Nonlinear Processes in Geophysics (1998) 5: 219–240

Nonlinear Processes in Geophysics

On the Hamiltonian approach: Applications to geophysical flows

V. Goncharov[1] **and V. Pavlov**[2]

[1]Institute of Atmospheric Physics, Russian Academy of Sciences, 109017 Moscow, Russia
[2]DMF, UFR de Mathématiques Pures et Appliquées, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

Received: 26 September 1996 – Accepted: 14 February 1999

Abstract. This paper presents developments of the Hamiltonian Approach to problems of fluid dynamics, and also considers some specific applications of the general method to hydrodynamical models.

Nonlinear gauge transformations are found to result in a reduction to a minimum number of degrees of freedom, i.e. the number of pairs of canonically conjugated variables used in a given hydrodynamical system. It is shown that any conservative hydrodynamic model with additional fields which are in involution may be always reduced to the canonical Hamiltonian system with three degrees of freedom only. These gauge transformations are associated with the law of helicity conservation. Constraints imposed on the corresponding Clebsch representation are determined for some particular cases, such as, for example, when fluid motions develop in the absence of helicity.

For a long time the process of the introduction of canonical variables into hydrodynamics has remained more of an intuitive foresight than a logical finding. The special attention is allocated to the problem of the elaboration of the corresponding regular procedure.

The Hamiltonian Approach is applied to geophysical models including incompressible (3D and 2D) fluid motion models in curvilinear and lagrangian coordinates. The problems of the canonical description of the Rossby waves on a rotating sphere and of the evolution of a system consisting of N singular vortices are investigated.

1 Introduction

In the last several decades, the Hamiltonian approach (HA) to description of fluid motions for fundamental hydrodynamical models has been very intensively developed. Previously, the field versions of the Hamiltonian approach were considered only in connection with the needs of quantum field theory (see, for example, Dirac 1950, Bogoljubov and Shirkov 1980). The important features of the Hamiltonian formalism are its versatility and adaptability which manifest themselves not only in the techniques of problem formulation and solution, but also in the *conceptual approach.*

This paper presents developments of the HA[1] to problems of fluid dynamics, and also considers specific applications of the general method to hydrodynamical models.

The paper was motivated by the following questions: what is the general structure of the resulting forces permitted by hydrodynamical models? What is the physical interpretation of fields that could play roles of canonical variables? What are the general requirements of the relations between canonical variables and the Clebsch representation for momentum density?

It is shown in particular that the existence of the Clebsch representation is not a privilege of only hydrodynamic systems for which the variational principle of least action is possible, but is a typical consequence of the construction of global canonical variables for the description of any Hamiltonian systems with degenerated Poisson brackets.

In the present paper the Hamiltonian description is not absolutely axiomatically constructed, but rather is directly driven from physically-based presumptions about the type of evolution of hydrodynamical systems and their internal properties.

Nonlinear gauge transformations are found to result in a reduction to a minimum number of degrees of freedom, i.e. the number of pairs of canonically conjugated variables used in a given hydrodynamical system. It is shown that any conservative hydrodynamic model with additional fields which are in involution may be always reduced to the canonical Hamiltonian system with three degrees of freedom only. These gauge transformations are associated with the law of helicity con-

[1]Below, discussing the Hamiltonian method or the Hamiltonian approach, we will imply the version defined by (13). The version (13) of the Hamiltonian method of the description of dynamical phenomena occupies a prominent place in modern theoretical physics and has proved itself to be a powerful tool for investigations of various problems in a wide range of applications.

Correspondence to: V. Pavlov

servation. Constraints imposed on the corresponding Clebsch representation are determined for some particular cases, such as, for example, when fluid motions develop in the absence of helicity.

For a long time the process of the introduction of canonical variables into hydrodynamics has remained more of an intuitive foresight than a logical finding. The special attention is allocated to the problem of the elaboration of the corresponding regular procedure.

The Hamiltonian Approach is applied to geophysical models including incompressible (3D and 2D) fluid motion models in curvilinear and lagrangian coordinates. The problems of the canonical description of the Rossby waves on a rotating sphere and of the evolution of a system consisting of N singular vortices are investigated.

The content of this paper is determined by the authors' intent, on one hand, to present some original results, and, on the other, to give an intact view on the specifics of the subject in an intelligible and relatively complete form.

2 Hamiltonian Approach for Hydrodynamical Systems

2.1 Background

Any conservative hydrodynamic model describing a fluid motion possesses two principal field attributes - hydrodynamic velocity $\mathbf{v}(\mathbf{x}, t)$ and mass density $\rho(\mathbf{x}, t)$, which satisfy the Euler equation and the continuity equation, respectively:

$$\dot{\pi}_i + \partial_j(v_j \pi_i) = F_i, \tag{1}$$

$$\dot{\rho} + div(\rho \mathbf{v}) = 0. \tag{2}$$

Here $i = 1, 2, 3,\ \dot{u} \equiv \partial_t u$. The first equation (1) expresses the law of change of the momentum density $\vec{\pi} = \rho \mathbf{v}$ under the influence of the resulting force $\mathbf{F}$. The second equation (2) expresses the mass conservation law.

In hydrodynamic models of a broader class, in addition to $\mathbf{v}$ and ρ there can appear a number of new physical fields $s_\alpha(\mathbf{x}, t)$ ($\alpha = 1, 2, ..., N$) describing the additional properties of the medium (e.g., magnetic, thermal, *etc.*) with corresponding evolution equations

$$\dot{s}_\alpha + \mathcal{L}(\mathbf{v}, \rho, s_\beta) = 0. \tag{3}$$

The additional field variables may have a scalar, tensor or spinor nature and their presence means that the model of continuous medium possesses intrinsic degrees of freedom.

We remind here some well-known types of hydrodynamical equations that appear in fluid dynamics.

a) The system of equations for compressible barotropic fluid is given by the equations

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\rho^{-1}\nabla p, \tag{4}$$

$$\dot{\rho} + div(\rho \mathbf{v}) = 0, \tag{5}$$

and completed by the equation of state $p = p(\rho)$. Here the force $\mathbf{F}[\rho] = -\nabla p$.

b) For the compressible (non-barotropic) fluid in the gravity field of the potential χ, we can write

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\rho^{-1}\nabla p - \nabla\chi, \tag{6}$$

$$\dot{\rho} + div(\rho \mathbf{v}) = 0, \tag{7}$$

$$\dot{s}_1 + (\mathbf{v} \cdot \nabla)s_1 = 0. \tag{8}$$

Here p is the pressure, the internal energy is a functional of density and entropy, so that $p = p(\rho, s_1)$, s_1 is the entropy per mass unit[2]. When the fluid flow is isentropic, i.e., there is no dissipation of any form, the evolution operator in (3) is defined by the expression $\mathcal{L}(\mathbf{v}, \rho, s_k) \equiv (\mathbf{v} \cdot \nabla)s_1$. In this case, we have $M = 1$ and the fluid motion is defined by a set of five functions $\vec{\pi}$, ρ, s_1 under the condition that the equation of state is fixed. In this case we obtain that $\mathbf{F}[\rho, s_1] = -\nabla p - \rho\nabla\chi$.

c) One can consider another widely known example - the perfect MHD equations with the lagrangian field s_1, and the frozen-in[3] vector field $\mathbf{B}/\rho$

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} =$$

$$= -\rho^{-1}\nabla p - \nabla\chi + (4\pi\rho)^{-1}[rot\, \mathbf{B}, \mathbf{B}], \tag{9}$$

$$\dot{\rho} + div(\rho \mathbf{v}) = 0, \tag{10}$$

$$\dot{s}_1 + (\mathbf{v} \cdot \nabla)s_1 = 0, \tag{11}$$

$$\partial_t(\frac{\mathbf{B}}{\rho}) + (\mathbf{v} \cdot \nabla)\frac{\mathbf{B}}{\rho} - (\frac{\mathbf{B}}{\rho} \cdot \nabla)\mathbf{v} = 0. \tag{12}$$

completed by the equation of state $p = p(\rho, s_1)$. Here the resulting force has the form $\mathbf{F}[\rho, s_1, \mathbf{B}] = -\nabla p - \rho\nabla\chi + (4\pi)^{-1}[rot\, \mathbf{B}, \mathbf{B}]$.

A list of similar examples can be continued.

The questions of (i) what is the general structure of $\mathbf{F}$ admitted by hydrodynamical models and (ii) what is the general form of evolution equations (3) of the system arise immediately.

Let us remind some definitions.

The Hamiltonian structure of hydrodynamical models consists of the hamiltonian functional given by the total energy $\mathcal{H}$ and the functional Poisson bracket $\{\,,\}$. The Hamiltonian systems evolve according to the law

$$\dot{u}_i = \{u_i, \mathcal{H}\} = \int d\mathbf{x}'\, \mathcal{U}^{ij}[u; \mathbf{x}, \mathbf{x}']\, \frac{\delta\mathcal{H}}{\delta u_j(\mathbf{x}')}, \tag{13}$$

where the Hamiltonian $\mathcal{H}$ of the system is the quantity functionally depending on the fields u_i.

Conservation of energy follows from the given formulation, since $\{\mathcal{H}, \mathcal{H}\}$.

The Poisson bracket of quantities $\mathcal{F}$ and $\mathcal{G}$ given on a phase space of the differentially - independent field variables $u_i(t, \mathbf{x})$

[2] The function s_1 is a so-called Lagrangian invariant. By definition, the Lagrangian invariant is governed by the equation similar to (8). The physical meaning of Lagrangian invariants is reduced to their advection by the flow. For an example of such invariants, the Ertel invariant $I = \rho^{-1}\nabla s_1 \cdot rot\, \mathbf{v}$ (Ertel 1942) can be considered. This invariant is of great importance in geophysical hydrodynamics and dynamical meteorology.

[3] It is well known that the field $rot\, \mathbf{v}$ for incompressible fluids, $\mathbf{B}/\rho$ for MHD models of compressible fluid, the fields $J_{e,i} = n_{e,i}^{-1}(rot\, \mathbf{v}_{e,i} \pm (c/m_{e,i}c)\mathbf{H})$ into the electron (ion) fluid, *etc.* are frozen.

$i = 1, 2, \ldots, N$ is specified with the tensor field $\mathcal{U}^{ij}[u; \mathbf{x}, \mathbf{x}']$ by the rule

$$\{\mathcal{F}, \mathcal{G}\} = \int d\mathbf{x}d\mathbf{x}' \frac{\delta \mathcal{F}}{\delta u_i(\mathbf{x})} \mathcal{U}^{ij}[u; \mathbf{x}, \mathbf{x}'] \frac{\delta \mathcal{G}}{\delta u_j(\mathbf{x}')}. \quad (14)$$

The notation $\mathcal{U}^{ij}[u; \mathbf{x}, \mathbf{x}']$ means that the so-called Poisson tensor $\mathcal{U}^{ij}$ is, generally speaking, not only a function of the coordinates $\mathbf{x}$ and $\mathbf{x}'$, but also a functional of the fields u_i. The substitution of $\mathcal{F} = u_i(t, \mathbf{x})$, $\mathcal{G} = u_j(t, \mathbf{x}')$ in (14) brings about the relation

$$\{u_i(\mathbf{x}), u_j(\mathbf{x}')\} \equiv \mathcal{U}^{ij}[u; \mathbf{x}, \mathbf{x}']. \quad (15)$$

It follows from (15) that the Poisson tensor is given if a complete set of the fundamental Poisson brackets $\{u_i(\mathbf{x}), u_j(\mathbf{x}')\}$ are given. Using (15), we can rewrite (14) in the form

$$\{\mathcal{F}, \mathcal{G}\} = \int d\mathbf{x}d\mathbf{x}' \frac{\delta \mathcal{F}}{\delta u_i(\mathbf{x})} \{u_i(\mathbf{x}), u_j(\mathbf{x}')\} \frac{\delta \mathcal{G}}{\delta u_j(\mathbf{x}')} \quad (16)$$

By definition, the Poisson brackets, (i), possess the property of skew-symmetry

$$\{u_i(\mathbf{x}), u_j(\mathbf{x}')\} = -\{u_j(\mathbf{x}'), u_i(\mathbf{x})\}, \quad (17)$$

that is equivalent to the requirement of skew-symmetry for the tensor field $\mathcal{U}^{ij}$, and, (ii), satisfy the Jacobi identity

$$\mathcal{T}^{ijk} = \{u_i(\mathbf{x}), \{u_j(\mathbf{x}'), u_k(\mathbf{x}'')\}\} + c.p. = 0. \quad (18)$$

Here the abbreviation *c.p.* denotes the terms derived from the first term by a cyclic permutation of the indices and the arguments.

Whenever we speak about the Hamiltonian structure of the equations describing dynamics of a continuous (i.e. infinite-dimensional) system, we shall in essence imply that there are specific forms of writing these equations. The canonical form of writing is the most familiar of all. In this case, a system is described with an *even* number of equations

$$\dot{q}_i = \frac{\delta \mathcal{H}}{\delta p_i}, \quad \dot{p}_i = -\frac{\delta \mathcal{H}}{\delta q_i} \quad (19)$$

for two groups of field variables - the generalized coordinates $q_1, \ldots, q_N$ and the momenta $p_1, \ldots, p_N$ which are functions of time t and space coordinate $\mathbf{x}$. The variables ensuring such a structure of equations are named canonical and the quantity $\mathcal{H}$ under functional derivatives which depends functionally on those variables, is named the Hamiltonian.

Other versions of Hamiltonian systems are also possible. As example of systems with the *odd* number of fields one can present the Korteweg-de Vries equation

$$\dot{u} = \partial_x \frac{\delta \mathcal{H}}{\delta u}. \quad (20)$$

The canonical Hamiltonian formalism (19) corresponds to the Poisson brackets of the following form:

$$\{q_i, q_j'\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j'\} = \delta_{ij}\delta(\mathbf{x} - \mathbf{x}'). \quad (21)$$

In the second instance, the Hamiltonian structure is defined by the Poisson brackets of another type

$$\{u, u'\} = \partial_x \delta(\mathbf{x} - \mathbf{x}'). \quad (22)$$

Both examples are the simplest in respect to the fact that the corresponding Poisson tensor $\mathcal{U}^{ij}$ is functionally independent of the field variables and, therefore, condition (18) is satisfied automatically.

There are classes of nontrivial Hamiltonian systems with $\delta\mathcal{U}^{ij}/\delta u \neq 0$.

An example of a Hamiltonian system with the nontrivial Poisson bracket

$$\{\omega_i, \omega_j'\} = \varepsilon^{ipn}\varepsilon^{jkl}\varepsilon^{nls}\partial_p \omega_s \partial_k \delta(\mathbf{x} - \mathbf{x}'), \quad (23)$$

where ε^{ijk} is the Levi-Chivitta symbol, is the equation (Arnol[d] 1969)

$$\dot{\vec{\omega}} = rot \left[\vec{\omega}, rot \frac{\delta \mathcal{H}}{\delta \vec{\omega}}\right] \quad (24)$$

This equation describes an evolution of the vorticity field $\vec{\omega} = rot\, \mathbf{v}$ ($\mathbf{v}$ is hydrodynamic velocity field) in a homogeneous incompressible fluid.

For the Landau-Lifshitz equation (Lifshitz and Pitaevskii 1978)

$$\dot{\mathbf{n}} = [\mathbf{n}, \frac{\delta \mathcal{H}}{\delta \mathbf{n}}], \quad (25)$$

with ($\mathbf{n}^2(\mathbf{x}, t) = 1$), the brackets are determined by the expression

$$\{n_i, n_j'\} = \varepsilon^{ijk} n_k \delta(\mathbf{x} - \mathbf{x}'). \quad (26)$$

The transformation (Faddeev 1976) from the vector field of vorticity $\vec{\omega}$ to $\mathbf{n}$–field

$$\omega_\alpha = \mu\, \varepsilon^{\alpha\beta\gamma} \mathbf{n} \cdot [\partial_\beta \mathbf{n}, \partial_\gamma \mathbf{n}], \quad (27)$$

where μ is a dimensional constant, maps equations (25) and (24) one into another and, thus, sets up one-to-one correspondence between vortex and spin dynamics.

2.2 Resultant of The Hydrodynamic Forces

What is the general structure of the resulting forces $\mathbf{F}[\rho, s_n \ldots]$ permitted by hydrodynamical models?

From the point of view of the Hamiltonian formalism (HA), the dynamics of hydrodynamical systems is described in the phase space of fields $\vec{\pi}, \rho, s_1, \ldots, s_N$ and is determined by the complete set of the Poisson brackets

$$\begin{aligned}&\{\pi_i, \pi_j'\}, \{\pi_i, \rho'\}, \{\pi_i, s_\alpha'\},\\ &\{\rho, \rho'\}, \{\rho, s_\alpha'\}, \{s_\alpha, s_\beta'\},\end{aligned} \quad (28)$$

and by the Hamiltonian $\mathcal{H}$. Here $i, j = 1, 2, 3$; $\alpha, \beta = 1, 2, \ldots, N$.

