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Hamiltonian formulation for the description of interfacial solitary waves

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Abstract

We consider solitary waves propagating on the interface between two fluids, each of constant density, for the case when the upper fluid is bounded above by a rigid horizontal plane, but the lower fluid has a variable depth. It is well-known that in this situation, the solitary waves can be described by a variable-coefficient Korteweg-de Vries equation. Here we reconsider the derivation of this equation and present a formulation which preserves the Hamiltonian structure of the underlying system. The result is a new variable-coefficient Korteweg-de Vries equation, which conserves energy to a higher order than the more conventional well-known equation. The new equation is used to describe the transformation of an interfacial solitary wave which propagates into a region of decreasing depth.

§1 Introduction

Internal solitary waves are now a well-documented phenomenon, particularly in the coastal oceans or the lower atmosphere (see, for instance, Apel. 1995 and Ostrovsky and Stepanyants, 1989, or Christie, 1989, respectively). For weakly nonlinear waves the Korteweg-de Vries (KdV) equation, or a closely-related equation, is usually considered to be an appropriate theoretical model (see, for instance, the recent review by Grimshaw, 1996). For the case when the background environment is variable, the familiar KdV equation needs to be modified, and becomes instead a variable-coefficient KdV equation of the form

$$A_\tau + \frac{c_\tau}{2c} A + \frac{\mu}{c} AA_\theta + \frac{\lambda}{c^3} A_{\theta\theta\theta} = 0 . \qquad (1.1)$$

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Here the coefficients c, μ and λ are all functions of the time-like variable τ, while θ is a space-like variable in the frame of reference moving with the wave (see (1.3) below). The coefficient c is the local linear long-wave phase speed, while the nonlinear and dispersive coefficients, λ and μ respectively, have self-evident interpretations. The derivation of (1.1) in quite general circumstances is described by Grimshaw (1981), and summarized in Grimshaw (1997). A derivation for the special case of interfacial waves with variable depth of the lower fluid has been given by Pelinovsky and Shavratsky (1976), and Djordjevic and Redekopp (1978). We will also derive (1.1) for this special case later in this paper, and in the process will define the scalings used.

Recently, however, van Groesen and Pudjaprasetya (1993) (see also Pudjaprasetya and van Groesen, 1995) have pointed out, for the case of water waves, that the underlying fluid system is Hamiltonian, with a conserved Hamiltonian functional representing energy. They exploited this to derive an alternative to the variable variable-coefficient KdV equation (1.1), namely

$$A_T = -\tfrac{1}{2}\left\{c\,\frac{\partial}{\partial X} + \frac{\partial}{\partial X}\,c\right\}\frac{\delta \hat{H}}{\delta A} , \qquad (1.2a)$$

where $$\hat{H} = \int_{-\infty}^{\infty} \hat{J}\,dX , \qquad (1.2b)$$

and $$\hat{J} = \tfrac{1}{2}A^2 + \varepsilon^2\left\{-\frac{\lambda}{2}A_X^2 + \frac{\mu}{6}A^3\right\}. \qquad (1.2c)$$

The skew-symmetric operator in (1.2a) ensures that this equation is Hamiltonian, and conserves the Hamiltonian $\hat{H}$, which is an asymptotic approximation to the energy of the underlying system. The two alternative forms of variable-coefficient KdV equation are related under the transformation

$$\tau = \varepsilon^2 X, \quad \theta = \frac{1}{\varepsilon^2}\int_0^{\tau}\frac{d\tau'}{c(\tau')} - T, \qquad (1.3)$$

where ε is the governing small parameter in the derivation of either of these equations. Thus ε^2 is a measure both of the amplitude of the wave, and of the linear dispersion. Under the change of variables (1.3), the equation (1.2a) asymptotically transforms to (1.1), with an error of $0(\varepsilon^2)$. This issue is explored in some detail later in this paper.

The purpose of this paper is to derive the Hamiltonian form (1.2) of the variable-coefficient KdV equation for interfacial waves, that is, waves on the interface between two fluids, each of constant density, where the upper fluid is bounded above by a rigid horizontal plane, and the lower fluid has variable depth. We show that the outcome is indeed an equation of the form (1.2), thus lending support to the notion that equation (1.2) is a convenient Hamiltonian version of (1.1) in more general situations.

In §2 we present a Hamiltonian formulation for interfacial waves, and then in §3 introduce a long wave scaling to reduce the full fluid system to a pair of equations of Boussinesq type, which are also in Hamiltonian form. Then in §4 we introduce the notion of uni-directional waves in order to derive (1.2). In §5 we outline two principal applications of (1.2), first to the transformation of a solitary wave as the depth of the lower fluid varies, and second to an estimation of the possibility that a second solitary wave may be generated in this process. The paper concludes with some discussion in §6.

§2. Hamiltonian formulation

We consider a two-layer fluid bounded above by a rigid horizontal plane, $z = h_1$, and below by a rigid but horizontally-varying boundary, $z = -h_2(x)$. For simplicity, we shall suppose that $h_2 \to h_\pm$ as $x \to \pm\infty$ respectively. The configuration and co-ordinate system are shown in Figure 1. Each layer consists of incompressible, inviscid fluid of constant density $\rho_{1,2}$ for the upper, lower layers respectively. The free interface between the layers is denoted by $z = \eta(x, t)$. Assuming irrotational flow in each layer, the governing equations are

$$\nabla^2\varphi_1 = 0, \quad \text{in} \quad -\eta < z < h, \qquad (2.1a)$$

$$\nabla^2\phi_2 = 0, \quad \text{in} \quad -h_2(x) < z < \eta, \qquad (2.1b)$$

where $\phi_{1,2}$ is the velocity potential in each layer so that the fluid velocity is $\nabla\phi_{1,2}$ in each layer. At the rigid boundaries the normal component of the velocity must vanish so that

$$\phi_{1z} = 0, \quad \text{at} \quad z = h_1, \qquad (2.2a)$$

$$\phi_{2z} = -\phi_{2x}h_{2x} \quad \text{at} \quad z = -h_2. \qquad (2.2b)$$

At the interface, the kinematic conditions that the interface is a material surface for each fluid are,

$$\eta_t + \phi_{ix}\eta_x = \phi_{iz} \quad \text{at} \quad z = \eta,\ i = 1, 2. \qquad (2.3)$$

