}\,\vec{k} \cdot \vec{B} = 0 \tag{18}$$

where $\vec{B}_0$ is a constant vector. Hence, either $\vec{k} \cdot \vec{B} = 0$ allowing B^2 and ρ to be arbitrary functions of ζ, or $\vec{k} \cdot \vec{B} \neq 0$ leading to constant values for B^2 and ρ; the latter case corresponding to the well known nonlinear planar Alfvén waves (see for example Landau and Lifshitz (1960)).

3 Kamchatnov's field and generalizations

3.1 Kamchatnov's original field

The vector potential constructed by Kamchatnov (1982) is conveniently defined through its poloidal and toroidal components $\vec{A}_p$ and $\vec{A}_t$ and is written as follows, although it was not originally written like this:

$$\vec{A}_K = \vec{A}_{K,p} + \vec{A}_{K,t} \tag{19}$$

$$\vec{A}_{K,p} = \frac{B_0 R^3}{4(R^2 + r^2)^2}\left[2(\vec{r} \cdot \vec{u})\,\vec{r} + (R^2 - r^2)\,\vec{u}\right] \tag{20}$$

$$\vec{A}_{K,t} = \frac{B_0 R^4}{2(R^2 + r^2)^2}\,\vec{u} \times \vec{r} \tag{21}$$

where B_0 and R are constants which respectively determine the intensity and size of this localized and solenoidal field, and $\vec{u}$ is a constant unit vector. The poloidal and toroidal components of the magnetic field are consequently given by equations:

$$\vec{B}_K = \vec{B}_{K,p} + \vec{B}_{K,t} \tag{22}$$

$$\vec{B}_{K,p} = \vec{\nabla} \times \vec{A}_t = \frac{B_0 R^4}{(R^2 + r^2)^3}\left[2\,(\vec{r} \cdot \vec{u})\,\vec{r} + (R^2 - r^2)\,\vec{u}\right] \tag{23}$$

$$\vec{B}_{K,t} = \vec{\nabla} \times \vec{A}_{K,p} = \frac{2 B_0 R^5}{(R^2 + r^2)^3}\,\vec{u} \times \vec{r} \tag{24}$$

The vector potential and the magnetic field appear to be linked together by the equation:

$$\vec{A}_K = \frac{R^2 + r^2}{4R}\vec{B}_K \tag{25}$$

The total helicity of this field, defined by equation (1), is:

$$H_K = \frac{\pi^2}{16} B_0^2 R^4$$

3.2 A simple derivation of Kamchatnov's field

The magnetic field created by the dipole $\vec{\mathcal{M}} = \mathcal{M}\vec{u}$, where $\vec{u}$ is a constant unit vector, is poloidal and is written, with the origin at the dipole location:

$$\vec{B} = \frac{\mathcal{M}}{r^5}\left[3(\vec{r}\cdot\vec{u})\,\vec{r} - r^2\,\vec{u}\right]$$

The singularity of the field at $r = 0$ is related to the negative sign of the magnetic flux everywhere in the plane $\vec{r}\cdot\vec{u} = 0$ except at $r = 0$. Allowing the flux to be positive inside the disk of radius R should remove the singularity, thus let us try the following form of the poloidal component:

$$\vec{B}_p = f(r)(\vec{r}\cdot\vec{u})\,\vec{r} + g(r)(R^2 - r^2)\vec{u} \tag{26}$$

This field is solenoidal provided that:

$$4rf + r^2\frac{df}{dr} = 2rg - (R^2 - r^2)\frac{dg}{dr} \tag{27}$$

One of these *regularized dipolar solutions* corresponds to the poloidal component (23) of Kamchatnov's field, it is determined by the following functions f and g:

$$f(r) = 2g(r) = \frac{2B_0R^4}{(R^2+r^2)^3}$$

The solenoidality condition (27) is not affected by adding to (26) any toroidal component written in the following form:

$$\vec{B}_t = \omega(r, \vec{r}\cdot\vec{u})\;\vec{u}\times\vec{r}$$

and looking for colinear fields $\vec{A}$ and $\vec{B}$ gives:

$$\omega(r, \vec{r}\cdot\vec{u}) = \frac{2B_0R^5}{(R^2+r^2)^3}$$

which leads to the toroidal component (24) of Kamchatnov's field and to the relation (25).

3.3 Generalizations

Sagdeev et al. (1986) have generalized Kamchatnov's field and proposed a class of solutions of incompressible MHD which can be written:

$$\vec{B} = \omega_1\vec{B}_{K,t} + \omega_2\vec{B}_{K,p} \tag{28}$$

where ω_1 and ω_2 are real constants and $\vec{B}_{K,p}$ and $\vec{B}_{K,t}$ are respectively the poloidal and toroidal components of Kamchatnov's field given by equations (23,24). Unfortunately none of these solutions is consistent with the energy constraint (14) which means that their validity is limited to incompressible MHD. Sagdeev et al. (1986) also gave a larger class of solutions obtained by considering ω_1 in (28) as an arbitrary function of variable:

$$1 + r^2 + \sqrt{r^2 - (\vec{r}\cdot\vec{u})^2}$$

for which we have not yet checked wether or not they are consistent with (14.)