The general structure of $\mathcal{H}$ is taken from physical considerations in the form

$$\mathcal{H} = \mathcal{T} + \mathcal{U} = \int d\mathbf{x} \frac{\vec{\pi}^2}{2\rho} + \mathcal{U}(\rho, s_1, \ldots, s_N). \quad (29)$$

Here $\mathcal{T}$, $\mathcal{U}$ are the functionals representing kinetic and potential energies[4].

Keeping in mind the structure of the Hamiltonian (29), we write the motion equations that describe evolution of the fields ρ, s_1, ..., s_N, $\vec{\pi}$ in terms of the Poisson brackets

$$\dot{\rho} = \int d\mathbf{x}'[v'_k\{\rho, \pi'_k\} + +(\frac{\delta\mathcal{U}}{\delta\rho'} - \frac{{v'_k}^2}{2})\{\rho, \rho'\} + \frac{\delta\mathcal{U}}{\delta s'_k}\{\rho, s'_k\}], \tag{31}$$

$$\dot{s}_i = \int d\mathbf{x}'[v'_k\{s_i, \pi'_k\} + +(\frac{\delta\mathcal{U}}{\delta\rho'} - \frac{{v'_k}^2}{2})\{s_i, \rho'\} + \frac{\delta\mathcal{U}}{\delta s'_k}\{s_i, s'_k\}], \tag{32}$$

$$\dot{\pi}_i = \int d\mathbf{x}'[v'_k\{\pi_i, \pi'_k\} + +(\frac{\delta\mathcal{U}}{\delta\rho'} - \frac{{v'_k}^2}{2})\{\pi_i, \rho'\} + \frac{\delta\mathcal{U}}{\delta s'_k}\{\pi_i, s'_k\}]. \tag{33}$$

Here, as above, the prime denotes that field variables depend on the space coordinate $\mathbf{x}'$ and the summation convention is implied for repeated indices.

By comparing (31) with (2), it is easy to find that (31) reproduces the continuity equation (2) if

$$\{\rho\,,\,\rho'\} = \{\rho\,,\,s'_\alpha\} = 0, \tag{34}$$

$$\{\rho\,,\,\pi'_k\} = -\partial_k(\rho\delta). \tag{35}$$

Here the symbol δ defines the Dirac-function: $\delta \equiv \delta(\mathbf{x}-\mathbf{x}')$.

Using (35) and comparing the Euler equation (1) with the equation (33), we obtain the condition for the resultant of hydrodynamic forces

$$F_i = \{\pi_i\,,\,\mathcal{U}\} + + \int d\mathbf{x}' v'_k[\{\pi_i, \pi'_k\} - \partial'_i(\pi'_k\delta) + \partial_k(\pi_i\delta)], \tag{36}$$

where

$$\{\pi_i, \mathcal{U}\} = \int d\mathbf{x}'[\frac{\delta\mathcal{U}}{\delta\rho'}\{\pi_i\,,\,\rho'\} + \frac{\delta\mathcal{U}}{\delta s'_n}\{\pi_i\,,\,s'_n\}]. \tag{37}$$

Like all mechanical systems obey laws of classical mechanics, hydrodynamical systems should satisfy the basic principles of hydrodynamics including Galileo's relativity principle. In accordance with the latter principle the equations of motion should be invariant with respect to space-time translations, spatial rotations, and changes to an arbitrarily-moving coordinate frame.

[4] We note that in the relativistic case we should take the following expression for the kinetic energy

$$\mathcal{T}[\vec{\pi}] = \int d\mathbf{x}\, \rho c^2[1 + (\vec{\pi}/\rho c)^2]^{1/2}, \tag{30}$$

where c is the velocity of light. The relation of hydrodynamic velocity and momentum is given by the universal formula $\mathbf{v} = \delta\mathcal{T}/\delta\vec{\pi}$, which leads to relationship $\rho\mathbf{v} = \vec{\pi}$ in the case of nonrelativistic fluid dynamics and to $\rho\mathbf{v} = \vec{\pi}[1 + (\vec{\pi}/\rho c)^2]^{-1/2}$ in the relativistic case.

However, Galileo's relativity principle does not fix all necessary Poisson brackets, but rather merely defines more precisely their general structure and gives the zero-th order term into the brackets with the momentum:

$$\{\pi'_k\,,\,\pi_i\} = \delta\,\partial_k\pi_i + \ldots,$$
$$\{\pi'_k\,,\,s_n\} = \delta\,\partial_k s_n + \ldots. \tag{38}$$

Here dots denote the additional terms including the delta function derivatives of the first and higher orders .

Thus, the Hamiltonian description remains incomplete until the Poisson brackets

$$\{\pi_i, \pi'_j\},\ \{\pi_i, s'_\alpha\},\ \{s_\alpha, s'_\beta\}$$

are fixed.

To solve this problem let us adopt a number of assumptions regarding the physical character of the evolution of additional fields and the hydrodynamic forces.

First, we consider the comparatively simple models with additional fields s_α evolving only due to hydrodynamic transfer. In other words, if in (32) $\vec{\pi} = 0$, then independently of specifics of the functional $\mathcal{U}$ it must follow that $\dot{s}_\alpha = 0$. According to (32), such a statement implies that

$$\{s_l\,,\,s'_n\} = 0. \tag{39}$$

Therefore, the fields describing the additional properties of the medium evolve according to the law

$$\dot{s}_l = \{s_l, \mathcal{H}\} = \int d\mathbf{x}' v'_k\{s_l\,,\,\pi'_k\}. \tag{40}$$

Secondly, let us restrict the analysis to the systems whose resultant of the hydrodynamic forces $\mathbf{F}$ is functionally independent of the velocity or the momentum[5], i.e. $\delta F_i/\delta v'_k = 0$. It can be easily shown that for such systems the bracket $\{F_i\,,\,\rho'\}$ turns to zero. Taking account of relations (34), (35), we obtain

$$\{F_i, \rho'\} = \partial'_k\,\frac{\delta F_i}{\delta v'_k}. \tag{41}$$

On the other hand, using (36) we can calculate the same bracket directly

$$\{F_i\,,\rho'\} = = \partial'_k\,[\{\pi_i, \pi'_k\} - \partial'_i(\pi'_k\delta) + \partial_k(\pi_i\delta)]. \tag{42}$$

In the class of local Poisson brackets, the condition (41) is satisfied if the expression between square brackets is zero. Thus, for the hydrodynamic systems under consideration, the following relations take place

$$\{\pi_i\,,\,\pi'_k\} = \partial'_i(\pi'_k\,\delta) - \partial_k\,(\pi_i\delta), \tag{43}$$

$$F_i = \{\pi_i\,,\,\mathcal{U}\}. \tag{44}$$

Thus, the first relation, (43), defines the Poisson bracket for the momentum components and the second, (44), means that the resultant of the hydrodynamic forces is determined in full by the functional $\mathcal{U}$, i.e. by the intrinsic properties of medium.

[5] The case $\delta F_i/\delta v'_k \neq 0$ will be considered below.

Note that a good many traditional conservative models of continuous media fall within the framework of the adopted restrictions on the character of the additional fields and the forces.

2.3 Canonical Variables. Clebsch Representation for Hydrodynamic Momentum Density

The comprehensive answer to the questions about the *general relation between canonical variables and the Clebsch representation for momentum density,* and about choice of Poisson brackets $\{s_m, \pi_k'\}$ may be obtained within the framework of the procedure of constructing the canonical variables.

Let us assume the possibility of the canonical description for hydrodynamic systems determined by the Hamiltonian (29) and the brackets (34), (35), (39), (43). In essence, there are canonical variables – generalized coordinates q_n and generalized momenta p_n, $(n = 1, .., N)$, so that:

a) the following canonical conditions are held

$$\{q_i, q_k'\} = \{p_i, p_k'\} = 0,$$
$$\{q_i, p_k'\} = \delta_{ik}\,\delta(\mathbf{x} - \mathbf{x}'). \tag{45}$$

b) all physical fields $\vec{\pi}$, ρ, s_n may be expressed in terms of q_n, p_n and the equations of motion take the form of (19).

Moreover, as variables ρ, s_n are in involution and as there is a certain freedom in a choice of the canonical variables q_n, p_n, we can always proceed to a more convenient canonical basis in which the first $N+1$ canonical coordinates are identified by the set of commutative physical variables

$$q_0 \equiv \rho,\; q_i \equiv s_i, \quad i = 1, 2, ..., N. \tag{46}$$

Such a choice considerably simplifies the problem since it essentially reduces the problem to a search of the functional dependence $\vec{\pi}[q\,,\,p]$ expressing the hydrodynamic momentum in terms of the canonical variables. For historical reasons (see, for example, Clebsch 1859, Lamb 1932, Serrin 1959, Seliger and Whitham 1968) this dependence is often called the *Clebsch representation* for momentum density, and the canonical variables q_n, p_n are called the *Clebsch potentials.*

The choice of (46) indicates that only the first $N+1$ generalized coordinates make physical sense. The others canonical variables are *unphysical* and arise in the Hamiltonian implicitly through the intermediary of the dependence $\vec{\pi}[q, p]$. It is obvious that such a reformulation guarantees the gauge invariance of the theory since the canonical transformations which leave the physical quantities invariant $\vec{\pi}$, ρ, s_n do not change the Hamiltonian $\mathcal{H}$.

The relation (43) may be considered as the functional equation in which the bracket $\{\pi_i\,,\,\pi_k'\}$ is written in the canonical basis q_n, p_n according to the rule

$$\{\pi_i, \pi_k'\} = \int d\mathbf{x}''\Big(\frac{\delta\pi_i}{\delta q_n''}\frac{\delta\pi_k'}{\delta p_n''} - \frac{\delta\pi_i}{\delta p_n''}\frac{\delta\pi_k'}{\delta q_n''}\Big). \tag{47}$$

Let us now suppose the *scalar* or *vector* character of the fields forming the canonical basis. In other words, we assume that the original canonical basis V is a direct sum of mutually-orthogonal phase subspaces V_l, $l = 0, 1, 2, \ldots$. Each of these subspaces has its own independent canonical basis formed by either a scalar or a vector pair of the canonically conjugated field variables, so that in every subspace V_l the equation (43) has an independent solution π^l possessing the property

$$\{\pi^l, \pi^m\} = 0, \quad l \neq m. \tag{48}$$

Such solutions, here and below called fundamental, satisfy the superposition principle, it means that the general solution of the equation (43) is derived by the formula

$$\vec{\pi} = \sum_l \vec{\pi}^l. \tag{49}$$

Thus, suffice it to consider the fundamental solutions of two types, namely, those that are realized on a *scalar* canonical basis q , p, and those that are realized on a *vector* canonical basis $\mathbf{q}$, $\mathbf{p}$.

Let us consider first the fundamental solution on the *scalar* basis. We shall seek those solutions in the form

$$\pi_i = a_{ik}\, q\, \partial_k p + b_{ik}\, p\, \partial_k q, \tag{50}$$

where a_{ik}, b_{ik} are tensor constants, and the indices i, k run from 1 to 3.

Ansatz (50) can be justified in the following way. Let the required dependence $\vec{\pi}[q\,,\,p]$ represent the power-type functional that has orders of m, n and l with respect to the variables q, p and the differential operator ∂. Let us express it in the symbolic formula $\vec{\pi} \sim q^m p^n \partial^l$. Then, carrying out the operations in accordance with (43), (47), we obtain the power functional $q^{2m-1}p^{2n-1}\partial^{2l}$ on the left side of (43) and $q^m p^n \partial^{l+1}$ on the right side of (43). Comparing the exponents, we can find that $m,\; n,\; l = 1$.

In order to find the tensor constants a_{ik}, b_{ik}, we substitute (50) in equation (47). If we compare the similar terms and carry out the property calculations, we obtain the system of equations. As the analysis shows (see in more detail (Goncharov and Pavlov 1993)), the solutions of this system have a very simple form

$$a_{ik} = c\,\delta_{ik}, \quad b_{ik} = (c-1)\,\delta_{ik}, \tag{51}$$

where the constant c takes only two values: 0 and 1.

Note that constants a_{ik}, b_{ik} can be chosen based on symmetry. It follows from the invariance of the hydrodynamic equations with respect to the spatial rotations of the coordinate frame that the representation (50) must possess the same symmetry property. Obviously, the latter will be provided, if a_{ik}, b_{ik} are isotropic tensors. From here, taking into account that δ_{ik} is a unique isotropic tensor of the second order with an accuracy to any scalar multiplier, we can easily reproduce formulae (51).

Since in the case $c = 1$ the motion equation for the canonical variable q coincides with the continuity equation (2), it is convenient to rename the canonical variables. We shall assume here that $q \equiv \rho$, $p \equiv \varphi$ if $c = 1$, and $q \equiv \sigma$, $p \equiv \lambda$ for notation of alternative scalar fields if $c = 0$. Then, due to the superposition principle, we may write down the Clebsch representation on the *scalar* sector of canonical basis in the form

$$\vec{\pi} = \rho \nabla \varphi + \vec{\pi}^0. \qquad (52)$$

Here $\rho \nabla \varphi$ is the contribution stipulated by the field of density ($c = 1$), and the summand

$$\vec{\pi}^0 = -\lambda \nabla \sigma \qquad (53)$$

is the fundamental solution for case $c = 0$, and is the contribution stipulated by the alternative scalar fields.

Let us consider now the fundamental solutions realized on the *vector* canonical basis. We shall seek those solutions in the form

$$\pi_i = a_{iknm}\, q_n\, \partial_k p_m + b_{iknm}\, p_m\, \partial_k q_n, \qquad (54)$$

where a_{iknm}, b_{iknm} are tensor constants, the explicit form of which is to be determined and indices i, k, m, n run from 1 to 3.

In the case of the vector canonical basis, the ansatz (54) is justified in the same manner as that (49) for the scalar basis. Substitution of (54) in (46) leads to the system of the form

$$a_{iknm} - b_{iknm} = \delta_{ik}\delta_{nm}, \qquad (55)$$

$$a_{ikmp}\, a_{ljpn} - a_{ikpn}\, a_{ljmp} + + a_{ikmn}\, \delta_{ij} - a_{ijmn}\, \delta_{kl} = 0. \qquad (56)$$

The invariance requirement for the representation (54) with respect to rotations in the three dimensional space denotes that a_{iknm} , b_{iknm} must be the isotropic tensors of the fourth order. By virtue of this, a_{iknm} has representation of the general form

$$a_{iknm} = a\, \delta_{im}\, \delta_{kn} + b\, \delta_{in}\, \delta_{km} + c\, \delta_{ik}\, \delta_{nm}. \qquad (57)$$

As the analysis shows (Goncharov and Pavlov 1993), three versions of solutions for a and b are possible:

$$1)\, a = b = 0, \quad 2)\, a = 0,\, b = 1, \quad 3)\, a = -1,\, b = 0. \qquad (58)$$

At the same time the constant c remains free and from the very beginning may be equated to zero without loss of generality and any restrictions. Such a possibility had been shown by Goncharov and Pavlov (1993) to be a consequence of the parametric arbitrariness in the choosing of a constant c, which may be interpreted as the freedom in choosing the variables, playing the role of canonical coordinates, with an accuracy to a multiplier that is proportional to an arbitrary power of the density ρ. Thus, we have three possible types of fundamental solutions

$$\vec{\pi}^1 = -a_k \nabla I_k, \qquad (59)$$

$$\vec{\pi}^2 = -b_k \nabla S_k + \partial_k (b_k\, \mathbf{S}), \qquad (60)$$

$$\vec{\pi}^3 = -g_i \nabla J_k - \partial_k (g_k\, \mathbf{J}). \qquad (61)$$

Each momentum representation is defined on the corresponding sector of the vector canonical basis: $(\mathbf{a}, \mathbf{I})$, $(\mathbf{b}, \mathbf{S})$, $(\mathbf{g}, \mathbf{J})$, and is characterized by the choice of constants a, b.

The general solution for the Clebsch representation that may be constructed by the complete set of the fundamental solutions (*minimal Clebsch representation*) obviously takes on the form

$$\vec{\pi} = \rho \nabla \varphi + \vec{\pi}^0 + \vec{\pi}^1 + \vec{\pi}^2 + \vec{\pi}^3, \qquad (62)$$

where each type of fundamental solutions is accounted for only once.