Finally the dynamic condition for continuity of pressure is, on using the Bernoulli relation for each fluid,

$$\rho_1(\phi_{1t} + \tfrac{1}{2}|\nabla\phi_1|^2 + g\eta) = \rho_2(\phi_{2t} + \tfrac{1}{2}|\nabla\phi_2|^2 + g\eta),$$

at

$$z = \eta. \qquad (2.4)$$

Next we cast these equations into a Hamiltonian form. Here the Hamiltonian is

$$H(\phi, \eta) = \int_{-\infty}^{\infty}(K + V)\,dx \qquad (2.5)$$

where K is the kinetic energy density

$$K = \int_{-h_2}^{\eta}\tfrac{1}{2}\rho_2|\nabla\phi_2|^2 dz + \int_{\eta}^{h_1}\tfrac{1}{2}\rho_1|\nabla\phi_1|^2 dz, \qquad (2.6)$$

and V is the potential energy density

$$V = \tfrac{1}{2}g(\rho_2 - \rho_1)\eta^2. \qquad (2.7)$$

Here we are anticipating that ϕ and η are canonical variables, where

$$\phi = \rho_2\phi_2 - \rho_1\phi_1, \quad \text{at} \quad z = \eta, \qquad (2.8)$$

where it is understood that $\phi_{1,2}$ are determined by solving the boundary value problems defined by (2.1), (2.2), (2.8) and the further boundary condition,

$$\phi_{1z} - \phi_{1x}\eta_x = \phi_{2z} - \phi_{2x}\eta_x \quad \text{at} \quad z = \eta. \qquad (2.9)$$

which is obtained from (2.3) on eliminating η_t. It can then readily be verified that the remaining kinematic condition (2.3), and the dynamic condition (2.4), may be expressed as a Hamiltonian system,

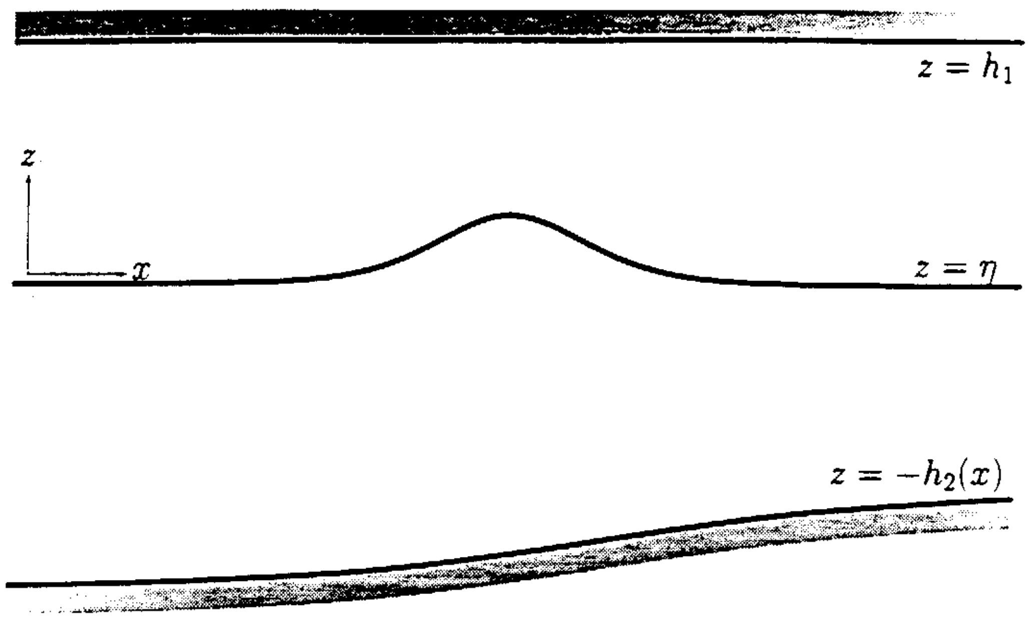

Fig. 1. The configuration and co-ordinate system.

$$\phi_t = -\frac{\delta H}{\delta \eta}, \qquad \eta_t = \frac{\delta H}{\delta \phi}. \tag{2.10}$$

Here δ denotes the variational derivative. The derivation is analogous to that for water waves (Zakharov, 1968). In the sequel it will be more convenient to introduce a new variable $u = \phi_x$, so that (2.10) are replaced by,

$$u_t = -\frac{\partial}{\partial x}\left(\frac{\delta H}{\delta \eta}\right), \qquad \eta_t = -\frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right). \tag{2.11}$$

Next we note that the system (2.11) possesses the conservation laws.

$$M = \int_{-\infty}^{\infty} \eta\, dx = \text{constant}, \tag{2.12a}$$

$$C = \int_{-\infty}^{\infty} u\, dx = 0, \tag{2.12b}$$

and

$$E = \int_{-\infty}^{\infty} (K + V)\, dx = \text{constant}. \tag{2.12c}$$

Here M and C are Casimirs of the Hamiltonian system (2.11), and represent conservation of mass and circulation respectively. Note that we are assuming here that all variables vanish as $x \to \pm\infty$, so that the circulation C is zero. E is the energy, and is conserved here since the Hamiltonian H is explicitly independent of the time t. Indeed, E is just the Hamiltonian itself. However, due to the variable bottom profile, the Hamiltonian H does depend explicitly on the horizontal variable x, and so the horizontal momentum, P, is not conserved. Here P is given by

$$P = \int_{-\infty}^{\infty} \left\{ \int_{-h_2}^{\eta} \rho_2 \phi_{2x}\, dz + \int_{\eta}^{h_1} \rho_1 \phi_{1x}\, dz \right\} dx, \tag{2.13a}$$

$$\text{or} \quad P = \int_{-\infty}^{\infty} \eta u\, dx - \int_{-\infty}^{\infty} \rho_2 \phi(z = -h_2) h_{2x}\, dx. \tag{2.13b}$$

Instead, we have the relation

$$\frac{dP}{dt} = \int_{-\infty}^{\infty} p_2(z = -h_2) h_{2x}\, dx, \tag{2.14a}$$

$$\text{where} \quad p_2 = -\rho_2 \phi_{2t} - \tfrac{1}{2}\rho_2 \left|\nabla \phi_2\right|^2 - \rho_2 g z. \tag{2.14b}$$

Here p_2 is the pressure in the lower layer.