4 Localized Alfvénic solutions

In this section we will construct an axisymmetric and localized Alfvénic solution of nondissipative and compressible MHD defined in a mixed way by the toroidal component of its vector potential $\vec{A} = A_\phi(r,z)\vec{e}_\phi$ and its toroidal component $\vec{B}_\phi = B_\phi(r,z)\vec{e}_\phi$, where $\vec{e}_\phi$ is the unit vector associated to the angular coordinate ϕ:

$$\vec{B} = -\partial_z A_\phi\;\vec{e}_r + \frac{1}{r}\partial_r(rA_\phi)\;\vec{e}_z + B_\phi\;\vec{e}_\phi \tag{29}$$

This field is solenoidal and constraint (14) reduces to:

$$\left(-\partial_z A_\phi\;\vec{e}_r + \frac{1}{r}\partial_r(rA_\phi)\;\vec{e}_z\right)\cdot\vec{\nabla}B^2 = 0$$

because $\vec{\nabla}B^2$ is poloidal. This condition can be restated as follows:

$$\vec{\nabla}B^2 = \lambda_1(r,z)\left(\frac{1}{r}\partial_r(rA_\phi)\;\vec{e}_r + \partial_z A_\phi\;\vec{e}_z\right) \tag{30}$$

with λ_1 an arbitrary scalar function of (r,z). Equation (30) can be rewritten:

$$\vec{\nabla}B^2 = \frac{\lambda_1(r,z)}{r}\vec{\nabla}(rA_\phi) \tag{31}$$

Such a condition is satisfied as soon as B^2 is an arbitrary function Λ of rA_ϕ. The poloidal component of the field (29) being:

$$\vec{B}_p = -\partial_z A_\phi\;\vec{e}_r + \frac{1}{r}\partial_r(rA_\phi)\;\vec{e}_z$$

is such that

$$B_p^2 = r^{-2}\left(\vec{\nabla}(rA_\phi)\right)^2$$

eventually leading to the following necessary relation between B_ϕ and A_ϕ:

$$\begin{aligned} B_\phi^2 &= B^2 - B_p^2 \\ &= \Lambda(rA_\phi) - r^{-2}\left(\vec{\nabla}(rA_\phi)\right)^2 \end{aligned} \tag{32}$$

The right-hand side of equation (32) has to be greater than or equal to zero. In conclusion, an axisymmetric localized Alfvénic solution can be constructed from a given function $A_\phi(r,z)$ through formulas (29 ,32) provided one can find a scalar function Λ such that:

$$\Lambda(rA_\phi) - r^{-2}\left(\vec{\nabla}(rA_\phi)\right)^2 \geq 0 \tag{33}$$

In a final step let us show that we can find at least one family of solutions. Let us start with:

$$r\,A_\phi = B_0\frac{r^q}{a^{q-2}}\exp\left(-\frac{\rho^2}{2a^2}\right)$$

with $\rho^2 = r^2+z^2$ and (B_0, a) two real constants. This choice leads to:

$$\frac{B_p^2}{rA_\phi} = B_0\frac{r^{q-4}}{a^{q+2}}\left((qa^2 - r^2)^2 + r^2z^2\right)\exp\left(-\frac{\rho^2}{2a^2}\right)$$

For $q \geq 4$ the above quantity is bounded by a positive real number K and an acceptable solution is determined by:

$$\Lambda(rA_\phi) = KrA_\phi$$

For these solutions B^2 is a rapidly decreasing function of ρ and the magnetic field lines are winding around torii of axis z. This construction does not provide the two components of the vector potential, hence in contrast with Kamchatnov's approach it does not allow to calculate the magnetic helicity.

5 Conclusions

Considering the full set of nondissipative MHD equations, including the energy equation, we have derived constraints that have to be satisfied by a magnetic field in order to built an Alfvénic solution of compressible MHD. It has been shown that various solutions of incompressible MHD proposed in the past do not satisfy the constraint coming from the energy equation when compressibility is taken into account. New localized fields consistent with all the derived constraints have been explicitly constructed. This work will be extended to define such solutions by their vector potential to allow the investigation of their helicity. The question of observability of such solutions in space plasmas is an open question but those solutions are easily simulated with compressible MHD codes and can be used for testing codes.

Acknowledgements. The author thanks Drs B. Tsurutani and T. Hada and the Organizing Commitee of Nonlinear Waves 99 Workshop (held in Carlsbad, California, March 1-5, 1999) for the invitation to attend this meeting. The author also thanks the referees and the Editor whose questions and criticisms have helped to clarify the presentation.

References

Biskamp, D., *Nonlinear Magnetohydrodynamics*, Cambridge University Press, Cambridge, 1993.

Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.

Kamchatnov, A., Topological solitons in magnetohydrodynamics, *Sov. Phys.–JETP, 55(1)*, 69–73, 1982.

Landau, L. and Lifshitz, E., *Electrodynamics of Continuous Media*, Addison-Wesley, Reading, Mass., 1960.

Sagdeev, R., Moiseev, S., Tur, A., and Yanovskii, V., Problems of the theory of strong turbulence and topological solitons, in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, edited by R. Sagdeev, pp. 137–182, MIR, Moscou, 1986.