In order to find the equations that describe evolutions of so-called *commutative* hydrodynamic fields σ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$ playing the role of canonical coordinates, we calculate the mutual Poisson brackets, which involve momentum (62), and find that the following evolution equations are admitted only

$$\dot{\sigma} + (\mathbf{v} \cdot \nabla)\sigma = 0, \quad \dot{\mathbf{I}} + (\mathbf{v} \cdot \nabla)\mathbf{I} = 0,$$
$$\dot{\mathbf{S}} + (\mathbf{v} \cdot \nabla)\mathbf{S} + (\mathbf{S} \cdot \nabla)\mathbf{v} + [\mathbf{S}, rot\, \mathbf{v}] = 0,$$
$$\dot{\mathbf{J}} + (\mathbf{v} \cdot \nabla)\mathbf{J} - (\mathbf{J} \cdot \nabla)\mathbf{v} = 0. \qquad (63)$$

In the above analysis, for simplicity, we restrain ourselves to the minimal set of different types of canonical coordinates one scalar field σ and three vector fields $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, making up together ρ the sector of canonical coordinates. To generalize the results on an arbitrary number of fields it is enough to consider that the variables σ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, and conjugate to them λ, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{g}$ depend of not only $\mathbf{x}$ but also of discrete indices, which number the fields of the same type.

The above-obtained results can be extended to the case of more complicated hydrodynamic systems whose force resultant $\mathbf{F}$ has a linear functional dependence of velocity $\mathbf{v}$ (Goncharov 1990; Goncharov and Pavlov 1993). Such a situation can be realized in the presence of a magnetic field for plasma motions (the Lorenz force) or for hydrodynamical motions in rotating frames (the Coriolis force), i.e. $\mathbf{F} \sim [\omega, \mathbf{v}]$.

With this purpose we slightly generalize the Clebsch representation for momentum (62) by adding to it the term $-\rho\mathbf{A}$

$$\pi_i = \rho \partial_i \varphi + {\pi_i}^0 + {\pi_i}^1 + {\pi_i}^2 + {\pi_i}^3 - \rho\, A_i, \qquad (64)$$

Here the vector field $\mathbf{A}$ is a function of $\mathbf{x}, t$ and a functional of the fields that play a role of physical canonical coordinates. (Note that $\mathbf{A}$ is assumed to possess the same intrinsic characteristic of the medium as the potential energy $\mathcal{U}$. The choice of this characteristic is determined by the medium model in each specific case.)

We shall follow the above-stated assumptions regarding the Hamiltonian structure of $\mathcal{H}$ and the physical character of the fields σ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, $\vec{\pi}$ that compose together a natural phase space for traditional description of fluid dynamics.

If earlier in order to define the Clebsch representation (62) we specified the character of hydrodynamic forces (i.e. assumed their independence of velocity), now we will address the inverse problem, i.e. we will define the character of hydrodynamic forces from the given representation (64).

Above all, let us note that modification of the Clebsch representation does not change the Poisson brackets for the fields $\rho, \sigma, \mathbf{I}, \mathbf{S}, \mathbf{J}$. Therefore, their evolution is described by equations (2), (63) as before. However, this modification changes the Poisson bracket $\{\pi_i, \pi'_k\}$ that now takes the form

$$\{\pi_i, \pi'_k\} = \partial'_i(\pi'_k\delta) - \partial_k(\pi_i\delta) + \\ +\rho'\{A'_k, \pi_i\} - \rho\{A_i, \pi'_k\} + \\ + \rho(\partial_i A_k - \partial_k A_i)\delta. \quad (65)$$

Using this result, we find directly from Euler equation (1), the expression for the hydrodynamic force

$$F_i = \rho([\mathbf{v}, rot\,\mathbf{A}]_i - \dot{A}_i) + \\ + \int d\mathbf{x}'\pi'_k\{A'_k, \pi_i\} + \{\pi_i, \mathcal{U}\}. \quad (66)$$

If $\mathbf{A}$ is not dependent on the field $\sigma, \ \rho, \ \mathbf{I}, \ \mathbf{S}, \ \mathbf{J}$ and is a function of $\mathbf{x}$ and t only then we arrive at the force of Lorentz type

$$F_i = \rho([\mathbf{v}, rot\,\mathbf{A}]_i - \dot{A}_i) + \{\pi_i, \mathcal{U}\}, \quad (67)$$

where $\mathbf{A}$ plays the role of the usual vector potential.

2.4 Physical Meaning of Commutative Fields Playing Role of Canonical Coordinates

It is well known that in fluid dynamics many laws of conservation are formulated as assertions that the integral of a field or a field combination over an arbitrary domain moving together with the fluid and, therefore, consisting of the same fluid particles, remains constant in time, i.e., is invariant. Such integrals are called substantial in order to emphasize the specificity of the integration domain.

There are four topologically-different domains of integration - a volume, a surface, a circuit and a point, which can be consequently described by generalized theta functions.

Let us consider a closed volume moving with a fluid. This volume can be characterized by the function $f(\mathbf{x}, t)$ possessing by the following properties: $f(\mathbf{x}, t) > 0$, if point $\mathbf{x}$ is inside the volume; $f(\mathbf{x}, t) < 0$, if point $\mathbf{x}$ is outside the volume; and $f(\mathbf{x}, t) = 0$ on its surface. Besides, if this surface is to move with fluid, as known (Serrin 1959), function f has to satisfy equation

$$\dot{f} + (\mathbf{v} \cdot \nabla)f = 0. \quad (68)$$

The substantial theta function, corresponding to the volume under consideration, is defined as $\theta[f] = 0$, if $f < 0$, and $\theta[f] = 1$, if $f \geq 0$, where θ is known as the Heaviside function and has the property

$$\partial\theta/\partial f = \delta(f). \quad (69)$$

With due account of this property and (68), the substantial theta function satisfies the equation

$$\dot{\theta} + (\mathbf{v} \cdot \nabla)\theta = 0. \quad (70)$$

If the theta function defines a finite fluid volume, then the closed surface bounding this volume is naturally defined as a gradient of the theta function

$$\nabla\theta = \delta(f)\,\nabla f. \quad (71)$$

The geometrical meaning of this formula consists of the fact that the function $\delta(f)$ marks the surface $f(\mathbf{x}, t) = 0$, and $\mathbf{n} = -\nabla f/|\nabla f|$ specifies the outer normal to this surface.

Using the geometrical interpretation of the gradient of the theta function, one can construct the quantity describing the oriented closed fluid circuit

$$[\nabla\theta_1, \nabla\theta_2] = \delta(f_1)\delta(f_2)[\nabla f_1, \nabla f_2], \quad (72)$$

where $\theta_{1,2} = \theta(f_{1,2})$ are the theta functions for two intercrossing fluid volumes, one of which must be finite. The geometrical sense of (72) lies in the fact that $\nabla\theta_1$ and $\nabla\theta_2$ give two surface $f_1(\mathbf{x}, t) = 0$ and $f_2(\mathbf{x}, t) = 0$, the crossing of which originates the circuit such that $[\nabla f_1, \nabla f_2]$ is the tangent vector to it.

The crossing of the fluid circuit and the surface is obviously the point moving with fluid. By means of (71) and (72), we can construct the quantity

$$(\nabla\theta \cdot [\nabla\theta_1, \nabla\theta_2]) = \\ = \delta(f)\delta(f_1)\delta(f_2)(\nabla f \cdot [\nabla f_1, \nabla f_2]), \quad (73)$$

which expresses this fact mathematically.

Note that the formulae (71), (72) apply only to the closed surfaces and circuits or of an infinite extent. In order to produce the quantities describing the isolated elements of those topological objects, the left sides of (71) and (72) are to be multiplied by a theta function θ_3 that would cut out corresponding elements from the original objects.

According to the type of integration domain - a volume, a surface element, a circuit element or a point, the four types of substantial integrals conserved in time may be constructed:

$$\mathcal{C}_1 = \int d\mathbf{x}\, D\theta, \\ \mathcal{C}_2 = \int d\mathbf{x}\, \rho\theta_3(\mathbf{J} \cdot \nabla\theta), \\ \mathcal{C}_3 = \int d\mathbf{x}\, \theta_3(\mathbf{S} \cdot [\nabla\theta_1, \nabla\theta_2]), \\ \mathcal{C}_4 = \int d\mathbf{x}\, \sigma(\nabla\theta \cdot [\nabla\theta_1, \nabla\theta_2]). \quad (74)$$

Here $\mathcal{C}_i$ are some constants ($i = 1, 2, 3, 4$), and the physical meaning of fields $D, \ \sigma, \ \mathbf{J}, \ \mathbf{S}$ will be established below.

Differentiating (74) with respect to time, and using the property of substantial theta functions (70) when integrating by parts, one can determine that equalities (74) are equivalent to the following equations:

$$\dot{D} + div(\mathbf{v}D) = 0, \\ \dot{\mathbf{J}} + (\mathbf{v} \cdot \nabla)\mathbf{J} - (\mathbf{J} \cdot \nabla)\mathbf{v} = 0, \\ \dot{\mathbf{S}} + (\mathbf{v} \cdot \nabla)\mathbf{S} + (\mathbf{S} \cdot \nabla)\mathbf{v} + [\mathbf{S}, rot\,\mathbf{v}] = 0, \\ \dot{\sigma} + (\mathbf{v} \cdot \nabla)\sigma = 0. \quad (75)$$

The first of equations (75) describes the fields D that evolve like the field ρ.

The second equation describes the behavior of the vector field $\mathbf{J}$ whose force lines are frozen into the fluid. The magnetic field and the vorticity field are examples of the frozen fields in an incompressible, inviscid and homogeneous fluid.

The third equation, which is not so well known, describes the vector fields S evolving along lagrangian trajectories similar to oriented elements of fluid surface (Batchelor 1967). The Lamb's momentum density can exemplify the fields of the given type in homogeneous ideal fluid (Kuzmin 1983).

The forth equation describes the law of conservation of the scalar quantity σ when it is conveyed by fluid particles along their Lagrangian trajectories. For this reason, such fields are called the Lagrangian invariants. There exist two types of the invariants - scalar, σ, and vector, **I**.

Thus, we can draw a conclusion that in addition to field density ρ there exist other *four* kinds of the commutative fields σ, **I**, **J**, **S** whose physical meaning has been established above and, which may be realized in the Hamiltonian systems under consideration.

2.5 Invariance of Hydrodynamic Systems to Gauge Transformations

As mentioned above, gauge symmetry or invariance of any Hamiltonian theory implies a constancy of all its physical characteristics under canonical transformation and permits variations of only those canonical variables that have no physical meaning. As applied to the Hamiltonian systems of the hydrodynamic type, gauge invariance of a Hamiltonian theory is equivalent to the invariance of the Clebsch representation for momentum. This conclusion follows directly from the obtained results, in accordance with the fact that the hydrodynamic velocity v is a unique physical quantity depending on the unphysical canonical variables in the theory.

If all canonical momenta are unphysical by construction, the classification of the canonical coordinates as physical and unphysical cannot occur unambiguously without additional information. Namely, only those canonical coordinates are physical which explicitly define the functional $\mathcal{U}$ - potential energy of the hydrodynamic system. Clearly such a classification of the canonical coordinates cannot be carried out without a detailed specification of the medium model.

Let us first examine the gauge transformation that varies solely the canonical momenta.

We write the suitable generating functional in the following general form

$$\mathcal{F} = -\int d\mathbf{x}(\rho\phi + \sigma\kappa + \mathbf{I}\cdot\mathbf{a} + \\ + \mathbf{S}\cdot\mathbf{b} + \mathbf{J}\cdot\mathbf{g}) + \mathcal{G}[\rho,\, \sigma,\, \mathbf{I},\, \mathbf{S},\, \mathbf{J}], \tag{76}$$

where $\mathcal{G}$ is a functional depending on the canonical coordinates. Then, the new momenta ϕ', κ', $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{g}'$ are defined as

$$\phi' = \phi - \frac{\delta\mathcal{G}}{\delta\rho}, \quad \kappa' = \kappa - \frac{\delta\mathcal{G}}{\delta\sigma}, \\ \mathbf{a}' = \mathbf{a} - \frac{\delta\mathcal{G}}{\delta\mathbf{I}}, \ \mathbf{b}' = \mathbf{b} - \frac{\delta\mathcal{G}}{\delta\mathbf{S}}, \ \mathbf{g}' = \mathbf{g} - \frac{\delta\mathcal{G}}{\delta\mathbf{J}}. \tag{77}$$

Using (77) we obtain that in the terms of new variables, the Clebsch representation of hydrodynamical momentum takes the form

$$\vec{\pi} = \vec{\pi}' + \{\vec{\pi},\, \mathcal{G}\}, \quad \vec{\pi}' = \vec{\pi}|_{\phi=\phi',\kappa=\kappa',\ldots}. \tag{78}$$

Obviously, the gauge invariance implying the equality $\vec{\pi} = \vec{\pi}'$ will take place if

$$\{\vec{\pi},\, \mathcal{G}\} = 0. \tag{79}$$

The relation (79) is obtained from the condition that $\mathcal{G}$ is a generator of gauge transformation and therefore is conserved by virtue of the Noether theorem. To find $\mathcal{G}$ by directly solving the functional equation (79) is quite difficult except for some particular cases.

There is a more judicious way to achieve this goal. In view of the invariance of the quantity $\mathcal{G}$, i.e. $\dot{\mathcal{G}} = 0$, it is logical to seek this quantity in the form

$$\mathcal{G} = \int d\mathbf{x}\, \rho\, \Phi, \tag{80}$$

where on one hand Φ is the scalar Lagrangian invariant and, on the other hand, Φ is the sought functional of the fields ρ, σ, **I**, **S**, **J**.

Thus, to construct the functional Φ it will be necessary to develop a procedure for constructing Lagrangian invariants that are functionally dependent on fields ρ, σ, **I**, **S**, **J**. This problem is a special case of the more general approach (Gavrilin and Zaslavskii 1970; Moissev *et al.*1982; Kuzmin 1983) that is stated below.

Let us consider four types of fields - density ρ, scalar Lagrangian invariant I, frozen field **J** and Lamb's type field **S**. We shall use primes or indices when necessary to distinguish a few named alike fields.

The aforesaid types of fields are connected by the so-called reciprocal relations which can be obtained in a few steps. On the first step we have the most obvious relations at our disposal:

$$\mathbf{J}' = \mathbf{J}\, I, \quad \mathbf{S} = \mathbf{S}\, I, \quad I = I(I',\, I'',\, I''',\ldots). \tag{81}$$

These formulae are easily verified and imply that multiplication on the Lagrangian invariant does not change the type of fields and that an arbitrary function of Lagrangian invariants is itself a Lagrangian invariant. Besides, due to quasilinearity of the equations (75), the linear combination of the same type fields gives rise to a field of just the same type.

On the second step, one can make sure by a direct check that there are relations

$$I = \frac{1}{\rho}\nabla(\rho\,\mathbf{J}), \quad \mathbf{J} = \frac{1}{\rho}\, rot\,\mathbf{S}, \quad \mathbf{S} = \nabla I, \tag{82}$$

in which not only do the fields themselves come into play, but so do their derivatives.

By substituting (81) in (82) we obtain

$$I = \mathbf{J}\cdot\nabla I', \ \mathbf{J}' = \frac{1}{\rho}[\mathbf{S},\, \nabla I], \tag{83}$$

If we use that $\mathbf{S}' = \nabla I'$, it follows

$$I = \mathbf{J}\cdot\mathbf{S}, \quad \mathbf{J} = \frac{1}{\rho}[\mathbf{S},\, \mathbf{S}'], \quad \mathbf{J} = \frac{1}{\rho}[\nabla I,\, \nabla I']. \tag{84}$$

Combining (82) - (84) we can obtain the following generation of the reciprocal relations

$$\mathbf{S} = \rho[\mathbf{J}, \mathbf{J}'],$$
$$I = \frac{1}{\rho}\mathbf{S}\cdot[\nabla I', \nabla I''],$$
$$I = \frac{1}{\rho}\nabla I'\cdot[\nabla I'', \nabla I''']. \quad (85)$$

In the general case the role of Φ can be played by an arbitrary function of any Lagrangian invariants that only may be constructed by formulae (81) - (85). In the given situation, when originally there are fields ρ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, we can obtain the following Lagrangian invariants of the first generation:

$$I_0' = \mathbf{J}\cdot\mathbf{S},\ I_{ik}' = \frac{1}{\rho}\mathbf{S}\cdot[\nabla I_i, \nabla I_k],$$
$$I_k' = (\mathbf{J}\cdot\nabla)I_k,\ I' = \frac{1}{\rho}\nabla I_1\cdot[\nabla I_2, \nabla I_3]. \quad (86)$$

The Lagrangian invariants of the second generation can be found by substituting the primed Lagrangian invariants of the first generation for I_1, I_2, I_3 in (86). If this procedure is applied repeatedly one can obtain the Lagrangian invariants of a higher and higher order in both fields ρ, σ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$ and their derivatives.