§3. Derivation of Boussinesq equations

Since we wish to consider weakly nonlinear long waves, we introduce a small parameter ε and the normalized variables as follows,

$$\eta = \varepsilon^2 A(X, T), \quad u = \varepsilon^2 U(X, T), \tag{3.1a}$$

where
$$X = \varepsilon x, \quad T = \varepsilon t. \tag{3.1b}$$

We shall also suppose that $h_2 = h_2(X)$, although later it will be necessary to insist that $h_2(X)$ vary even more slowly. Using (3.1) we next solve (2.1) subject to the boundary conditions (2.2) in the form of an asymptotic series in powers of ε^2. Thus

$$\phi_1 = \varepsilon F_1(X, T) - \tfrac{1}{2}\varepsilon^3 F_{1XX}(z - h_1)^2 + \frac{1}{24}\varepsilon^5 F_{1XXXX}(z - h_1)^4 + \ldots, \tag{3.2a}$$

$$\phi_2 = \varepsilon F_2(X, T) - \tfrac{1}{2}\varepsilon^3 \frac{\partial}{\partial X}\left\{F_{2X}(z + h_2)^2\right\} + \frac{1}{24}\varepsilon^5 \frac{\partial^3}{\partial X^3}\left\{F_{2X}(z + h_2)^4\right\} + \ldots . \tag{3.2b}$$

Then with $u = \phi_x$ we readily find from (2.8) that

$$U = \rho_2 F_{2X} - \rho_1 F_{1X} + \varepsilon^2\left\{-\tfrac{1}{2}\rho_2 \frac{\partial^2}{\partial X^2}(h_2^2 F_{2X}) + \tfrac{1}{2}\rho_1 h_1^2 F_{1XXX}\right\} + \ldots . \tag{3.3}$$

Next we use the kinematic condition (2.9) together with (3.1) to show that

$$h_1 F_{1X} + h_2 F_{2X} = \varepsilon^2 \left\{A(F_{1X} - F_{2X}) + \tfrac{1}{6}h_1^3 F_{1XXX} + \frac{1}{6}\frac{\partial^2}{\partial X^2}\left(h_2^3 F_{2X}\right)\right\} + \ldots . \tag{3.4}$$

Then, from the relations (3.3) and (3.4) we can iteratively determine both F_1 and F_2 in terms of U and A. We find that

$$(\rho_1 h_2 + \rho_2 h_1)F_{1X} = -h_2 U + \varepsilon^2\left\{-\frac{\rho_2(h_1 + h_2)}{(\rho_1 h_2 + \rho_2 h_1)} AU - \frac{(\frac{1}{3}\rho_2 h_1 h_2^3 + \frac{1}{6}\rho_2 h_1^3 h_2 + \frac{1}{2}\rho_1 h_1^2 h_2^2)}{(\rho_1 h_2 + \rho_2 h_1)} U_{XX}\right\} + \ldots, \tag{3.5a}$$

$$(\rho_1 h_2 + \rho_2 h_1)F_{2X} = h_1 U + \varepsilon^2\left\{-\frac{\rho_1(h_1 + h_2)}{(\rho_1 h_2 + \rho_2 h_1)} AU + \frac{(\frac{1}{3}\rho_1 h_1^3 h_2 + \frac{1}{6}\rho_1 h_2^3 h_1 + \frac{1}{2}\rho_2 h_1^2 h_2^2)}{(\rho_1 h_2 + \rho_2 h_1)} U_{XX}\right\} + \ldots . \tag{3.5b}$$

Here, in the {...} terms we have omitted expressions involving derivatives of h_2 as in the sequel we shall in fact require that h_{2X} is $0(\varepsilon^2)$, that is, $h_2 = h_2(\tau)$ where $\tau = \varepsilon^2 X$.

The final step in this section is the approximation of the Hamiltonian (2.5) using the asymptotic expressions described above. We find that, using (3.1) and (3.2),

$$H = \varepsilon^3 \mathcal{H} = \varepsilon^3 \int_{-\infty}^{\infty} J\, dX, \tag{3.6a}$$

where
$$J = \tfrac{1}{2}g(\rho_2 - \rho_1)A^2 + \tfrac{1}{2}\rho_1 h_1 F_{1X}^2 + \tfrac{1}{2}\rho_2 h_2 F_{2X}^2 + \frac{\varepsilon^2}{3}\rho_1 h_1^3 F_{1XX}^2 + \frac{\varepsilon^2}{3}\rho_2 h_2^3 F_{2XX}^2 + \frac{\varepsilon^2}{2}(\rho_2 F_{2X}^2 - \rho_1 F_{1X}^2)\, A + \ldots . \tag{3.6b}$$

Next we use the relations (3.5) to rewrite $\mathcal{H}$ in terms of the canonical variables U and A. The result is

$$J = \tfrac{1}{2}g(\rho_2 - \rho_1)A^2 + \tfrac{1}{2}\frac{h_1 h_2}{(\rho_1 h_2 + \rho_2 h_1)} U^2 + \varepsilon^2\left\{-\beta U_X^2 + \nu A U^2\right\} + \ldots, \tag{3.7a}$$

where
$$\beta = \frac{h_1^2 h_2^2}{6}\frac{(\rho_1 h_1 + \rho_2 h_1)}{(\rho_1 h_2 + \rho_2 h_1)^2}, \tag{3.7b}$$

and
$$\nu = \tfrac{1}{2}\frac{(\rho_2 h_1^2 - \rho_1 h_2^2)}{(\rho_1 h_2 + \rho_2 h_1)^2} . \tag{3.7c}$$

With the rescaling of (3.1) and (3.6a), the Hamiltonian system (2.11) becomes

$$U_T = -\frac{\partial}{\partial X}\left(\frac{\delta \mathcal{H}}{\delta A}\right), \qquad A_T = -\frac{\partial}{\partial X}\left(\frac{\delta \mathcal{H}}{\delta U}\right). \tag{3.8}$$

Then, on using the expressions (3.6) and (3.7a) for $\mathcal{H}$, we obtain the Boussinesq equations,

$$U_T + \left\{ g(\rho_2 - \rho_1)A + \varepsilon^2 \nu U^2 \right\}_X + \ldots = 0 ,$$

$$A_T + \left\{ \frac{h_1 h_2}{(\rho_1 h_2 + \rho_2 h_1} U + 2\varepsilon^2 \nu A U + 2\varepsilon^2 \beta U_{XX} \right\}_X + \ldots = 0. \quad (3.9)$$

Note that putting $\rho_1 = 0$ reduces the present theory to that for water waves, and in this case it is readily verified that the Hamiltonian (3.6) and the Boussinesq equations (3.9) agree with those obtained by van Groesen and Pudjaprasetya (1993).

This system possesses the conservation laws,

$$\mathcal{M} = \int_{-\infty}^{\infty} A\, dX , \quad (3.10a)$$

$$\mathcal{C} = \int_{-\infty}^{\infty} U dX = 0 , \quad (3.10b)$$

and

$$\mathcal{E} = \int_{-\infty}^{\infty} J dX , \quad (3.10c)$$

where J is defined in (3.7a). There are just rescaled asymptotic versions of the conservation laws (2.12a), (2.12b) and (2.12c) for mass, circulation and energy respectively. The counterpart of the relation (2.13b) for horizontal momentum is

$$\mathcal{P} = \int_{-\infty}^{\infty} \left\{ AU + \frac{\rho_2 h_1 h_2}{(\rho_1 h_2 + \rho_2 h_1} U \right\} dX . \quad (3.11)$$

Note that if h_2 is a constant, then the second term in (3.11) is zero by virtue of (3.10b). It can now be show that

$$\frac{\partial P}{\partial T} = \int_{-\infty}^{\infty} \left(\frac{h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \right)_X \left\{ -\tfrac{1}{2} U^2 + g(\rho_2 - \rho_1)\rho_2 A \right\} dX. \quad (3.12)$$

Note that since h_{2X} is $0(\varepsilon^2)$, the right-hand side of (3.12) is also $0(\varepsilon^2)$ and we have omitted terms of $0(\varepsilon^4)$ here. It can be verified that (3.12) is the counterpart of the relation (2.14).

§4. Derivation of variable-coefficient Korteweg-de_Vries equations

The Boussinesq equations (3.9) describe waves which can propagate both to the left and to the right. The next step is to consider just the waves propagating to the right. The procedure we follow is very similar to that described by van Groesen and Pudjaprasetya (1993) for water waves (see also Pudjaprasetya and van Groesen, 1995), and hence we shall only give a brief outline here. In place of A and U we introduce the new variables R and S as follows,

$$\left. \begin{aligned} A &= R - S \\ U &= \frac{g(\rho_2 - \rho_1)}{c}(R+S) \end{aligned} \right\} \quad (4.1a)$$

where $$c^2 = \frac{g(\rho_2 - \rho_1) h_1 h_2}{(\rho_1 h_2 + \rho_2 h_1)} . \quad (4.1b)$$

Here c is the linear long wave phase speed for a flat bottom, and R, S are the Riemann invariants for the linearised Boussinesq equations in that case. Substituting (4.1a) into the Hamiltonian (3.6) we find that $\mathcal{H} = 2g(\rho_2 - \rho_2)\hat{\mathcal{H}}$ where

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{J}\, dX \quad (4.2a)$$

and $$\hat{J} = \tfrac{1}{2}(R^2 + S^2) + \varepsilon^2 \frac{g(\rho_2 - \rho_1)}{2c^2} \ldots$$
$$\left\{ -\beta(R_X + S_X)^2 + \nu(R+S)^2(R-S) \right\} + \ldots \quad (4.2b)$$

The counterpart of the Hamiltonian system (3.8) is now obtained by substituting (4.1a) into (3.8) and it is readily found that

$$\begin{pmatrix} R_T \\ S_T \end{pmatrix} = \begin{pmatrix} -\Gamma \;, & -\tfrac{1}{2}c_X \\ \tfrac{1}{2}c_X \,, & -\Gamma \end{pmatrix} \begin{pmatrix} \delta\hat{\mathcal{H}}/\delta R \\ \delta\hat{\mathcal{H}}/\delta S \end{pmatrix}, \quad (4.3a)$$

where $$\Gamma = \tfrac{1}{2}\left\{ c\frac{\partial}{\partial X} + \frac{\partial}{\partial X} c \right\}. \quad (4.3b)$$

Note that the operator on the right-hand side of (4,3a) is skew-symmetric, and so (4.3a) is skew-symmetric, and so (4.3a) is a Hamiltonian system which conserves the Hamiltonian $\hat{\mathcal{H}}$.