The use of gauge invariants is methodically advantageous in quite a number of applications. These applications include, for instance, the nonlinear stability theory (Arnold 1969, Holm *et al.* 1985, Abarbanel *et al.* 1986, McIntyre *et al.* 1987) and the research of wave motions evolving on the background of a given equilibrium flow regime, *et al.* (Salmon 1988).

Let us dwell in more detail on the application of the gauge invariants for studying the motions in hydrodynamic systems (Arnold 1969). We shall assume that the equilibrium regime is defined by the stationary distributions of velocity $\mathbf{v}_s$, density ρ_s as well as the remaining physical fields playing the role of canonical coordinates. The characteristic property of such systems is that the equilibrium values of the unphysical canonical variables, for example, momenta, may exhibit secular growth, i.e. may possess a linear time dependence. Such situation is clearly undesirable since it leads to certain technical difficulties arising when an expansion of the Hamiltonian in the perturbation theory series around equilibrium is performed.

Now we describe in brief the procedure by which one can exclude the effect of secular growth. For short we shall use purely nomenclative notation q_i, p_i for the fields making up the canonical basis. Next we carry out a point canonical transformation to new momenta p_i' (without changing the coordinates q_i) with the help of the generating functional

$$\mathcal{F} = \int d\mathbf{x}\, q_i p_i' + t\mathcal{G}[q] + \mathcal{Q}[q], \quad (87)$$

where $\mathcal{G}$ is a gauge invariant satisfying the relation (79). The functional $\mathcal{Q}$ is chosen such that

$$\{\vec{\pi}, \mathcal{Q}\}_s = \vec{\pi}_s. \quad (88)$$

Here and below index s marks the equilibrium values of corresponding fields.

We can find the transformation formulae for the momenta and the Hamiltonian:

$$p_i = \frac{\delta\mathcal{F}}{\delta q_i} = p_i' + t\frac{\delta\mathcal{G}}{\delta q_i} + \frac{\delta\mathcal{Q}}{\delta q_i}, \quad (89)$$

and

$$\mathcal{H}' = \mathcal{H} + \mathcal{G}. \quad (90)$$

At the same time, the Clebsch representation transforms to the form

$$\vec{\pi}' = \vec{\pi} - \{\vec{\pi}, \mathcal{Q}\}. \quad (91)$$

Thus it is easy to see from (88) - (91) that a corresponding choice of the functional $\mathcal{Q}$ and the gauge invariant $\mathcal{G}$ can ensure a transition to a new momenta which satisfies the conditions $p_s' = 0$, $\vec{\pi}_s' = 0$ in the equilibrium state.

2.6 Reduction of Degrees of Freedom

We can note that the fluid motion is defined if at any moment we know the relation between the initial and the instantaneous positions of the particles making up the fluid:

$$\vec{\xi} = \vec{\xi}(\mathbf{x}, t). \quad (92)$$

Coordinates $\vec{\xi}$ are called the Lagrangian coordinates and satisfy the equation

$$\dot{\vec{\xi}} + (\mathbf{v}\cdot\nabla)\vec{\xi} = 0. \quad (93)$$

This shows that the field $\vec{\xi}(\mathbf{x}, t)$ is a vector Lagrangian invariant convected with the fluid. Certain freedom in choosing the coordinates $\vec{\xi}$ can be eliminated through the initial condition $\vec{\xi}(\mathbf{x}, t) = \mathbf{x}$, which unambiguously extracts the field $\vec{\xi}$ from manifold of all the Lagrangian invariants.

It is obvious, that if we know the field $\vec{\xi}(\mathbf{x}, t)$, i.e., the motion of each fluid particle, we have thereby exhaustive information about the evolutions of all the hydrodynamic fields, which evolve due to hydrodynamic transfer only. Consequentl in principle the hydrodynamic fields ρ, σ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$ must be reconstructed uniquely by their own initial values and the field $\vec{\xi}$.

Indeed, by using the reciprocal relations (81) - (85), we can establish the following equalities

$$\rho = \rho^0\nabla\xi_1\cdot[\nabla\xi_2, \nabla\xi_3], \quad (94)$$
$$\sigma = \sigma^0(\vec{\xi}),\ \ \mathbf{I} = \mathbf{I}^0(\vec{\xi}),\ \ \mathbf{S} = S_k^0\nabla\xi_k, \quad (95)$$
$$\mathbf{J} = \frac{1}{2\rho}\rho^0 J_i^0\varepsilon^{ikl}[\nabla\xi_k, \nabla\xi_l], \quad (96)$$

where ρ^0, σ^0, $\mathbf{I}^0$, $\mathbf{S}^0$, $\mathbf{J}^0$ are functions of the Lagrangian coordinate $\vec{\xi}$ and have a physical meaning of the initial spatial distributions, i.e., are defined from the initial conditions

$$\rho^0 = \rho(\mathbf{x}, 0),\ \sigma^0 = \sigma(\mathbf{x}, 0),$$
$$\mathbf{I}^0 = \mathbf{I}(\mathbf{x}, 0),\ \mathbf{S}^0 = \mathbf{S}(\mathbf{x}, 0),\ \mathbf{J}^0 = \mathbf{J}(\mathbf{x}, 0). \quad (97)$$

The relations (94)-(96) allow us to reduce the number of degrees of freedom, i.e. to reduce the number of pairs of the canonically conjugated variables used for the Hamiltonian description of hydrodynamic systems.

As can be shown (Goncharov 1990a; Goncharov and Pavlov 1993), the relations (94) - (96) can be presented as the result of the point canonical transformation to only one canonical coordinate $\vec{\xi}$. In this case all others pairs of canonical variables

$$(\rho,\ \varphi),\ (\sigma,\ \kappa),\ (\mathbf{I},\ \mathbf{a}),\ (\mathbf{S},\ \mathbf{b}),\ (\mathbf{J},\ \mathbf{g})$$

are eliminated from description, and the Clebsch representation (62) takes the form

$$\vec{\pi} = -\lambda_i \nabla \xi_i \tag{98}$$

In using (98) it is necessary to take into account that, in the Hamiltonian

$$\mathcal{H} = \int d\mathbf{x}\, \frac{\vec{\pi}^2}{2\rho} + \mathcal{U}\left[\rho, \sigma, \mathbf{I}, \mathbf{S}, \mathbf{J}\right] \tag{99}$$

everywhere except in the fields $\rho,\ \sigma,\ \mathbf{I},\ \mathbf{S},\ \mathbf{J}$ playing the role of canonical coordinates, we must substitute their expressions (94) - (96) in terms of $\vec{\xi}$.

In principle, that implies that any conservative hydrodynamic model with additional fields which are in involution may be always reduced to the canonical Hamiltonian system with *three degrees of freedom.* This result does not depend on the number of fields of the same type in the original Hamiltonian system.

Besides the maximal reduction when the number of degrees of freedom dwindles to three, we may carry out the partial reduction. In this case not all pairs of the canonical variables $(\rho,\ \varphi),\ (\sigma,\ \kappa),\ (\mathbf{I},\ \mathbf{a}),\ (\mathbf{S},\ \mathbf{b}),\ (\mathbf{J},\ \mathbf{g})$ are eliminated from the description, but only some of them.

The reduction by means of the transformations (95), (96), which do not change the density ρ, can be one such example. As a result, we can eliminate all the fields playing the role of the canonical coordinates from the description, except for $\rho,\ \vec{\xi}$. In this case the corresponding Clebsch representation takes on the form

$$\vec{\pi} = \rho\, \nabla \varphi - \lambda_i\, \nabla \xi_i. \tag{100}$$

The Clebsch representation in the form (100) proved to be preferable in studying the gauge symmetries of a more general type than those which were considered in Section 2.5. The appearance of such symmetries should be expected in the case when it turned out after the transformations (95), (96) that $\mathcal{U}$ - the potential part of Hamiltonian - does not depend on all or some components of the field $\vec{\xi}$. This means that the corresponding components $\vec{\xi}$ are nonphysical. Thus, it opens up the possibility to carry out the additional gauge transformations varying not only the canonical momenta φ, λ_i, but also the coordinates ξ_i.

Analogous to Section 2.5, the corresponding generating functional can be presented in the form

$$\mathcal{F} = -\int d\mathbf{x}\,(\rho\,\varphi + \lambda_i \xi_i') + \mathcal{G},\ \ \mathcal{G} = \int d\mathbf{x}\,\rho\,\Phi, \tag{101}$$

where $\vec{\xi}'$ is a new coordinate and Φ is a function of all Lagrangian invariants possible for given model.

The gauge transformation is performed by the formulae

$$\begin{aligned} \xi_i &= \xi_i' - \int d\mathbf{x}\,\rho\,\frac{\delta\Phi}{\delta\lambda_i}, \\ \lambda_i' &= \lambda_i - \int d\mathbf{x}\,\rho\,\frac{\delta\Phi}{\delta\xi_i}, \\ \varphi' &= \varphi - \frac{\delta}{\delta\rho}\int d\mathbf{x}\,\rho\Phi, \end{aligned} \tag{102}$$

where $\varphi',\ \lambda_i'$ are new momenta.

Let all the components $\xi_i,\ (i\ =\ 1, 2, 3)$ be *nonphysical,* i.e. $\delta\mathcal{U}/\delta\xi_i = 0$.

In this case,

$$\dot{\lambda}_i = -\frac{\delta\mathcal{H}}{\delta\xi_i} = -\int d\mathbf{x} v_j \frac{\delta\pi_j}{\delta\xi_i} = -\nabla(\lambda_i \mathbf{v}). \tag{103}$$

Consequently, the quantities $\mu_i = \lambda_i/\rho$ have the meaning of Lagrangian invariants as well as ξ_i. In the absence of fields of $\mathbf{S}$ - and $\mathbf{J}$ - types, one of the Lagrangian invariants generated by way of recursive employment of the formula (85), in which the fields $\xi_i,\ \mu_i$ are used as a starting point, has the form

$$\Phi = \frac{\varepsilon}{\rho}\mu_i \nabla\mu_k \cdot \left[\nabla\xi_k,\ \nabla\xi_i\right], \tag{104}$$

where ε is some constant with a corresponding dimension.

By virtue of the Noether's theorem, the invariance under transformation (102) implies that the generator of this transformation, i.e. integral

$$G = \int d\mathbf{x}\rho\Phi = \varepsilon\int d\mathbf{x}\,\mu_i\,\nabla\mu_k \cdot \left[\nabla\xi_k,\ \nabla\xi_i\right] \tag{105}$$

is invariant of the motion.

Using the Clebsch representation we can transform the integral helicity

$$G = \varepsilon\int d\mathbf{x}(\mathbf{v}\cdot rot\,\mathbf{v}), \tag{106}$$

to the form of Eq. 105. The topological meaning of the integral helicity has been repeatedly discussed in literature (e.g., Moffatt 1969, 1978, 1990).

The gauge transformations corresponding to the law of helicity conservation can be written using formula (102) in the explicit form

$$\begin{aligned} \xi_i &= \xi_i' + \frac{2\varepsilon}{\rho}\left[\nabla\xi_k',\ \nabla\xi_i'\right]\cdot\nabla\mu_k, \\ \lambda_i' &= \lambda_i + 2\varepsilon\left[\nabla\xi_k',\ \nabla\mu_k\right]\cdot\nabla\mu_i, \\ \varphi' &= \varphi - \frac{2\varepsilon}{\rho}\mu_i\left[\nabla\xi_i',\ \nabla\xi_k'\right]\cdot\nabla\mu_k. \end{aligned}$$

It can be easily proven that these transformations, in spite of their complex character, do not change the hydrodynamic momentum, i.e.,

$$\vec{\pi}[\rho,\ \varphi,\ \vec{\lambda},\ \vec{\xi}] = \vec{\pi}[\rho,\ \varphi',\ \vec{\lambda}',\ \vec{\xi}'].$$

Let us now answer the question what kind of restrictions are imposed on the class of flows and, therefore, on the Clebsch representation, under the requirement of the helicity absence $G = 0$. In this case, it follows from (105) that

$$\mu_i \nabla \mu_k \cdot [\nabla \xi_k, \nabla \xi_i] = 0. \tag{107}$$

To satisfy this condition, the following relation is an obvious necessary and sufficient condition

$$\vec{\lambda} = \eta \vec{\xi}, \tag{108}$$

where η is a scalar function of $\mathbf{x}$, t.

Relation (108) can be regarded as a point canonical transformation to the new momentum η with the generating functional

$$F = \frac{1}{2} \int d\mathbf{x} \eta \vec{\xi}^2. \tag{109}$$

Such a transformation allows us to go from the vector pair of the canonically conjugated variables $\vec{\xi}, \vec{\lambda}$ to a scalar pair ζ, η, where the new canonical variable ζ is determined as

$$\zeta = \frac{\delta F}{\delta \eta} = \frac{1}{2} \vec{\xi}^2. \tag{110}$$

Substituting (108) in (100) and taking account of (110), we find a modification of the Clebsch representation

$$\vec{\pi} = \rho \nabla \varphi - \eta \nabla \zeta, \tag{111}$$

which corresponds to the *flow with zero helicity.* It is important to note that within the scope of the representation (111) a more general class of flows can be described only at the cost of using the *multivalued* Clebsch potentials (Zakharov and Kuznetsov 1982, Kuznetsov and Mikhailov 1980).

3 Hamiltonian Approach: Applications to Fluid Motions

Let us turn to some applications of the general ideas presented above. This section is motivated by the following points:

1) The development of the theory would remain incomplete if no practical application of the theoretical analysis were given. As one of the important examples we consider a geophysical flow of incompressible fluid. Such a model is of special interest to atmospheric and planetary physicists, meteorologists, and specialists in fluid dynamics.

2) The questions regarding how the Hamiltonian looks and what is the structure of the canonical equations in concrete situations are not as trivial as they may appear at first glance. Indeed they must be addressed at the very beginning of the analysis of any practical application.

3) Another motivating point is that the necessity of using the approximate analytical or numerical methods imposes special requirements on the structure of the Poisson bracket. It is evident that such methods are merely realized in the framework of the Hamiltonian (canonical) formulation with the Poisson tensor independent of field variables. In this case, there is really the sole object for approximation - the Hamiltonian, and corresponding calculations, which, as a rule, could have a cumbersome, recurrent character, are not replicated according to the number of equations. It also should be kept in mind that in using the approximation methods, a formal application of finite-difference methods to systems with Poisson brackets depending on fields (i.e. in non canonical form) can lead to equations which are not conservative in contrast to initial equations. A similar attempt in hydrodynamics was the introduction of 3D dimensional "vortons" (E. Novikov 1983), which by analogy with 2D dimensional point vortices would serve as basic elements in constructing finite-dimensional models describing 3D dimensional vortex flows in an ideal fluid.

In all such cases the loss of conservativity is easily explained if we call attention to the fact that the violation of the Jacoby property (18) is made possible, e.g., after using the finite difference approximations.

This remark is of particular significance because theoretical and computing physics widely uses discrete models with adequate corresponding continuous analogies. In this light the canonical formulation should be considered as one of most methodically developed version of the Hamiltonian formalism not only corresponding to the requirement of the structure simplicity, but also possessing standard approaches and tools to solve various hydrodynamical (geophysical) problems.

3.1 Hamiltonians and Canonical Variables for Incompressible Fluid Flows

We consider here *incompressible stratified fluid in the gravity field* of potential $\chi = gz$, characterized in equilibrium by vertical profiles of horizontal velocity $\mathbf{v}_s = \mathbf{u}(z)$ and density $\rho_s = \rho_0(z)$. Then the integral of the total energy, and correspondingly, Hamiltonian, is defined as

$$\mathcal{H} = \int d\mathbf{x} \left(\frac{\vec{\pi}^2}{2\rho} + \rho \chi \right), \tag{112}$$

and the Clebsh representation for the momentum $\vec{\pi}$ is determined by the expression

$$\vec{\pi} = \rho \nabla \varphi + \vec{\pi}^1 = \rho \nabla \varphi - \lambda_i \nabla \xi_i.$$

Here ξ_i are the Lagrangian coordinates identifying the positions of fluid particles.

In the case of incompressible fluid, i.e. $div\, \mathbf{v} = 0$, the density ρ is also a Lagrangian variable and it may be written:

$$\rho(\vec{\xi}) = \rho_0(z = \xi_3). \tag{113}$$

If we consider (113) as a canonical transformation, where ρ is a function of ξ_3, it appears that (113) describes a point transformation, which simply redefines momentum λ_3 :

$$\lambda_3' = \frac{\delta \mathcal{F}}{\delta \xi_3} = \lambda_3 + \varphi \frac{\partial \rho}{\partial \xi_3}, \tag{114}$$

not changing other canonical variables.