Next we consider the equation for the left going wave S, which to leading order is

$$S_T = cS_X + \tfrac{1}{2}c_X(R+S) + 0(\varepsilon^2) . \quad (4.4)$$

But now we recall that we are assuming that h_{2X} is $0(\varepsilon^2)$ and, to leading order, we can now effectively put $S = 0$ in (4.3a) and in the Hamiltonian density $\hat{J}$ (4.2b). Thus we can now replace (4.2b) with

$$\hat{J} = \tfrac{1}{2}R^2 + \varepsilon^2 g(\rho_2 - \rho_1)(-\beta R_X^2 + \nu R^3), \tag{4.5}$$

while the equation (4.3a) for R becomes

$$R_T = -\Gamma \frac{\delta \hat{\mathcal{H}}}{\delta R}. \tag{4.6}$$

This is the desired variable-coefficient KdV-equation. Using the expressions (3.7b, c) for ρ and ν respectively, and (4.1b) for c^2, the expression (4.5) for $\hat{J}$ can be further simplified to

$$\hat{J} = \tfrac{1}{2}R^2 + \varepsilon^2\left\{-\frac{\lambda}{2}R_X^2 + \frac{\mu}{6}R^3\right\} + \dots, \tag{4.7a}$$

where $$\lambda = \frac{h_1 h_2}{6}\frac{(\rho_1 h_1 + \rho_2 h_2)}{(\rho_1 h_2 + \rho_2 h_1)}, \tag{4.7b}$$

and $$\mu = \frac{3}{2h_1 h_2}\frac{(\rho_2 h_1^2 + \rho_1 h_2^2)}{(\rho_1 h_2 + \rho_2 h_1)}. \tag{4.7c}$$

Finally, we note that, as shown by van Groessen and Pudjaprasetya (1993) we can replace R with A in (4.6) and (4.7c) with an error which is $0(\varepsilon^4)$. Henceforth we shall use (4.6) and (4.7c) with the variable A, as $\eta = \varepsilon^2 A$ is the interface displacement. The result is the variable-coefficient KdV equation (1.2). Note that putting $\rho_1 \equiv 0$ reduces the present results to that for water waves obtained by van Groesen and Pudjaprasetya (1993) and Pudjaprasetya and van Groesen (1995).

The variable coefficient KdV-type equation (1.2) conserves the Hamiltonian $\hat{\mathcal{H}}$, that is the energy

$$\hat{\mathcal{E}} = \int_{-\infty}^{\infty} \hat{J}\, dX \tag{4.8}$$

which is the counterpart of the conservation law (3.10c) for the Boussinesq system (3.9). It also conserves the mass-like expression

$$\hat{\mathcal{M}} = \int_{-\infty}^{\infty} \frac{A}{\sqrt{c}}\, dX \tag{4.9}$$

which is a Casimir for the Hamiltonian system (1.2). But note that $\hat{\mathcal{M}}$ differs from the exact expression for the mass $\mathcal{M}$ (3.10a) by $0(\varepsilon^2)$ terms, which arise from the generation of $0(\varepsilon^2)$ left-going waves, as described above. Further, it can be shown that the conservation law (4.9) is a consequence of the exact conservation laws (3.10a, b), after again accounting for the $0(\varepsilon^2)$ left-going waves (cf. Miles, 1979, and Akylas and Prasad, 1997). The counterpart of the expression $\mathcal{P}$ (3.11) for the horizontal momentum is

$$\hat{P} = \int_{-\infty}^{\infty}\left\{\frac{A^2}{2c} + \frac{\rho_2 c}{2g(\rho_2 - \rho_1)}(A+S)\right\} dX \tag{4.10}$$

Note that it is necessary here to include the term S representing the left-going waves. Then it can be shown that

$$\frac{\partial \hat{P}}{\partial T} = \int_{-\infty}^{\infty} \frac{c_X}{c}\left\{-\tfrac{1}{2}A^2 + \frac{\rho_2 c^2 A}{g(\rho_2 - \rho_1)}\right\} dX, \tag{4.11}$$

which is just the counterpart of (3.12). In defining (4.10) and deriving (4.11) we are consistently ignoring $0(\varepsilon^4)$ terms, but retaining all $0(\varepsilon^2)$ terms.

Finally, in this section, we consider the transformation of the variable-coefficient KdV-equation (1.2) into the conventional form (1.1). To do this, we recall that $h_2 = h_2(\tau)$ where $\tau = \varepsilon^2 X$, and then define a new variable θ by (1.3). Then (1.2) transforms to (1.1), with an error term of $0(\varepsilon^2)$. Thus (1.1) and (1.2) are only asymptotically equivalent. Equation (1.1) is also Hamiltonian in the sense that

$$A_\tau + \frac{c_\tau}{2c}A = -\frac{1}{c}\frac{\partial}{\partial\theta}\frac{\delta \mathcal{J}}{\delta A}, \tag{4.12a}$$

where $$\mathcal{J} = \int_{-\infty}^{\infty} I\, d\theta, \tag{4.12b}$$

and $$I = -\frac{\lambda}{2c^2}A_\theta^2 + \frac{\mu}{6}A^3. \tag{4.12c}$$

However, because I depends explicitly on τ, the Hamiltonian $\mathcal{J}$ is not conserved here. Instead, equation (1.1) has the two invariants,

$$\mathcal{M}^* = \int_{-\infty}^{\infty} \sqrt{c}\, A\, d\theta, \tag{4.13a}$$

and $$\mathcal{E}^* = \int_{-\infty}^{\infty} \tfrac{1}{2}cA^2 d\theta. \tag{4.13b}$$

Note that here $\mathcal{M}^*$ is the counterpart of the conserved expression (4.9) for equation (4.2), while $\mathcal{E}^*$ in (4.13b) is the leading order term in the energy (4.8), since from (1.3) we have that $dX = c\,d\theta$ to leading order. It is interesting to observe that the Hamiltonian density I in (4.12b) is just the higher-order part of the Hamiltonian density $\hat{J}$ for equation (1.2a). Of course, the full expression for the energy $\hat{\mathcal{E}}$ should remain a conserved quantity under the transformation (1.3), but in order to achieve this it is necessary to include higher-order terms of $0(\varepsilon^2)$ both in equation (1.1), and in the transformation of the expression (4.8) for $\hat{\mathcal{E}}$. Finally we note that the momentum density $\hat{\mathcal{P}}$ transforms to

$$\mathcal{P}^* = \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2} A^2 + \frac{\rho_2 c^2 A}{2g(\rho_2 - \rho_1)} \right\} d\theta. \qquad (4.14)$$

Like $\hat{\mathcal{P}}$ this is not a conserved quantity, and we find that

$$\mathcal{P}_\tau^* = \int_{-\infty}^{\infty} \frac{c_\tau}{c} \left\{ -\tfrac{1}{2} A^2 + \frac{3\rho_2 c^2}{4g(\rho_2 - \rho_1)} A \right\} d\theta, \qquad (4.15)$$

which is just the counterpart of (4.11).