As a result, by introducing new variable $\Phi = \rho \varphi$ and omitting the prime in the new momentum λ_3', we lead Clebsh transformation to the form

$$\vec{\pi} = \nabla \Phi - \lambda_i \nabla \xi_i. \tag{115}$$

It should be noted that although the variable Φ stays formally in the description it is no longer a canonical variable, as the variable ρ has lost its meaning as an independent canonic coordinate. The presence of such variable Φ indicates the existence of the connection, which cab be explicitly formulated as the incompressibility condition

$$\frac{\delta\mathcal{H}}{\delta\varphi} = div\, \mathbf{v} = div\, (\frac{1}{\rho}\, (\nabla\Phi - \lambda_i\, \nabla\xi_i)) = 0. \qquad (116)$$

Expressing Φ from (116) in terms of ξ_i and λ_i, we may exclude Φ from the description. Appearance of this kind of problems (possibly complemented by corresponding boundary conditions) is a typical in constructing the canonical variables for different models of incompressible fluid (see Goncharov and Pavlov and refs. therein).

If we know the equilibrium values of the physical fields, which describe the incompressible fluid in the state of equilibrium, then from the canonical equations we can find the equilibrium values for corresponding fields:

$$\vec{\xi}_s = \mathbf{x} - t\mathbf{u},$$

$$\vec{\lambda}_s = -\rho_0\mathbf{u} - tgz(\frac{\partial\rho_0}{\partial z})\mathbf{n}, \qquad (117)$$

$$\Phi_s = t(\rho_0\frac{\mathbf{u}^2}{2} - \int_0^z dz(gz + \frac{\mathbf{u}^2}{2})\,(\frac{\partial\rho_0}{\partial z})). \qquad (118)$$

Here $\mathbf{n}$ is a unit vertical vector, $\mathbf{n} = (0,\, 0,\, 1)$.

The canonical transformation to new variables $\vec{\xi}'$, $\vec{\lambda}'$, satisfying the conditions $\vec{\xi}' = \vec{\lambda}' = 0$ in equilibrium state, is realized according to the standard scheme with the help of the generating functional:

$$\mathcal{F} = \int d\mathbf{x}\, \lambda_i\xi_i + t\mathcal{G} + \mathcal{Q},$$

where

$$\mathcal{G} = -\int d\mathbf{x}(g\int_0^{\zeta_3} dz\, z\,(\frac{\partial\rho_0}{\partial z}) + \frac{(\vec{\lambda}' - \rho\mathbf{u})^2 - {\lambda'_3}^2}{2\,\rho}) \qquad (119)$$

$$\mathcal{Q} = -\int d\mathbf{x}(\rho\mathbf{u}\cdot(\xi - \mathbf{x}) + \mathbf{x}\cdot\vec{\lambda}'). \qquad (120)$$

Note that the density ρ is anywhere considered as a function of ξ_3, determined by (113), and ρ_0 is a function of z.

In the new canonical variables the Hamiltonian and Clebsh representation are transformed as

$$\mathcal{H} \Rightarrow \mathcal{H} + \mathcal{G} =$$
$$= \int d\mathbf{x}(\frac{\vec{\pi}^2}{2\rho} - \rho\, g\xi'_3 + g\int_0^{\xi'_3+z} dz\rho_0(z) -$$
$$-\frac{(\vec{\lambda}' - \rho\mathbf{x})^2 - (\lambda'_3)^2}{2\rho}), \qquad (121)$$

$$\vec{\pi} = \rho\mathbf{u} + \nabla\Phi' - \vec{\lambda}' - \rho\xi'_i\nabla u_i - \lambda'_i\nabla\xi'_i, \qquad (122)$$

where the variable Φ', which appeared as the result of combination of gradient terms, may be excluded by the incompressibility condition

$$div\, \left(\frac{1}{\rho}\nabla\Phi'\right) = div\, \left(\xi'_i\nabla u_i + \frac{\vec{\lambda}' + \lambda'_i\nabla\xi'_i}{\rho}\right) \qquad (123)$$

Let us consider now several cases very important in practice. The simplest one, $\mathbf{u} = const$, which indicates the absence of velocity shear in equilibrium, is analyzed absolutely in the same manner. In the result, for the model of incompressible fluid there remains only one pair of canonically conjugated variables $(\xi_3,\ \lambda'_3)$, and the expressions (121) - (123) become more simple:

$$\mathcal{H} = \int d\mathbf{x}\,[\frac{\vec{\pi}^2}{2\rho} - \rho\, g\,\xi'_3 + g\int_0^{\xi'_3+z} dz\,\rho_0(z)\,], \qquad (124)$$

$$\vec{\pi} = \nabla\Phi' - \lambda'_3\mathbf{n} - \lambda'_3\nabla\xi'_3, \qquad (125)$$

$$div\,(\frac{1}{\rho}\nabla\Phi') = div\,(\frac{\lambda'_3\mathbf{n} - \lambda'_3\nabla\xi'_3}{\rho}). \qquad (126)$$

It should be noted that the presented Hamiltonian formulation of the dynamics of incompressible inhomogeneous fluid differs from the similar formulations, developed in some publications, by the choosing of much more convenient canonical variables. The transformation from one to another is connected with the point canonical transformation where ξ_3 is expressed via the function ρ. Such a description in terms of ξ_3 appears to be more adequate and has an advantage for the models with non-monotonous profile $\rho_0(z)$, because in the latter case the inverse function to $\rho = \rho_0(\xi_3)$ is not unique. Such a profile of density may be meaningful for the models with stratification in the plane perpendicular to the gravity vector, or for the models with no gravity at all.

Let us consider the Hamiltonian formulation of the incompressible inhomogeneous fluid motion with a velocity shear, i.e. with the constant vector characteristics $\mathbf{e} = \mathbf{u}/|\mathbf{u}|$, where $u = |\mathbf{u}(z)|$. Then, we come to the conclusion that the motion is described by already two pairs of canonical variables $(\xi'_3,\ \lambda'_3)$ and $(\xi,\ \lambda)$, where the latter are defined by

$$\lambda'_\alpha = e_\alpha\, \lambda,$$

$$\xi = \frac{\delta}{\delta\lambda}\int d\mathbf{x}\,(\vec{\xi}'\cdot\mathbf{e})\lambda = (\vec{\xi}'\cdot\mathbf{e}). \qquad (127)$$

Here $\alpha = 1,\, 2$ and $\lambda = ({\lambda'}_1^2 + {\lambda'}_2^2)^{1/2}$. In terms of these canonical variables the Hamiltonian and the Clebsh transformations for the momentum are reformulated as follows:

$$\mathcal{H} = \int d\mathbf{x}\,[\frac{\vec{\pi}^2}{2\rho} - \rho g\,\xi'_3 +$$
$$+ g\int_0^{\xi_3+z} dz\,\rho_0(z) - \frac{(\lambda - \rho u)^2}{2\rho}], \qquad (128)$$

$$\vec{\pi} = \rho\,\mathbf{u} + \nabla\Phi' - \lambda\mathbf{e} - \lambda'_3\mathbf{n} -$$
$$-\rho\xi\nabla u - \lambda\nabla\xi - \lambda'_3\nabla\xi'_3, \qquad (129)$$

where Φ' is determined from the incompressibility equation

$$div\,(\frac{1}{\rho}\,\nabla\Phi') = div\,(\xi\nabla u + \frac{1}{\rho}\,(\lambda\mathbf{e} +$$
$$+\lambda'_3\mathbf{n} + \lambda\nabla\xi + \lambda'_3\nabla\xi'_3). \qquad (130)$$

The formulation (128)-(130) becomes more simple in the *homogeneous fluid* approximation, i.e. $\rho = const$. In this case there remains only one pair of canonical variables $(\xi,\ \lambda)$

in the description of the fluid dynamics. All others are excluded due to the relation $\lambda'_3 = 0$, which follows from the lagrangectivity of the variable λ'_3 and the condition $\lambda'_{3_s} = 0$. Finally, assuming $\rho = 1$, we obtain the following result

$$\mathcal{H} = \frac{1}{2}\int d\mathbf{x}\left(\vec{\pi}^2 + (\lambda - u)^2\right), \tag{131}$$

$$\vec{\pi} = \nabla\Phi' - (\lambda - u)\,\mathbf{e} - \xi\nabla u - \lambda\nabla\xi, \tag{132}$$

$$\Delta\Phi' = div\ (\xi\nabla u + \lambda\mathbf{e} + \lambda\nabla\xi)\,. \tag{133}$$

For the *2-dimensional motions* in the $(x,\ z)$-plane with the background z-stratified flow $\mathbf{u} = (u(z),\ 0,\ 0)$, one may restructure the description (131)-(133) by introducing the stream function ψ, defined as

$$\pi_1 - u = -\partial_3\psi, \quad \pi_3 = \partial_1\psi. \tag{134}$$

$\vec{\pi}$ and Φ' may be excluded from consideration by the use of the stream function ψ. The result is

$$\mathcal{H} = \frac{1}{2}\int dx\,dz\left(-\psi\Delta\psi + \lambda^2 + 2u\lambda\partial_1\xi\right), \tag{135}$$

$$\Delta\psi = \partial_3\lambda - u'\partial_1\xi + J\left(\xi,\ \lambda\right), \tag{136}$$

where $u' \equiv \partial_3 u$, and Jacobian $J(\xi,\ \lambda)$ is determined as

$$J\left(\xi,\ \lambda\right) = \left(\partial_1\xi\right)\left(\partial_3\lambda\right) - \left(\partial_3\xi\right)\left(\partial_1\lambda\right). \tag{137}$$

The equations for such a model obtain sufficiently simple form

$$\partial_t\xi = \frac{\delta\mathcal{H}}{\delta\lambda} = -u\partial_1\xi - \lambda + \partial_3\psi + J\left(\xi,\ \lambda\right), \tag{138}$$

$$\partial_t\lambda = -\frac{\delta\mathcal{H}}{\delta\xi} = -u\partial_1\lambda + u'\partial_1\psi + J\left(\lambda,\ \psi\right). \tag{139}$$

Directly, by differentiating with respect to t the Clebsch transformation for the stream function (136), it is easy to see that the system of canonical equations (138),(139) is equivalent to the traditional vortex equation

$$\partial_t\Omega + J\left(\Psi,\ \Omega\right) = 0, \tag{140}$$

where the absolute vortex $\Omega = \Delta\Psi$ and the total stream function Ψ are related by

$$\Psi = \psi + \int^z dz\,u, \quad \Omega = \Delta\psi + u'. \tag{141}$$

In conclusion let us note that when $u'' = 0$ there appears a particular class of flows which is characterized by partially-linear velocity profile, or, in other words, partially-constant profile of vorticity Ω. The specificity of such flows, called *layered* (Gossard and Hooke 1975), is that the description of fluid dynamics may be narrowed (due to condition $\Delta\psi = 0$ to the description of the motion of the inner boundaries, where the vorticity $\Omega = u'$ has discontinuity. Then, the corresponding equations of motion of the boundaries (equations of contour dynamics) may be formulated in the Hamiltonian form.

3.2 Hamiltonian Description in Curvilinear Coordinates

When the basic relations of Hamiltonian formalism in hydrodynamics are being derived and discussed, it is sufficient to limit our consideration by operating within a Cartesian coordinate system. When physical problems with certain types of symmetry of hydrodynamical motions are considered, it is convenient to use corresponding systems of curve-linear spatial coordinates. Thus, a method which allows us to transfer the Hamiltonian description in Cartesian coordinates to the Hamiltonian description in general curvilinear coordinates has to be shown.

Consider continuous transformation

$$\mathbf{x} = \mathbf{x}(\vec{\zeta}, t), \tag{142}$$

which generally depends on time t and describes transformation from Cartesian coordinates $\mathbf{x} = (x^1,\ x^2,\ x^3)$ to curvilinear coordinates $\vec{\zeta} = (\zeta^1,\ \zeta^2,\ \zeta^3)$. It is known that from the point of view of geometry, properties of space connected with curvilinear coordinates are characterized by the metric tensor

$$g_{ik} = \frac{\partial\mathbf{x}}{\partial\zeta^i}\cdot\frac{\partial\mathbf{x}}{\partial\zeta^k}, \tag{143}$$

or the countervariant tensor

$$g^{ik} = \frac{\partial\zeta^i}{\partial\mathbf{x}}\cdot\frac{\partial\zeta^k}{\partial\mathbf{x}}. \tag{144}$$

Taking into account transformation (142), in new curvilinear coordinates ξ the Clebsch transformation for the momentum may be rewritten as

$$\pi = \frac{\partial\zeta^k}{\partial\mathbf{x}}(\rho\partial_k\varphi - \lambda_l\partial_k\xi^l). \tag{145}$$

Here all field variables acting as Clebsh potentials are functions of $\vec{\zeta},\ t$ and correspondingly $\partial_k = \partial/\partial\zeta^k$ is a differentiating operator, it is a partial derivative with respect to the coordinate ζ^k.

If we assign

$$p_k = g^{1/2}(\rho\partial_k\varphi - \lambda_l\partial_k\zeta^l), \tag{146}$$

where g is the determinant of the metric tensor g_{ik}, the equation (145) may be rewritten in the more compact manner

$$\pi_i = g^{-1/2}\frac{\partial\zeta^k}{\partial x^i}\,p_k. \tag{147}$$

The obtained result allows us to rearrange the kinetic part of the hydrodynamical hamiltonian in the following form

$$\frac{1}{2}\int d\mathbf{x}\,\frac{\vec{\pi}^2}{\rho} = \frac{1}{2}\int d\vec{\zeta}\,(g^{1/2}\rho)^{-1}\,g^{ik}p_i p_k. \tag{148}$$

In the derivation of (148) we used, besides equation (147), the fact known from tensor analysis that in transformation (142) the volume element is transformed to

$$d\mathbf{x} = g^{1/2}d\zeta^1\,d\zeta^2\,d\zeta^3. \tag{149}$$

Also, equation (149) indicates that the quantity $g^{1/2}\rho$ has a meaning of density in the curvilinear coordinates. In fact, if an element of liquid mass dm in x-coordinates has volume $d\mathbf{x}$ and corresponding density $dm/d\mathbf{x} = \rho$, in $\vec{\zeta}$-coordinates, taking the volume $d\vec{\zeta}$, it will have a density of $g^{1/2}\rho$. Taking this into account and comparing left- and right-hand sides of equation (148), we may conclude that p_k are the covariant components of hydrodynamical momentum in the local basis of curvilinear coordinates.

It should be noted that where in Cartesian coordinates there exists a very simple relation between hydrodynamical momentum and velocity, in curvilinear non-stationary coordinates this relation is essentially modified.

Using the invariant definition of a hydrodynamical velocity

$$u^k = \frac{d\zeta^k}{dt}, \qquad (150)$$

which has a meaning for any coordinate systems, we may determine so-called countervariant components of the velocity u^k in a curvilinear coordinate system $\vec{\zeta}$.

Assuming that in (150) ζ^k is related to x and t by the transformation, reversed to (142), we may find

$$u^k = \dot{\zeta}^k + \frac{\partial \zeta^k}{\partial \mathbf{x}} \cdot \mathbf{v}, \qquad (151)$$

where $\mathbf{v} = d\mathbf{x}/dt$ is a Hamiltonian velocity in Cartesian coordinates.

Finally, taking into account that $\mathbf{v} = \vec{\pi}/\rho$ and (147), we obtain

$$u^k = \frac{d\zeta^k}{dt} = \dot{\zeta}^k + \frac{g^{ki}}{g^{1/2}\rho}\, p_i. \qquad (152)$$

This relation together with (148) allows us to express the hydrodynamical hamiltonian $\mathcal{H}$ in the form

$$\mathcal{H} = \frac{1}{2}\int d\vec{\zeta}\,(u^k - \dot{\zeta}^k)\, p_k + \mathcal{U}. \qquad (153)$$

It is known that while transforming from Cartesian to curvilinear coordinates the canonical basis is not conserved. Hence, it can be easily recalculated with the use of point canonical transformation.

Let us define new canonical variables φ^* and ξ^{*l} by expressing each of them only as follows

$$\varphi^* = \int d\mathbf{x}\,\varphi\delta(\mathbf{x} - \mathbf{x}(\vec{\zeta}, t)),$$

$$\xi^{*l} = \int d\mathbf{x}\,\xi^l\delta(\mathbf{x} - \mathbf{x}(\vec{\zeta}, t)). \qquad (154)$$

This means only a simple variable transformation.