§5. Applications

In this section we discuss two applications of the variable-coefficient KdV-equation (1.2a) which exploit the Hamiltonian form, and in particular, the conservation of the Hamiltonian $\hat{\mathcal{H}}$. First, we consider the slowly-varying solitary wave, and then the generation of a second solitary wave due to the variable depth. In both parts we will assume that the depth $h_2 = h_2(s)$ where

$$s = \sigma X \qquad (5.1)$$

and $\sigma << \varepsilon^2$. Recall that the derivation of (4.8) already requires h_{2X} to be $0(\varepsilon^2)$, so that now we are requiring that the variation is slower still. We adopt the point-of-view, that even though ε^2 is a small parameter, we shall regard (1.2) as a given "exact" equation. Strictly speaking, it should also be assumed that $\sigma >> \varepsilon^4$, so that the terms of $0(\sigma)$ due to the variable depth are larger than the $0(\varepsilon^4)$ error terms in (1.2).

(i) Slowly-varying solitary wave

The asymptotic procedure employed here is standard as far as the more conventional equation (4.14) is concerned (see, for instance, Grimshaw and Mitsudera, 1993), so here we give only a brief outline. We seek an asymptotic solution of (1.2) whose leading term is a solitary wave of variable amplitude $a(s)$ and variable speed $V(s)$. Thus we put

$$\phi = \frac{1}{\sigma} \int_0^s \frac{ds'}{V(s')} - T \qquad (5.2)$$

and seek a solution of (1.2) in the form,

$$A = A_0(\phi, s) + \sigma A_1(\phi, s) + \dots \qquad (5.3a)$$

$$V = V_0 + \sigma V_1 + \dots . \qquad (5.3b)$$

It is readily found that A_0 satisfies the equation

$$\frac{(V_0 - c)}{c} A_0 = \varepsilon^2 \left\{ \frac{\lambda}{V_0^2} A_{0\phi\phi} + \mu \frac{A_0^2}{2} \right\}, \qquad (5.4)$$

This has the well-known KdV-solitary wave as a solution,

$$A_0 = a \operatorname{sech}^2 \gamma\phi, \qquad (5.5a)$$

where
$$\frac{V_c - c}{c} = \varepsilon^2 \frac{\mu a}{3} = \varepsilon^2 \lambda \left(\frac{\gamma}{V_0} \right)^2. \qquad (5.5b)$$

At the next order we obtain the following equation for A_1,

$$\frac{c}{V_0} \frac{\partial}{\partial\phi} \left\{ -\frac{V_0 - c}{c} A_1 + \varepsilon^2 \left(\frac{\lambda}{V_0^2} A_{0\phi\phi} + \mu A_0 A_1 \right) \right\} + F_1 = 0 \qquad (5.6a)$$

where
$$F_1 = (V_0 A_0)_s - \frac{c_s}{2c}(V_0 A_0) + \varepsilon^2 \frac{c}{V_0} \frac{\partial}{\partial\psi} \left\{ \frac{2\lambda}{V_0} A_{0\phi s} + \left(\frac{\lambda}{V_0} \right)_s A_{0\phi} \right\}$$
$$+ V_1 \left\{ -\frac{c}{V_0^2} A_{0\psi} + \epsilon^2 \left(-\frac{3c\lambda}{V_0^4} A_{0\phi\phi\phi} - \frac{c\mu}{V_0^2} A_0 A_{0\phi} \right) \right\}. \qquad (5.6b)$$

Here we have used (5.4) to simplify the expression for F_1. The compatibility condition for (5.6a) is

$$\int_{-\infty}^{\infty} F_1 A_0 \, d\phi = 0. \qquad (5.7)$$

It can now be shown that this yields the equation

$$\frac{\partial}{\partial s} \left\{ \int_{-\infty}^{\infty} \frac{1}{2} \frac{V_0^2 A_0^2}{c} d\phi - \varepsilon^2 \int_{-\infty}^{\infty} \frac{\lambda}{V_0} A_{0\phi}^2 d\phi \right\} = 0, \qquad (5.8a)$$

and so

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{V_0^2 A_o^2}{c} d\phi - \varepsilon^2 \int_{-\infty}^{\infty} \frac{\lambda}{V_0} A_{0\phi}^2 \, d\phi = \text{constant}. \qquad (5.8b)$$