Taking into account the fact that φ and ξ^l are of different types (φ is a canonical momentum, ξ^l are canonical coordinates in a Cartesian system), we may establish that the corresponding generating functional has the form

$$\mathcal{F} = \int d\mathbf{x}\, d\vec{\zeta}\,(\rho\varphi^* - \lambda_l\xi^{*l})\delta(\mathbf{x} - \mathbf{x}(\vec{\zeta}, t)). \qquad (155)$$

As a result, all other canonical variables, conjugates to φ^* and ξ^{*l}, and the new hamiltonian of the hydrodynamical system may be found in a standard manner as

$$\rho^* = \frac{\delta\mathcal{F}}{\delta\varphi^*} = g^{1/2}\rho,\ \lambda_l^* = \frac{\delta\mathcal{F}}{\delta\xi_l^*} = g^{1/2}\rho. \qquad (156)$$

Here $\mathbf{x} = \mathbf{x}(\vec{\zeta}, t)$, and

$$\mathcal{H}^* = \mathcal{H} + \dot{\mathcal{F}} = \frac{1}{2}\int d\vec{\zeta}(u^k + \dot{\zeta}^k)p_k + \mathcal{U}. \qquad (157)$$

Here, in terms of canonical variables (ρ^*, φ^*), (ξ^l, λ^{*l}) for the vector of hydrodynamical velocity and momentum convector there are the following transformations which are generalizations of the Clebsch transformations in curvilinear coordinates

$$u^k = \dot{\zeta}^k + \frac{g^{ik}}{\rho^*}p_i,\ p_i = \rho^*\partial_i\varphi^* - \lambda_l^*\partial_i\xi^{*l}. \qquad (158)$$

Besides a general theoretical interest, such a covariant formulation of Hamiltonian method of description of (156)-(158) has a practical meaning. The set of problems, for which this formulation appears to be very useful, is broad. As an example we may mention problems connected with the studying of fluid motions developing on the basic (main) flow. Such motions are usually considered as perturbations on the basic flow and are convenient to be studied in the moving system of coordinates, "frozen" in the fluid in such a way that in each point in the steady state the fluid is locally in rest. The latter circumstance determines the following choice of curvilinear coordinates:

$$\zeta^i = \zeta_s^i(\mathbf{x},\ t), \qquad (159)$$

where ζ_s^i are lagrangian coordinates of the fluid parcels in a steady state.

3.3 Canonical Hamiltonian Description in Lagrangian Coordinates

Let us now formulate a canonical Hamiltonian description in lagrangian coordinates. The mathematical specifics of such a transition is that transition (142) represents now the law of motion of the fluid. This means that the function $\mathbf{x}(\vec{\zeta},\ t)$ determines a mutually unique continuous image, which corresponding to the equations of motion, transports each parcel of fluid from a certain location $\vec{\zeta}$, where it was initially, to another location $\mathbf{x}$ where it will be at moment t.

The transformation from canonical variables $(\rho,\ \varphi)$ and $(\lambda_l,\ \xi^l)$, in terms of which the Hamiltonian description in Cartesian coordinates is formulated, to the basis where the canonical variable is $\mathbf{x}(\vec{\zeta},\ t)$ may be performed as point transformation with the generating functional

$$\mathcal{F} = \int d\mathbf{x}'\,(\rho'\varphi' - \lambda_l'\,\xi^{l'}), \qquad (160)$$

where prime in the field variable means dependence on the primed argument $\mathbf{x}'$. Here the components of a new canonical momentum, conjugated to x^k, may be found according to the formula:

$$\beta_k(\vec{\zeta}, t) = \frac{\delta \mathcal{F}}{\delta x^k(\vec{\zeta}, t)}. \qquad (161)$$

It should be kept in mind that when operating $\delta/\delta x^k$ with respect to functional (160) the old canonical variables ρ' and λ'_l are considered to be fixed functions, so only the field variables φ' and $\xi^{l'}$ vary with $x^k(\vec{\zeta}, t)$. As a result, formula (161) may be rewritten as

$$\beta_k(\vec{\zeta}) = \int d\mathbf{x}'(\rho' \frac{\delta\varphi(\mathbf{x}')}{\delta x^k} - \lambda_l' \frac{\delta \xi^l(\mathbf{x}')}{\delta x^k(\vec{\zeta})}). \qquad (162)$$

Here, variational derivatives should be interpreted as the existence of functional dependence of φ' and $\xi^{l'}$, which occurs because the coordinate $\mathbf{x}'$ of these fields is considered as the result of image $\vec{\zeta}' \to \mathbf{x}'$ according to (142), i.e.

$$\mathbf{x}' = \mathbf{x}(\vec{\zeta}', t). \qquad (163)$$

For any field $f(\mathbf{x}')$, the argument $\mathbf{x}'$ which satisfies conditions (163), we will have

$$\frac{\delta f(\mathbf{x}')}{\delta x^k(\vec{\zeta})} = \frac{\delta f(\mathbf{x}(\vec{\zeta}'))}{\delta x^k(\vec{\zeta})} = \frac{\partial'}{\partial x^{k'}} \delta(\vec{\zeta}' - \vec{\zeta}). \qquad (164)$$

Taking into account (164),(149), we may find that

$$\beta_k(\vec{\zeta}) = g^{1/2} \pi_k(\mathbf{x} = \mathbf{x}(\vec{\zeta}, t)). \qquad (165)$$

Under Hamiltonian description of hydrodynamical systems in terms of $x^k(\vec{\zeta}, t)$, $\beta_k(\vec{\zeta}, t)$, which, evidently, have a direct meaning of physical coordinate and momentum of the fluid parcel, there is an obvious analogy between the fluid and a system of discrete particles with the only difference being that the role of the counting index is played by the continuous parameter $\vec{\zeta}$.

In conclusion, let us present an expression for the Hamiltonian and the equation of motion. Taking into account that the generating functional (160) (with fixed canonical variables) does not depend on time ($\partial_t \mathcal{F} = 0$), we obtain

$$\mathcal{H}^* = \mathcal{H} = \frac{1}{2} \int d\vec{\zeta} \frac{\beta_k^2}{\rho^*} + \mathcal{U}, \qquad (166)$$

where ρ^* is the density of the fluid in lagrangian coordinates. As it has been shown (Serrin 1959, Landau and Lifshitz 1987), ρ^* does not depend on time and is a function of only a lagrangian variable $\vec{\zeta}$.

The equations of motion obviously have the form

$$\dot{x}_i = \frac{\delta \mathcal{H}^*}{\delta \beta_i} = \frac{\beta_i}{\rho^*}, \quad \dot{\beta}_i = -\frac{\delta \mathcal{H}^*}{\delta x_i} = -\frac{\delta \mathcal{U}}{\delta x_i}. \qquad (167)$$

3.4 2-Dimensional Models of Incompressible Fluid. Canonical Equations for Rossby Waves on a Rotating Sphere

2-dimensional hydrodynamics studies a special class of motions, representing itself a 3-dimensional motion of a fluid stratified into motionless (stationary), non-crossing surfaces which fix the motion of the particles. A pai particular case of such motions is, for example, flat, or spherical, or axis-symmetric motions. Depending on the geometry of the 2-dimensional motion of fluid it is natural to study it in a corresponding system of curvilinear orthogonal coordinates $\zeta^1, \zeta^2, \zeta^3$ which possesses the following properties. The coordinate lines ζ^3 are orthogonal to the fluid surfaces $\zeta^3 = const$ in such a way that ζ^3 plays the role of a continuous parameter identifying these surfaces. Other coordinates ζ^1 and ζ^2 form on each such surface a system of curvilinear orthogonal coordinates which describe the motion of the fluid. The metric tensor g_{ik}, characterizing geometrical properties of space connected with this system of coordinates, has a diagonal form with components g_{11}, g_{22}, g_{33} and determinant $g = g_{11}g_{22}g_{33}$ not equal to zero.

Based on the results from the previous section, the initial rules of the hamiltonian description of 2-dimensional hydrodynamics in such a coordinate system may be written as

$$\mathcal{H} = \frac{1}{2} \int d\zeta^1 d\zeta^2 u^\alpha p_\alpha + \mathcal{U}[\rho], \qquad (168)$$

$$p_\alpha = \rho^* \partial_\alpha \varphi^* - \lambda^*_\beta \partial_\alpha \xi^{*\beta} \qquad (169)$$

where p_α and u^α are correspondingly the components of the covector of hydrodynamical momentum and vector of hydrodynamical velocity, which are related by

$$p_\alpha = \rho^* g_{\alpha\beta} u^\beta, \ (\rho^*, \varphi^*) , \ (\lambda^*, \xi^*)$$

are canonically conjugated pairs of variables, Greek indexes of vectors and tensors have the values α, $\beta = 1, 2$.

In the frame of description (168),(169) the model of incompressible inhomogeneous fluid is realized under the condition $\rho = \rho(\zeta^3)$, i.e. if fluid surfaces are surfaces of equal density. In this case, similar to the case of flat models, the potential part of the hamiltonian is excluded because $\delta\mathcal{U} = 0$. The parameter φ^* may be excluded via the incompressibility equation

$$\partial_\beta \left(g^{1/2} u^\beta\right) = 0. \qquad (170)$$

Condition (170) allows us to introduce a stream function Ψ

$$u^\beta = g^{-1/2} \varepsilon^{\alpha\beta} \partial_\alpha \Psi, \qquad (171)$$

where $\varepsilon^{\alpha\beta}$ is an antisymmetrical unit tensor of second order. Using equation (171), the hamiltonian $\mathcal{H}$ may be rearranged as

$$\mathcal{H} = -\frac{\rho}{2} \int d\zeta^1 \, d\zeta^2 \, \Psi \, g^{1/2} \, \Omega, \qquad (172)$$

$$\Omega = g^{-1/2} \, \varepsilon^{\alpha\beta} \partial_\alpha \, u_\beta = \Delta\Psi. \qquad (173)$$

Here the quantity Ω is a generalized vortex on a non-flat 2-dimensional flow, u_β are covariant components of the hydrodynamical velocity, $u_\beta = g_{\beta\alpha} \, u^\alpha$, Δ is a 2-dimensional operator similar to the Laplace operator:

$$\Delta = g^{-1/2}(\partial_1 g_{22} g^{-1/2} \partial_1 + \partial_2 g_{11} g^{-1/2} \partial_2) \qquad (174)$$

Notice that density ρ, which is a fixed function of ζ^3, may be excluded from the description by using the canonical transformation

$$\lambda_\alpha = \lambda^*_\alpha/\rho, \xi^\alpha = \xi^{*\alpha},\ \mathcal{H} = \mathcal{H}^*/\rho.$$

Taking this comment, (173) and (169) into account, we may formulate the Hamiltonian description on the canonical basic ξ^α , λ_α:

$$\mathcal{H} = -\frac{1}{2}\int d\zeta^1 d\zeta^2\, \Psi g^{1/2}\Omega, \tag{175}$$

$$\Omega = \Delta\Psi = g^{-1/2}J(\xi^\beta,\, g^{-1/2}\lambda_\beta) \tag{176}$$

$$J(a,\, b) = (\partial_1\, a)\,\partial_2\, b - (\partial_2\, a)\,\partial_1\, b. \tag{177}$$

Here $J(a,\, b)$ is a Jacobian of two functions, and relation (176) has a dual meaning. On one hand, it establishes a correspondence between Ψ and Ω, and on the other hand, it presents Clebsh transform for Ω by expressing a vortex field via canonical variables.

It is easy to see directly that the canonical equations of 2-dimensional hydrodynamics

$$\partial_t\xi^\beta = \frac{\delta\mathcal{H}}{\delta\lambda_\beta} = g^{-1/2}\, J(\xi^\beta,\, \Psi),$$
$$\partial_t\lambda_\beta = -\frac{\delta\mathcal{H}}{\delta\xi^\beta} = J(\Psi,\, g^{-1/2}\lambda_\beta) \tag{178}$$

are equivalent to the vortex evolution equation for *nonflat* 2D motion of incompressible fluid

$$\partial_t\Omega + g^{-1/2}J(\Psi,\, \Omega) = 0. \tag{179}$$

We may show that (179) which is written in terms of vorticity represents hamiltonian, but not canonical system, which evolves in phase space of only one field variable Ω and is determined by Poisson bracket $\{\Omega,\, \Omega'\}$. We can write

$$\partial_t\Omega = \{\Omega,\, \mathcal{H}\} =$$
$$= -\int d\zeta^{1'} d\zeta^{2'}\, g^{1/2'}\, \Psi'\, \{\Omega,\, \Omega'\}. \tag{180}$$

The Poisson bracket may be easily calculated, as we know the Clebsh transformation for the vortex (176):

$$\{\Omega,\, \Omega'\} = g^{-1/2}\, J\, (g^{-1/2}\delta(\vec{\zeta} - \vec{\zeta}'),\, \Omega). \tag{181}$$

As an illustration let us use the results for canonical hamiltonian description of fluid motions (for example, Rossby waves) on a *rotating with angular velocity* ω *sphere.* In this case, due to the symmetry of the problem, it is natural to use a spherical coordinate system $\zeta^1 = \theta,\ \zeta^2 = \phi,\ \zeta^3 = r$, assuming that r is the radius of the sphere, and θ and ϕ are correspondingly latitude and longitude.

A solid body rotation of a fluid with the angular velocity ω may be presented as a flow where lagrangian coordinates of fluid parcels change according to the law

$$\zeta^\alpha_s = \zeta^\alpha - u^\alpha_s\, t, \tag{182}$$

where u^α_s are the components of equilibrium velocity of the flow and have the values $u^1_s = 0,\ u^2_s = \omega$. The corresponding equilibrium values of other variables which determine the dynamics may be found from the conditions of equilibrium similar to the way it was done for flat models. In the result we have

$$\lambda_{1s} = 0,\ \lambda_{2s} = -r^4\omega\, \sin^3\theta,$$
$$\Psi_s = -r^2\,\omega\, \cos\theta,\ \Omega_s = 2\omega\, \cos\theta. \tag{183}$$

Due to lagrangevity of variable λ_1 it follows from the condition $\lambda_{1s} = 0$ that $\lambda_1 \equiv 0$. Thus, the pair (ξ^1, λ_1) may be excluded from the Hamiltonian description of the considered model.

In order to describe a flow of a fluid as perturbations on a stationary flow which simulates a solid body rotation of the sphere, first, following the developed method, transfer to the corresponding (rotating) coordinate system

$$\phi' = \phi - \omega\, t, \tag{184}$$

and, second, reformulate the description in terms of variables

$$\zeta = \zeta^2 - \zeta^2_s,\ \lambda = \lambda_2 - \lambda_{2s},$$
$$\psi = \Psi - \Psi_s,\ \Delta\psi = \Omega - \Omega_s, \tag{185}$$

which obviously mean the fluctuations from the equilibrium values.

Because the Jacobian of transformation (182) is equal to one, the canonical variables $(\xi,\, \lambda)$ will remain the same. The time-dependence of the transformation will modify only the hamiltonian. In particular, according to (157) we have

$$H' = \mathcal{H} - \omega r^2 \int d\theta\, d\phi'\, \sin^2\theta\, \partial_1\psi =$$
$$= -\frac{1}{2}\int d\theta\, d\phi'\, \psi(\Delta\psi), \tag{186}$$

$$\Delta\psi = g^{1/2}\, 2\omega\, \cos\theta\, \frac{\partial\xi}{\partial\phi'} -$$
$$\frac{\partial}{\partial\theta'}(g^{-1/2}\lambda) + J(\xi,\, g^{-1/2}\lambda), \tag{187}$$

where $g^{1/2} = r^2\, \sin\theta$ and

$$\Delta = \frac{1}{r^2}\left(\frac{\partial}{\partial\theta}\, \sin\theta\, \frac{\partial}{\partial\theta} + \sin^{-1}\theta\, \frac{\partial^2}{\partial\phi'^2}\right),$$
$$\Delta \equiv g^{1/2}\, \Delta_1. \tag{188}$$

The canonical equations of motion in this case will have the form

$$\partial_t\xi = \frac{\delta H}{\delta\lambda} =$$
$$= -g^{-1/2}\frac{\partial\psi}{\partial\theta} + g^{-1/2}J(\xi,\, \psi), \tag{189}$$

$$\partial_t\lambda = -\frac{\delta H}{\delta\xi} =$$
$$= g^{1/2}2\omega\cos\theta\frac{\partial\psi}{\partial\phi'} + J(\psi,\, g^{-1/2}\lambda) \tag{190}$$

and are fully equivalent to the equation

$$\Delta\dot{\psi} = -2\omega\, \sin\theta\, \frac{\partial\psi}{\partial\phi'} + J(g^{-1/2}\Delta\psi, \psi) \tag{191}$$

which is used for the analysis of planetary nondivergent Rossb waves.