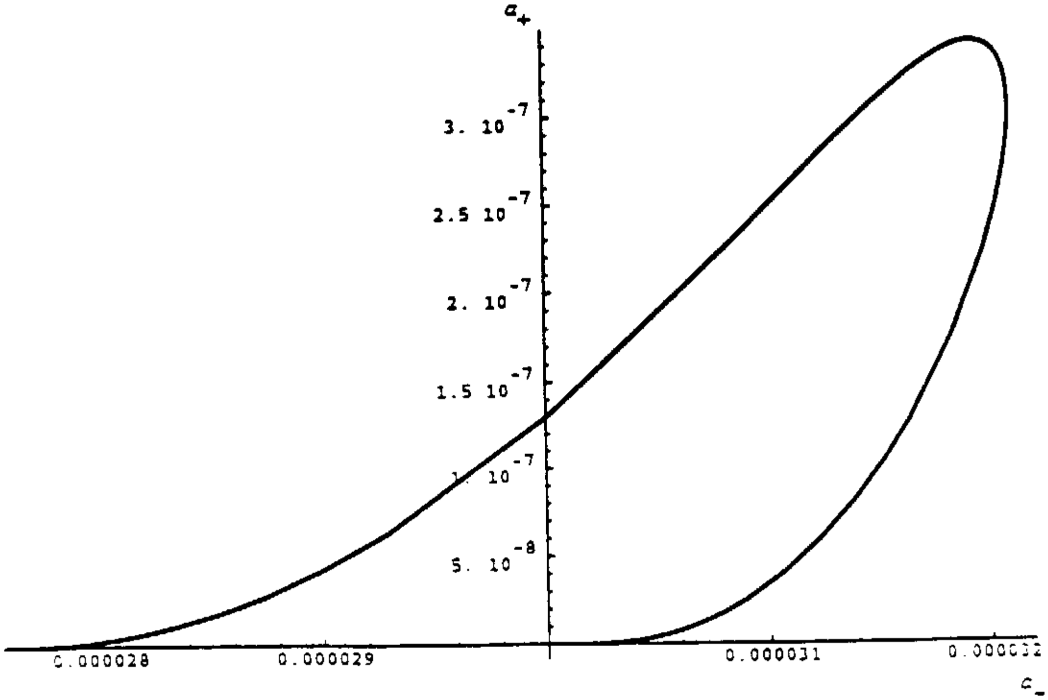

Fig. 2. Plot of $\alpha_\pm$ determined from (5.14). (a) Trajectors of the α_+ / α_- plane as h_2 decreases.

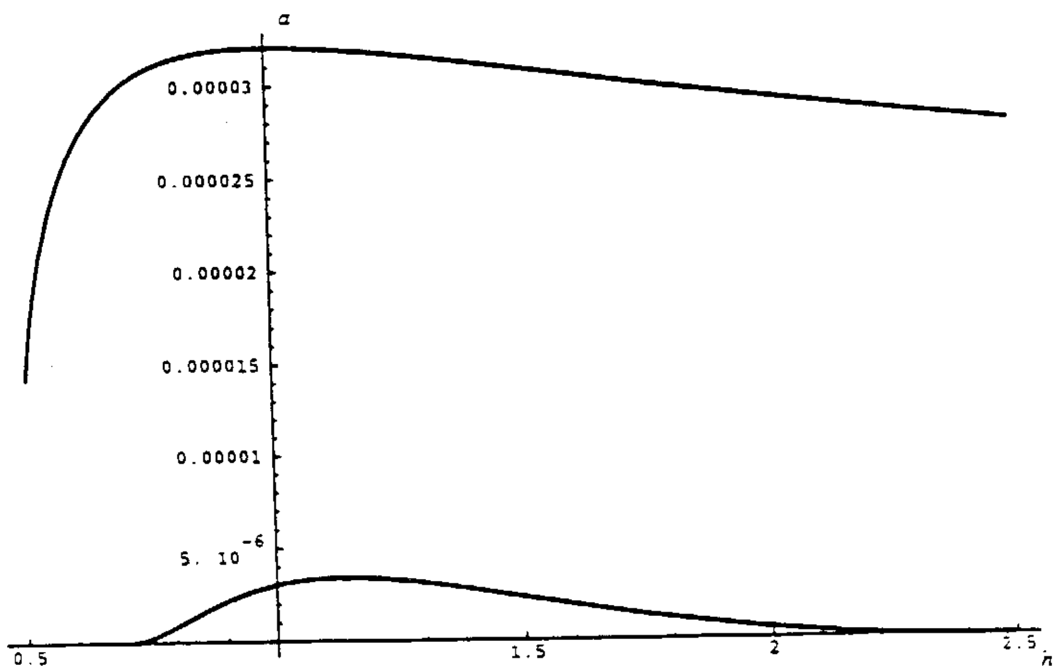

Fig. 2. (b) Plots of $\alpha_\pm$ as functions of h_2.

This last expression thus determines the variation of the amplitude $a(s)$. Before substituting (5.5) into (5.8b) to achieve this, we note that on using (5.4), (5.8b) can also be written as

$$\int_{-\infty}^{\infty} \tfrac{1}{2} V_0 A_0^2 \, d\phi \; .$$

$$+ \varepsilon^2 \int_{-\infty}^{\infty} \left(-\frac{\lambda}{2V_0^2} A_{0\phi}^2 + \frac{\mu}{6} A_0^3 \right) V_0 d\phi = \text{constant} \qquad (5.9)$$

Noting that $dX = V d\phi$ for a fixed value of T (see (5.2)), we see that the left-hand side of (5.9) (or (5.8b)) is just the energy $\hat{\mathcal{E}}$ (4.8) evaluated for the leading order term A_0 (see (1.2c)). Thus as expected, the variation of the solitary wave amplitude is determined by the conservation of energy.

The analogous theory for the conventional KdV-equation (1.1) leads to the same expression (5.9) but with the omission of the $0(\varepsilon^2)$ term on the left-hand side of (5.9), the replacement of V_0 with c, and $d\phi$ with $d\theta$, again with errors of $0(\varepsilon^2)$. Thus for both equations the variation of the solitary wave amplitude is determined by conservation of energy, but for the conventional equation (1.1), the energy is given by just the leading order term $\mathcal{E}^*$ (4.13b).

Next we comment on the well-known fact that although the slowly-varying solitary wave conserves energy, it cannot by itself, conserve the mass $\hat{\mathcal{M}}$. Instead, this is conserved by the creation of a trailing shelf of amplitude of $0(\sigma)$, but whose length is $0(\sigma^{-1})$. At the rear of the solitary wave, the amplitude of the trailing shelf is A_1^-, where $A_1 \to 0$ as $\phi \to -\infty$. We readily find from (5.6) that

$$\left(\frac{V_0 - c}{V_0} \right) A_1^- + \sqrt{c}\, \frac{\partial}{\partial s} \left\{ \int_{-\infty}^{\infty} \frac{V_0 A_0}{\sqrt{c}} d\theta \right\} = 0 \; . \qquad (5.10)$$

Also we note that the first-order speed correction term V_1 is not determined at this order, and it is necessary to proceed to second order to find it (Grimshaw and Mitsudera, 1993).