3.5 Evolution Equations for a System Consisting of N Singular Vortices

For now in our consideration in the frame of the method of hamiltonian formalism for different hydrodynamical models we based on the reduced Clebsh transformation. However, in certain cases the application of one of the nonreduced forms of Clebsh transformation may be more reasonable. Examples are 2-dimensional models, which permit the existence of singular point vortices, the equations of motion of which may be written in canonical form of Hamilton equations. In principle, this derivation from the point of view of canonical transformations has a methodical interest.

An expanded version of Clebsh transformation might have been introduced since the very beginning by generalizing relation (169) by simple expansion of the number of pairs of canonical variables ξ^i, β_i, and by assuming that the index i, counting these fields, has values $1, 2, \ldots N$, and that ξ^i are lagrangian variables of a general type. It is easy to see that all the previous results including the formulation (175)-(177) stay valid.

Evolution Equations

Consider now in the frame of such a modified hamiltonian formulation an evolution of a system consisting of N singular vortices. In other words, let us assume that the field Ω is characterized by the following distribution of vorticity

$$\Omega = g^{-1/2} \sum_i \kappa_i \delta(\vec{\zeta} - \vec{\zeta}_i(t)) \tag{192}$$

here κ_i are independent of time intensities of the vortices, and $\vec{\zeta}_i$ depend on time theirs coordinates $\vec{\zeta}_i = \left(\zeta_i^1,\, \zeta_i^2\right)$. Notwithstanding this, let us agree that the repetitive index in this chapter will not mean a summation, which will be shown as $\sum$.

Analysis of (176) with consideration of (192) shows that the necessary distribution of vorticity is realized if Clebsh potentials from the right-hand side of (176) are chosen as

$$a_i\, b_i = \kappa_i \tag{193}$$

where a_i, b_i are time-independent parameters which satisfy the condition

$$\xi^i = a_i \theta\left(\zeta^1 - \zeta_i^1\right), \quad \lambda_i = g^{1/2} b_i \theta\left(\zeta^2 - \zeta_i^2\right). \tag{194}$$

The relations (194) may be interpreted as a transition from the description of system dynamics in canonical basis ξ^i , λ_i to the description in phase space ζ_i^α $(\alpha = 1, 2)$. The evolution of the system in terms of variables ζ_i^α from the perspective of hamiltonian formalism is defined by the full set of Poisson brackets

$$\{\zeta_i^\alpha, \zeta_j^\beta\} = \sum_l \int d\vec{\zeta} \left(\frac{\delta \zeta_i^\alpha}{\delta \xi^l} \frac{\delta \zeta_j^\beta}{\delta \lambda_l} - \frac{\delta \zeta_i^\alpha}{\delta \lambda_l} \frac{\delta \zeta_j^\beta}{\delta \xi^l} \right). \tag{195}$$

Let us calculate Poisson brackets (195). Taking into account that the variables ζ_i^α and ξ^i , λ_i are related as

$$\zeta_i^\alpha = \frac{1}{a_i\, b_i} \int d\vec{\zeta} \zeta^\alpha \frac{\partial \xi^i}{\partial \zeta^1} \frac{\partial (g^{-1/2} \lambda_i)}{\partial \zeta^2}, \tag{196}$$

which follows directly from (194), we find variational derivatives

$$\frac{\delta \zeta_i^\alpha}{\delta \xi^i} = -\frac{\delta_{ij} \delta_{\alpha 1}}{a_i} \delta(\zeta^2 - \zeta_i^2),$$

$$\frac{\delta \zeta_i^\alpha}{\delta \lambda_j} = -g^{-1/2} \frac{\delta_{ij} \delta_{\alpha 2}}{b_i} \delta(\zeta^1 - \zeta_i^1). \tag{197}$$

In the result, the substitution of (197) into (195) yields

$$\{\zeta_i^\alpha, \zeta_j^\beta\} = \frac{\delta_{ij} \varepsilon^{\alpha\beta}}{\kappa_i g_i^{1/2}}, \tag{198}$$

where $g_i = g(\vec{\zeta} = \vec{\zeta}_i)$.

Thus, in terms of the variables ψ_i^α the dynamics of a system of singular vortices will be described by the equations

$$\partial_t \zeta_i^\alpha = \{\zeta_i^\alpha, H\} = \frac{\varepsilon^{\alpha\beta}}{\kappa_i g^{1/2}} \frac{\partial H}{\partial \zeta_i^\beta}, \tag{199}$$

where the hamiltonian H (175), equal to kinetic energy of the medium, must be expressed in terms of ζ_i^α.

In order to solve this problem, let us define Green's function $\mathcal{G}(\vec{\zeta}, \vec{\zeta}')$ as a function of two variables and which satisfies the equation

$$g^{1/2} \Delta_1 \mathcal{G}(\vec{\zeta}, \vec{\zeta}') = \delta(\vec{\zeta} - \vec{\zeta}') \tag{200}$$

and which, by expressing Ψ via Ω, allows us to write for the hamiltonian the expression

$$H = -\frac{1}{2} \int d\vec{\zeta} d\vec{\zeta}'\, \Omega\, \Omega' (g\, g')^{1/2} \mathcal{G}(\vec{\zeta}, \vec{\zeta}') \tag{201}$$

or

$$H = -\frac{1}{2} \sum_{i,j} \kappa_i \kappa_j \mathcal{G}(\vec{\zeta} - \vec{\zeta}'). \tag{202}$$

The final expression for H via implicit Green's function is obtained by substituting (192) into (201).

Problem of the Self-Action of Vortices

This expression, however, has a shortcoming by having an uncertainty which arises from turning into infinity of the terms of series (202) when $i = j$. These terms describe the proper energy of the point vortices. Obviously, only in the case when the character of these infinities does not depend on the location of the vortices, the self-action corresponding to the infinite energy which do not affect the evolution of the vortices, may be excluded from the hamiltonian (202).

The mathematical nature of these infinities is universal. It is defined by the fact that when $|\vec{\zeta} - \vec{\zeta}'| \to 0$ Green's function which satisfies (200) is described by an asymptotic expression

$$\mathcal{G} = \frac{{g_{33}}^{1/2}}{2\pi} \ln r(\vec{\zeta}, \vec{\zeta}'),$$

$$r^2 = g_{11}(\zeta^1 - \zeta^{1'})^2 + g_{22}(\zeta^2 - \zeta^{2'})^2 \tag{203}$$

which has a logarithmic divergence.

Because the above-described divergence appears only under the assumption of singular vortices, when the vorticity distribution is described by delta-function, finite objects should be considered where this problem does not occur. Let us consider that the vorticity distribution near the vertex center $\vec{\zeta}_i$ of i-vortex is determined by a local radially-symmetric function

$$\Omega_i = g^{-1/2} \frac{\kappa_i}{\pi\varepsilon^2} \theta(\varepsilon - r), \tag{204}$$

which in the limit, when the size of the vortex tends to zero, describes the singular distribution for the singular vortex.

Because the problem of divergence is related to the proper energy of the vortices, let us evaluate the integral (201).

Taking into account the local character of distribution of Ω, and using the asymptotic form for Green's function (203), we may show that

$$H_i = -\frac{\kappa_i^2}{4\pi}(\ln\varepsilon - \frac{1}{2}) g_{33}^{1/2} + O(\varepsilon). \tag{205}$$

It is easy to see that the effect of self-action is absent (at least in first order approximation), if $\partial g_{33}/\partial\zeta^\alpha = 0$. Strictly saying, only under this condition the regularization of the hamiltonian for the assumption of point vortices, which is based on truncating the infinite terms of the series (202), is valid. Spherical coordinates may be considered as an example of a system of curvilinear coordinates which satisfy this requirement.

Let us remind that the different aspects of evolution of vortices on a sphere has been studied in numerous works (for example, see Bogomolov 1977, Zabusky and McWilliams 1982, Reznik 1992 and Refs. therein).

4 Discussion

Generally speaking, there are different versions of the method of Hamiltonian Formalism. That is why when talking about this method it is necessary to fully realize which one of the versions is implied.

The point of the departure of one of the versions is the action integral taken in the form

$$S = \int dt\, \mathcal{L}[u_i, \dot{u}_i] \equiv$$
$$\int dt \{ \left(\int d\mathbf{x} \hat{A}_j[u; \mathbf{x}, \mathbf{x}_1] \dot{u}_j(\mathbf{x}_1) \right) - H[u] \}. \tag{206}$$

Variations of the action S with respect to the variables $u = (u_k)$ lead to the evolution equation[6]

$$\int d\mathbf{x}_1 \hat{\omega}_{ik}[u; \mathbf{x}, \mathbf{x}_1]\, \dot{u}_k(\mathbf{x}_1) = \frac{\delta\mathcal{H}}{\delta u_i(\mathbf{x})}. \tag{207}$$

Here $\mathcal{H}$ is the Hamiltonian of the considered hydrodynamical system, $\dot{u}_k \equiv \partial_t u_k$ is the partial derivative with respect to time, $\delta/\delta u_m$ are the functional derivatives and ω_{ik} is the symplectic form defined by the condition

$$\hat{\omega}_{ik}[\mathbf{x}, \mathbf{x}_1] = \delta\hat{A}_i[u(\mathbf{x}_1)]/\delta u_k(\mathbf{x}) - \delta\hat{A}_k[u(\mathbf{x})]/\delta u_i(\mathbf{x}_1).$$

The version of the Hamiltonian description which proceeds from the evolution equations

$$\dot{u}_i(\mathbf{x}) = \{u_i, \mathcal{H}\} \equiv$$
$$\equiv \int d\mathbf{x}_1 \{u_i(\mathbf{x}), u_j(\mathbf{x}_1)\} \frac{\delta\mathcal{H}}{\delta u_j(\mathbf{x}_1)} \tag{208}$$

has been considered in the present paper.

Equations (207) and (208) would have become equivalent if there existed a one-to-one transformation, i.e. if there existed the relation

$$\int d\mathbf{x}_2\, \hat{\omega}_{ij}[u; \mathbf{x}_1, \mathbf{x}_2] \{u_j(\mathbf{x}_2), u_k(\mathbf{x}_3)\} =$$
$$= \delta_{ik}\, \delta(\mathbf{x}_1 - \mathbf{x}_3). \tag{209}$$

Such a scenario is realized when Poisson brackets are nondegenerated. In this case, it would be absolutely irrelevant which of the formulations was taken as a point of departure. However, for degenerated brackets $\{u_j, u'_k\}$, transformation (209) are impossible for a great number of hydrodynamical models (examples can be found for the traditional hydrodynamics of compressible and incompressible fluids, magnetohydrodynamics, spine-fluid models, *etc.*).

In this case, if from the beginning one is forced to work within the class of models determined by the evolution equations (207), it is necessary to go through the process of not only searching for the canonical variables, but also ascertaing their connection with physically observed field quantities (for example, one needs to elucidate the sense of the multivalued Clebsch representations), and then inventing the models of hydrodynamic systems with unusual properties, and so on.

Even if the necessary structure of the Lagrangian is guessed or selected in some intuitive way, the use of variational principle (206) requires the formulation of additional postulates concerning latent constraints, the physical argumentation of which is not always obvious. In our paper the Hamiltonian description is not absolutely axiomatically constructed, but rather is directly driven from physically-based presumptions about the type of evolution of hydrodynamical systems and their internal properties.

5 Appendix

5.1 Transformation of Canonical Variables

The fact that Hamiltonian systems exhibit a stiffness of equations structure, i.e., keep the properties (17), (18) under various transformation, arrests our attention at once as a feature of great importance. In other words, new equations generated by the transformations remain a Hamiltonian character although not only Poisson brackets but also the Hamiltonian itself can be changed. This peculiar feature of Hamiltonian

[6] The approach based on the use of the equations (207), and which, by the way, may be reformulated in terms of 2-forms, has a wide dissemination (see Refs of the present paper).

systems opens up considerable possibilities for different applications.

From this standpoint, a significance of a Hamiltonian formulation with one or another structure may be different. A degree of its importance is defined by availability of such methods in its body as should be adequate to the problem under study. This circumstance must always be kept in mind when there are alternative possibilities for choosing of Hamiltonian formulation.

The important class of transformations, which do not change the structure of Poisson brackets and consequently the structure of equations (19), is concerned with the canonical formulation. For finite-dimensional systems, such transformations named canonical are fully described in any textbook of classical mechanics (Arnold 1978, Goldstein 1980).

As known,any canonical transformation is characterized by own generating functional F and can be found with the help of it. Corresponding procedure is formulated in the following way.

We give some examples of using the generating functionals. Every so often in application it is necessary to execute so-called point transformations, which express old coordinates in terms of new coordinate only

$$q_i = f_i\left[Q_k; \mathbf{x}, t\right]. \tag{210}$$

To such transformations correspond the generating functional of type

$$F = -\int d\mathbf{x}' p_i' f_i', \tag{211}$$

in compliance with which the new momenta and Hamiltonian are determined as

$$P_i = -\delta/\delta Q_i = \int d\mathbf{x}' p_n' \frac{\delta f_n'}{\delta Q_i},$$
$$\mathcal{H}' = \mathcal{H} + \partial_t F, \tag{212}$$

where $\mathcal{H}'$ is new Hamiltonian of the system, and $\partial_t F$ is partial derivative of generating functional with respect to time at fixed the field variables q_i, p_i, Q_i, P_i.

Canonical Variables for Surface Waves

In spite of relative simplicity, point transformations enable us to obtain sufficiently profound and beneficial results. The canonical variables for waves on surface of incompressible homogeneous fluid serve as an instructive illustration. The fact that the surface shape and the potential of velocity on the surface are these variables had been heuristically established by Zakharov in 1968 and remains exotic for a long time. However, this fact may be obtained by simple and, what is more important, by regular way.

First of all recall one result (Bateman 1932) preceded the work of Zakharov. Following this result, for a potential compressible flow of perfect unlimited fluid the density ρ is canonical coordinate and the hydrodynamic potential φ is momentum. Given rule is fit for homogeneous incompressible fluid limited by free surface $z = \eta\left(t, x, y\right)$ too, if the density distribution is characterized by the generalize function

$$\rho = \rho_0 \theta\left(\eta - z\right), \quad \rho_0 = const, \tag{213}$$

where θ is the Haviside function: $\theta(z) = 1$ if $z \geq 0$ and $\theta(z) = 0$ if $z < 0$.

The density, thus defined, allows us in mathematically well-posed manner to get around problem of medium finiteness and to take account of free surface properly (Goncharov *et al.* 1977, Goncharov 1980). Such extension of medium definition into whole space makes possible the using of Bateman rule. According to this rule, in describing a potential flow of incompressible fluid limited by free surface, the density $\rho_0\theta\left(\eta - z\right)$ acts as canonical coordinate and the velocity potential φ acts as canonical momentum.

Consider (213) as the point transformation to the new generalized coordinate η. Note in passing that the space dimension of problem is thereby reduced by one. Then, in according to (210)- (212), we find new momentum ψ:

$$\psi = \frac{\delta}{\delta\eta}\int dx dy \rho_0 \theta\left(\eta - z\right)\varphi = \rho_0\varphi\mid_{z=\eta}. \tag{214}$$

This result complains with that obtained by Zakharov up to constant factor ρ_0. Setting ρ_0 equal to unity or using the renormalization, ρ_0 can be excluded without loss of generality.

Transformation of Space Coordinates

Another example of using the point canonical transformation is of the transformation of space coordinates. As known, every so often the using of corresponding curvilinear coordinate system simplifies a solution of problem essentially. Because it is desirable to have a covariant formulation of the canonical Hamiltonian formalism. This is found to be not a particular problem if question had been solved in a frame of the Cartesian coordinates (Goncharov *et al.* 1977).

As an illustration, let us consider a one-to-one, continuous, time-depending transformation of space coordinates $\mathbf{x}$ and ξ:

$$\mathbf{x} = \mathbf{x}(\vec{\xi}, t). \tag{215}$$

Using (215), perform the canonical point transformation

$$P_i(\vec{\xi}, t) = \int d\mathbf{x}\, p_i(\mathbf{x}, t)\, \delta(\mathbf{x} - \mathbf{x}(\vec{\xi}, t)), \tag{216}$$

which expresses the simple fact that the new momenta P_i are obtained from the old p_i through ordinary change of variables (215).