Finally, we substitute the explicit expressions (5.5) for A_0 and V_0 into (5.8b) to obtain, after consistently omitting $0(\varepsilon^4)$ terms,

$$\frac{a^3 \lambda}{\mu} \left\{ 1 + \frac{2\varepsilon^2}{5} \mu a \right\} = \text{constant} \; . \qquad (5.11)$$

The explicit expressions for λ and μ (4.7b, c) can now be substituted into (5.11) to yield the variation of the amplitude a as a function of the depth h_2. First, however, we consider the special case when $\rho_1 = 0$ in which case the result (5.11) reduces to that for water waves obtained by Pudjaprasetya and van Groesen (1995). In this special case, in replacing h_2 with h, the total depth of water, for convenience, we see that $\lambda = h^2/6$ and $\mu = \tfrac{3}{2} h$ so that (5.11) becomes

$$(ah)^3 \left\{ 1 + \varepsilon^2 \frac{3a}{5h} \right\} = \text{constant} \qquad (5.12)$$

to leading order in ε^2, this gives the well-known result that $a \propto h^{-1}$. Here we see that the $0(\varepsilon^2)$ term will slightly diminish this result. For interfacial waves, on using the Boussinesq approximation, we get

$$\frac{a^3 h_1^3 h_2^3}{(h_1 - h_2)} \left\{ 1 + \frac{3}{5} \varepsilon^2 a \frac{(h_1 - h_1)}{h_1 h_2} \right\} = \text{constant} \, . \qquad (5.13)$$

This expression yields the well-known result that $a \to 0$ as $h_2 \to h_1$, (e.g. Pelinovsky and Shavratsky, 1976 or Djordjevic and Redekopp, 1978), and we see that the $0(\varepsilon^2)$ term in (5.13) does not effect this in any significant way. Of course as $a \to 0$ the present asymptotic theory fails, and the fate of the solitary wave as $h_2 \to h_1$ is unclear. We see from (4.7a) that μ changes sign at $h_1 = h_2$ and so solitary waves of depression which exist for $h_2 > h_1$ cannot exist for $h_2 < h_1$. A detailed study of this situation has recently been reported by Grimshaw et. al (1997).

(ii) Generation of a second solitary wave

Let us continue to consider the problem of an interfacial solitary wave, initially in a region where $h_2 > h_1$ and propagating towards the point where $h_2 = h_1$. In the previous subsection we showed that as $h_2 \to h_1$, the solitary wave amplitude tends to zero (see (5.13)). Here we use the technique described by Pudjaprasetya et al. (1997) for water waves to determine the amplitude of a second solitary wave possibly generated in this process. In essence the technique is to assume that there are two slowly-varying solitary waves, of amplitudes $a_\pm$ say where $a_- = 0$ at $h_2 = h_{20}$, and then to use the two conservation laws (4.8) and (4.9) for energy and mass respectively to determine how the amplitudes depend on h_2. Note that, in effect, the trailing shelf of the slowly-varying solitary wave is here replaced by the second solitary wave.

Invoking conservation of energy and mass,

$$(L_+ + L_-) = \frac{c_0 L_0}{c} \qquad (5.14a)$$

$$\left(L_+^{1/3} + L_-^{1/3}\right) = \left(\frac{c}{c_0}\right)^{1/6} \left(\frac{\mu\lambda_0}{\lambda\mu_0}\right)^{1/3} L_0^{1/3} . \qquad (5.14b)$$

Here the subscript "0" denotes conditions at $h_2 = h_{20}$, and for each wave, L is the momentum-like integral (see (4.11)),

$$L = \int_{-\infty}^{\infty} \tfrac{1}{2} A_0^2 \frac{V_0}{c}\, d\theta , \qquad (5.15a)$$

or $$L = \frac{2}{3}\left(\frac{12\lambda}{\mu}\right)^2 \left(\frac{\gamma}{V_0}\right)^3 \frac{1}{c} = \frac{2}{3}\left(\frac{12\lambda}{|\mu|}\right)^{1/2} \frac{|a|^{3/2}}{c} . \qquad (5.15b)$$

Here, we are ignoring the $0(\varepsilon^2)$ term in the energy since the analysis of the previous subsection shows that this term is not significant in the present application. Equations (5.14) are readily solved for $L_\pm$ and then (5.15b) determines $a_\pm$. Finally using the expressions (4.7b, c) for λ and μ we obtain $a_\pm$ as functions of h_2. The results are shown in Figure 2, where (a) plots the trajectory in the a_+/a_- plane and (b) plots a_+ and a_- as functions of h_2 up to the point where $h_2 = h_1$. We see that as h_2 decreases a_+ increases until approximately $h_2 = 3h_1/2$ and then a_+ decreases, while a_- at first increases and then decreases. These results are for the Boussinesq approximation for which

$$\lambda \approx h_1 h_2/6, \quad \mu \approx 3(h_1 - h_2)/2h_1 h_2$$

and $$c^2/c_0^2 \approx h_2(h_1 + h_2)/h_{20}(h_1 + h_{20}) .$$

We can conclude that a small amplitude second solitary wave is generated, but it vanishes again as $h_2 \to h_1$.

§6. Conclusion

The main purpose of this paper was to demonstrate that the Hamiltonian variable-coefficient KdV equation (1.2), previously derived for water waves by van Groesen and Pudjaprasetya (1993), can also be derived in a similar way for interfacial waves. This lends support to the notion that equation (1.2) is a convenient Hamiltonian reformulation of the more conventional variable-coefficient KdV equation (1.1), which is known to be valid for many physical systems. Further, we demonstrate that through the transformation (1.3), the two equations (1.1) and (1.2) are asymptotically equivalent (with an error of $0(\varepsilon^2)$ in (1.1)). The principal difference is that the Hamiltonian form (1.2) conserves the energy $\hat{H}$ (1.2b), whereas (1.1) only conserves the leading term in $\hat{H}$. Thus we conjecture that in situations where conserving the energy to a higher order is important, then equation (1.2) should be preferred. We have illustrated this issue by considering the transformation of an interfacial solitary wave propagating in a region where the lower fluid has variable depth.

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