Inasmuch the system (215) is resolvable with respect to the coordinates ξ and there exists the inverse transformation

$$\vec{\xi} = \vec{\xi}\left(\mathbf{x}, t\right), \tag{217}$$

the relation (216) can be rewritten in another form

$$p_i(\vec{\xi}, t) = \int d\vec{\xi}\, P_i(\vec{\xi}, t)\, \delta(\vec{\xi} - \vec{\xi}(\mathbf{x}, t)). \tag{218}$$

The transformation (218) corresponds to generating functional

$$F[q, P] = \\ = \int d\mathbf{x}\, d\vec{\xi}\, q_i(\mathbf{x}, t)\, P_i(\vec{\xi}, t)\, \delta(\vec{\xi} - \vec{\xi}(\mathbf{x}, t)). \quad (219)$$

Firstly, we find new generalized coordinates Q_i. Taking into account of known property of delta-function

$$\delta(\vec{\xi} - \vec{\xi}(\mathbf{x}, t)) = g^{1/2}\, \delta(\mathbf{x} - \mathbf{x}(\vec{\xi}, t))$$

where $g^{1/2}$ is the Jacobian of transformation (215), we obtain

$$Q_i = \frac{\delta F}{\delta P_i} = \int d\mathbf{x}\, q_i(\mathbf{x}, t)\, \delta(\vec{\xi} - \vec{\xi}(\mathbf{x}, t)) \\ = g^{1/2}\, q_i \,\big|_{\mathbf{x}=\mathbf{x}(\vec{\xi}, t)} . \quad (220)$$

To find the new Hamiltonian it is necessary to differentiate the generating functional with respect of time with q_i, P_i fixed. As a result we find

$$\mathcal{H}' = \mathcal{H} + \dot{F} = \mathcal{H} + \int d\vec{\xi}\, \dot{\vec{\xi}}\, Q_i \frac{\partial P_i}{\partial \vec{\xi}} \quad (221)$$

where $\dot{\vec{\xi}} = \left[\partial \vec{\xi} / \partial t\right]_{\mathbf{x}=\mathbf{x}(\vec{\xi}, t)}$.

From the standpoint of application not only canonical transformations deserve attention but noncanonical also. For example, for any noncanonical Hamiltonian system one can construct transformations of field variables, which do not change Poisson brackets and hence a structure of equations but can change Hamiltonian similar to canonical transformations. Conversely, there are transformations, which change Poisson brackets in preserving Hamiltonian.

An example of such transformation is the transformation from vector field of vorticity ω to so-called **n**-*field* (Faddeev 1976)

$$\omega_\alpha = A \varepsilon^{\alpha\beta\gamma} \mathbf{n} \cdot [\partial_\beta \mathbf{n}, \partial_\gamma \mathbf{n}], \quad (222)$$

where $\mathbf{n}^2 = 1$, A is a dimensional constant.

The transformation (222) maps equations (25) and (24) one into another and thus set up one-to-one correspondence between vortex and spin dynamics.

5.2 Symmetry Transformation and Conservation Laws

Of special place among continuous transformations occupy those realizing a variation of dynamical variables without changing motion equations of the system. Pointing to an existence of corresponding symmetry properties for the system, such symmetry transformations have profound physical meaning since the availability of symmetries is closely connected with conservation laws.

The tool, which enables one to derive explicit expressions for quantities conserved during time and called motion invariants, is just the Noether theorem (Arnold 1978, Bowman 1987).

In general case the symmetry transformations, which are admitted by Hamiltonian systems, are bound to satisfy two requirements. Firstly, they must not change the structure of equations and hence Poisson brackets. Secondly, they must provide an invariability of Hamiltonian.

Note that for canonical Hamiltonian systems the first requirement satisfies automatically if confine oneself to canonical transformation only. Because whether or not a canonical transformation is symmetry transformation depends on fulfillment of the second requirement.

Let us consider an infinitesimal canonical transformation of canonical variables q_i, p_i. It is evident that the generating functional of this transformation differs infinitesimally from the functional for identity transformation and thus has the forrr

$$F = \int d\mathbf{x}\, q_i P_i + \varepsilon\, G[q, P; t]. \quad (223)$$

Here ε is the infinitesimal parameter of the transformation. Then, we obtain

$$p_i = P_i + \varepsilon \frac{\delta G}{\delta q_i}, \\ Q_i = q_i + \varepsilon \frac{\delta G}{\delta P_i}, \\ \tilde{H} = H + \varepsilon \partial_t G, \quad (224)$$

where Q_i are new coordinates, P_i are new momenta, and H, $\tilde{H}$ are old and new Hamiltonian correspondingly.

Introduce the quantity

$$I\,[q, p; t] = G\,\big|_{P_i = p_i} . \quad (225)$$

As will be shown below this quantity called the generator of infinitesimal canonical transformation plays an important role in formulating Noether theorem.

Inasmuch old and new momenta differ infinitesimally from ones others, in equalities (224) we can replace G by I and $\delta G/\delta P_i$ by $\delta I/\delta p_i$ to within small quantities of the first order. As a result, instead of (224), we find

$$p_i = P_i + \varepsilon \frac{\delta I}{\delta q_i}, \quad q_i = Q_i - \varepsilon \frac{\delta I}{\delta P_i}, \quad (226)$$

$$\tilde{H} = H + \varepsilon \partial_t I, \quad (227)$$

Taking into account (226) and expanding H into a functional Taylor series in powers of ε, from (227) we obtain the approximate equality

$$\tilde{H} = H\,[Q, P; t] + \varepsilon\,(\partial_t I + \{I, H\}) . \quad (228)$$

Whence, considering that

$$\partial_t I + \{I, H\} = \frac{dI}{dt}, \quad (229)$$

we arrive at the conclusion that if an infinitesimal canonical transformation does not change Hamiltonian, i.e.,

$$\tilde{H} = H\,\big|_{q_i = Q_i,\ p_i = P_i}, \quad (230)$$

then the generator of transformation is invariant:

$$\frac{dI}{dt} = 0. \quad (231)$$

It is this conclusion that constitutes the dominant bulk of the subject matter of Noether theorem for canonical Hamiltonian systems.

As example, let us consider potential motion of unlimited compressible ideal fluid. In this case the density ρ acts as the canonical coordinate while the canonical momentum is φ - potential of hydrodynamic velocity $\mathbf{v}$. As a potential is defined within an accuracy of the constant, we can make the transformation to the new canonical variables

$$\rho' = \rho, \quad \varphi' = \varphi + \varepsilon, \quad \varepsilon = const. \tag{232}$$

This transformation is one of symmetry because it does not change Hamiltonian described by the same expression as the total energy of fluid

$$H = \int d\mathbf{x}\,[\rho\frac{\mathbf{v}^2}{2} + U(\rho)], \tag{233}$$

where $U(\rho)$ is the density of internal energy and $\mathbf{v} = \nabla\varphi$.

The generating functional, corresponding to (232), can be represented in the following form

$$F = \int d\mathbf{x}\,\rho\,\varphi' - \varepsilon \int d\mathbf{x}\,\rho. \tag{234}$$

From (234) in compliance with the Noether theorem follows a conservation of the generator of infinitesimal canonical transformation

$$I = \int d\mathbf{x}\,\rho, \tag{235}$$

which has evidently the meaning of the total mass of fluid.

The symmetry transformation can be conditionally divided on two basic types. First type transformations touch upon only space coordinate and time. It is commonly known (Arnold 1978, Goldstein 1980, Landau and Lifshitz 1982) that space-time symmetry properties dictate three fundamental laws of mechanics: law of conservation of energy, law of conservation of linear momentum and law of conservation of angular momentum. Energy conservation law is a consequence of invariance of theory in relation to a shift of the time origin and implies that time is uniform, i.e., laws of motion must be independent of a choice of the time origin. Just as from time uniformity follows energy conservation, so from space uniformity, which implies invariance of the theory in relation to space translation, follows conservation of the total momentum. Analogously, from invariance of theory in relation to 3D rotation group existing by virtue of an isotropy of the coordinate space, follows conservation of the total angular momentum.

Other important type of symmetry transformations is the gauge transformations, which do not touch on the space coordinates and thus characterize only internal properties of symmetry of dynamical system. In field theory transformations are called gauge in the widest sense if they vary unobservable field characteristics but therewith do not vary observable quantities making a physical sense.

As the simplest example we refer to (232) varying the velocity potential φ by a constant ε. Inasmuch as only the quantity $\nabla\varphi$ - velocity makes the physical sense in hydrodynamics, the transformation (232) is gauge.

According to the classification accepted in field theory, in the first example (232) we deal with so-called a global gauge transformation the parameter of which is a number ε. However, such parameter can be, generally speaking, an arbitrary function $\varepsilon(\mathbf{x})$. This sort of gauge transformations is called local.

Let us consider an example made it possible to trace that an appearance of the gauge symmetries is connected with certain ambiguity, which arises in changing the dimension of phase space in result of the canonical reformulation. We have faced the similar situation in constructing the canonical basis for the KdV equation. Recall that in this case the transformation linking the physical variable u with the canonical variables q and p takes the form

$$u = \partial_x q + \frac{1}{2}p. \tag{236}$$

As is easy to see, (236) remains invariant in relation to gauge transformation to new canonical variables Q and P:

$$Q = q + \frac{1}{2}\varepsilon,$$
$$P = p - \partial_x\varepsilon,$$
$$F = \int dx\,[qP + \varepsilon(\frac{1}{2}P - \partial_x q)], \tag{237}$$

where $\varepsilon = \varepsilon(x)$ is arbitrary function of x, F is corresponding generating transformation.

Consequently, by the Noether theorem the integral

$$J = \int dx\,\varepsilon(\frac{1}{2}p - \partial_x q) \tag{238}$$

is conserved.

From (238), we find the local relation

$$\partial_x q - \frac{1}{2}p = 0. \tag{239}$$

From the geometrical standpoint, the relation (239) fixes some surface into given simplectic phase space q, p. such that all motion trajectories of the system $\dot{q} = \delta H/\delta p, \dot{p} = -\delta H/\delta q$ must lie in this surface for they to reproduce the solutions of the KdV equation in compliance with (236).

References

Abarbanel, H.D.I., Holm D.D., Marsden J.E. and Ratiu T.S. a) 1986 Nonlinear stability analysis of stratified fluid equilibria. *Phyl. Trans. Roy. Soc. London*, A**318**, 349-409; b) 1987 Nonlinear stability analysis of inviscid flows in three dimensions: incompressible fluids and barotropic fluids. *Phys. Fluids*, **30**, 3369-3382.

Arnold, V.I. 1969 The Hamiltonian nature of the Euler equations in the dynamics of a rigid body and of a perfect fluid. *Russ. Math. Surveys*, **24** (3), 225-226.

Batchelor, G.K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press, 615 pp.

Bateman H. 1929 Notes on a differential equation which occurs in the two-dimensional motion of a compressible fluid and the associated variational problems. *Proc. Roy. Soc. London,* Ser **A**, Vol. **125.799,** 598-618.

Bogoljubov, N.N. and Shirkov, D.B. 1980 *Introduction to the Theory of Quantized Fields.* Ed.: Interscience, 1959; 3rd ed., Wiley 1980.

Bogomolov, V.A. 1977 The dynamics of vorticity on a sphere. *Izv. Akad. Nauk SSSR, Mekh. Ghid. Gaza* **13**, p.57.

Bretherton, F.P. 1970 A note on Hamilton's principle for perfect fluids. *J. Fluid Mech.*, **44**(1), 19-31.

Camassa, R. and Holm, D.D. 1992 Dispersive barotropic equations for stratified mesoscale ocean dynamics. *Physica D,* **60,** 1-15.

Clebsch, A. 1859 Uber die Integration der Hydrodynamschen Gleichungen. *J. Reine Angew. Math.,* **56**, 1-10.

Dirac, P.A.M. 1950 On generalized Hamiltonian dynamics. *Can. J. Math.,* **2,** 129-148.

Ertel, H. 1942 Ein Neuer Hydrodynamischer Wirbelsatz. *Meteorol. Z.,* **B 59,** 277-281.

Faddeev, L.D. 1976 Some comments on many-dimensional solitons. *Lett. Math. Phys.,* **1,** no. 4, 289-293.

Gavrilin, B.L. and Zaslavskii, M.M. 1970 [O lagranjevykh invariantakh dynamiki nevjazkoii sjimaemoii jidkosti.] *Sov. Math. - Dokl.,* **19,** (1), 48-51.

Goncharov, V.P., Krasil'nikov, V.A. and Pavlov, V.I. 1976 Canonical variables in the magnetohydrodynamics of a nonuniform medium. *Sov. Phys. - Dokl.,* **21** (5), 255-256.

Goncharov, V.P., Krasil'nikov, V.A. and Pavlov, V.I. 1977 Introduction of canonical variables in the hydrodynamic equations of an ideal liquid in the general theory of relativity. *Sov. Phys. - Dokl.,* **22** (9), 495-497.

Goncharov, V.P. 1990 Constructing canonical variables and the problem of "gauge" freedom in hydrodynamics. *Sov. Phys. - Dokl.,* **35**(7), 616-618.

Goncharov, V.P. 1990 Poisson brackets for hydrodynamic models with internal degrees of freedom. *Sov. Phys. - Dokl.,* **35**(8), 728-729.

Goncharov, V.P. and Pavlov, V.I. 1993 *Problems of Hydrodynamics in Hamiltonian Approach.* Ed.: Moscow Univ., Moscow (in Russian), 197 pp.

Gossard, E.E. and Hooke, W.H. 1975 *Waves in the Atmosphere.* Elsevier, Amsterdam, 456 pp.

Holm, D.D., Marsden, J.E. Ratiu, T. and Weinstein, A. 1985 Nonlinear stability of fluid and plasma equilibria. Phys. Rep., **123,** 1-116.

Kuzmin, G.A. 1983 Ideal incompressible hydrodynamics in terms of the vortex momentum density. *Phys. Lett.,* **A96,** 88-90.

Kuznetsov, E.A., and Mikhailov, A.V. 1980 On the topological meaning of canonical Clebsch variables. *Phys. Lett.,* **77A**(1), 37-38.

Lamb, H. 1932 *Hydrodynamics,* 6th ed., Cambridge: Cambridge University Press (reprinted New York : Dover, 1945.)

Landau, L.D. and Lifshitz, E.M. 1982 *Physique theorique: Mecanique.* IYeme ed., Ed.: Mir, Moscou, 242p.

Lifshitz, E.M. and Pitaevskii, L.P. 1978 *Statistical Physics*, Part. II, Ed.: Nauka, Moscow.

Lin, C.C 1961 *Hydrodynamics of helium II.* In: Rend. Scuola. Internaz. "Enrico Fermi", 1961. New York-London: Acad. Press.,**21**, 93-146.

L'vov, V.S. 1994 *Wave Turbulence Under Parametric Excitation*, Springer-Verlag Berlin Heidelberg 1994, Springer Series in Nonlinear Dynamics.

McIntyre, M.E. and Shepherd, T.G. 1987 An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnold's stability theorems. *J. Fluid Mech.,* **181,** 527-565.

Moffatt, H.K. a) 1969 The Degree of Knottedness of Tangled Vortex' Lines. *J. Fluid Mech.*, **35**, 117-129; b) 1990 The topological (as opposite to the analytical) approach to fluid and plasma flow problems. In *Topological Fluid Mechanics* (ed. H.K. Moffat and A. Tsinober), 1-10. Cambridge University Press.

Novikov, E.A. 1983 Generalized dynamics of three-dimensional vorticial singularities (vortons). *Sov. Phys. - JETP,* **57,** (3), 566-569.

Reznik, G.M. 1992 Dynamics of singular vortices on a beta-plane. *J. Fluid Mech.,* **240**, 405-432.

Salmon, R. 1988 Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.* **20,** 225-256.

Seliger, R.L. and Whitham, G.B. 1968 Variational principles in continuum mechanics. *Proc. Roy. Soc. London,* Ser. **A 305**: 1-25.

Serrin, J. 1959 *Mathematical principles of classical fluid mechanics.* Handbuch der Physik YIII-I, Berlin: Springer-Verlag, 125-263.

Zabusky, N.J, and McWilliams, J.S. 1982 A modulated point vortex for geostrophic beta-plane dynamics. *Phys. Fluids* **25**, 2175.

Zakharov, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* , **2**, 190-194; (see also [*Prikladnaja Mekhanika i Tekhnicheskaja Fizika*], **2,** 86-94 (in Russian)).

Zakharov, V.E. and Faddeev, L.D. 1971 The Korteweg - de Vries equation: a completely integrable Hamiltonian system. *Funct. Anal. Appl.,* **5**, no. 4, 280 - 287.

Zakharov, V.E. and Kuznetsov, E.A. a) 1982 The Hamiltonian formalism for systems of a hydrodynamical type. [Gamiltonovskii formalism dlja sistem hydrodinamicheskogo typa.] Preprint no. **186,** Inst. AE SO AN USSR, Novosibirsk (in Russian); b) 1984 *Sov. Sc. Rev.,* Sec.C, ed. S.Novikov (Harwood Acad. Publ., N.Y.), 167-.