Nonlinear Processes in Geophysics (1998) 5: 105–110

Nonlinear Processes in Geophysics

Three-wave coupling in a stratified MHD plasma

P. Axelsson

Department of Plasma Physics, Umeå University, S-90187 Umeå, Sweden

Received: 18 May 1998 – Accepted: 25 August 1998

Abstract. A general coupling coefficient for three wave interactions in an ideal MHD plasma is presented. Using a special stratified background atmosphere, an explicit symmetric form of the coupling coefficient is derived.

1 Introduction

The dynamics of planetary atmospheres includes many different nonlinear phenomena. Various model equations governing the low-frequency fluid motion have been derived. They can have solitary and vorticity like solutions (Stenflo, 1986, 1987a). In particular, Stenflo and Stepanyants have shown that strongly localized two-dimensional solitary acoustic-gravity vortices can exist in regions with strong temperature gradients (Stenflo, 1987b; Stepanyants, 1989, 1991; Gryanik and Dobritsyn, 1990). Similar structures can also exist in dusty gases (Shukla and Shaikh, 1998). Furthermore, the transport properties of turbulent media consisting of such solitary structures are changed (Pavlenko and Stenflo, 1992). Numerical studies result in chaotic solutions (Zhou et al., 1997a, 1997b).

It is well-known that similar nonlinear structures occur in magnetized media. Two-dimensional solitary Alfvén wave dipolar vortices have thus been studied (e.g. Shukla and Stenflo, 1997). They can account for the different nonlinear localized electromagnetic structures that have been observed by the Freja satellite in the ionosphere of the Earth.

A fundamental nonlinear process in fluids and plasmas is the resonant three-wave interaction process. Calculation of three-wave coupling coefficients for weakly nonlinear, resonant interactions between waves in fluids and plasmas have received much attention by numerous authors (e.g. Dong and Yeh, 1988, 1991; Lindgren, 1982; Stenflo, 1994; Yeh and Liu, 1981). Due to the algebraic difficulties, the coupling coefficients have in general not been presented in a explicit symmetric form. The Hamiltonian approach to wave coupling has however recently proved to be very useful for deriving general and symmetric coupling coefficients for interactions between atmospheric waves (Axelsson et al., 1996a, 1996b).

In the present paper we will consider resonant interactions between MHD waves in an inhomogeneous media. We then take advantage of the Hamiltonian property of the MHD equations to derive general expressions for the wave coupling coefficients. One particular case of special interest, relevant for the ionosphere of the Earth and the atmospheres of stars, is the case with an exponentially stratified background state. For a special stratified magnetic field distribution the magneto-acoustic-gravity waves can be described by a global dispersion relation with constant coefficients. Then the simplified coupling coefficients can be written in an explicit Manley-Rowe symmetric form.

2 Hamiltonian models and wave coupling

Ideal MHD can be written in Hamiltonian form in terms of a generalised noncanonical Poisson bracket (Morrison and Greene, 1980). In the present paper we choose to start from a corresponding generalisation of Hamiltons canonical equations (e.g. Olver, 1993)

$$\partial_t u = X(u) \equiv J_u \frac{\delta H}{\delta u}(u). \quad (1)$$

Here, the Hamiltonian H(u) of the fluid is the system energy, u is the vector of field variables and $\delta/\delta u$ is the functional derivative. The Poisson structure J_u is a linear operator such that the Poisson bracket is

$$\{F,G\} = \left\langle \frac{\delta F}{\delta u}, J_u \frac{\delta G}{\delta u} \right\rangle. \quad (2)$$

Thus, the Poisson structure must be antisymmetric

$$\langle f, J_u g\rangle = -\langle J_u f, g\rangle. \tag{3}$$

The Jacobi identity now follows by requiring

$$\left\langle f, J_u \frac{\delta}{\delta u}\langle g, J_u h\rangle\right\rangle + \left\langle h, J_u \frac{\delta}{\delta u}\langle f, J_u g\rangle\right\rangle + \left\langle g, J_u \frac{\delta}{\delta u}\langle h, J_u f\rangle\right\rangle = 0 \tag{4}$$

Here F and G are arbitrary functionals and f, g and h are arbitrary fields. Now we want to study the small amplitude expansion of the right-hand side of equation (1). Thus, assuming that the unperturbed stationary state u_0 is given by

$$\partial_t u_0 = 0,\ X(u_0) = 0, \tag{5}$$

we write

$$\partial_t \delta u = X^{(1)}_{u_0}\delta u + \frac{1}{2}X^{(2)}_{u_0}(\delta u, \delta u) + \dots. \tag{6}$$

Furthermore, we consider three linear independent resonant normal modes u_1, u_2 and u_3, i.e.

$$iX^{(1)}_{u_0}u_j = \omega_j u_j,\quad j = 1,2,3 \tag{7}$$

$$\omega_1 + \omega_2 + \omega_3 = 0, \tag{8}$$

where the waves are assumed to be Hamiltonian perturbations of the background state (cf. Morrison and Pfirsch, 1992; Larsson, 1996, 1998a). Then the conjugate fields ξ_j will be defined in terms of the linearised field variations as

$$u_j = J_{u_0}\xi_j. \tag{9}$$

The concept of dynamical accessible perturbations is also discussed in Morrison (1998). Now using the the Ansatz

$$u = u_0 + \sum_{j=1}^{3}\left[C_j(t)u_j \exp(-i\omega_j t) + \text{comp. conj.}\right], \tag{10}$$

the slowly time varying amplitudes C_j can be shown to satisfy the coupled mode equations (Larsson, 1998b)

$$\frac{d}{dt}\overline{C}_1 = -i\omega_1 \frac{V}{W_1}C_2C_3,\quad \frac{d}{dt}\overline{C}_2 = -i\omega_2 \frac{V}{W_2}C_1C_3,$$
$$\frac{d}{dt}\overline{C}_3 = -i\omega_3 \frac{V}{W_3}C_1C_2, \tag{11}$$

where

$$W_j = -\left\langle \overline{\xi}_j, X^{(1)}_{u_0}u_j\right\rangle, \tag{12}$$

and

$$V = -\left\langle \xi_1, X^{(2)}_{u_0}(u_2, u_3)\right\rangle, \tag{13}$$

where $\langle , \rangle$ denotes an inner product and the overbar the complex conjugate.

The coupling coefficient (13) is written in terms of the operators

$$H'_u \equiv \frac{\delta H}{\delta u}, \tag{14}$$

$$H''_u(a) \equiv \frac{\delta}{\delta u}\langle a, H'_u\rangle, \tag{15}$$

$$H'''_u(a,b) \equiv \frac{\delta}{\delta u}\langle a, H''_u(b)\rangle \dots, \tag{16}$$

$$J'_u(a,b) \equiv \frac{\delta}{\delta u}\langle a, J_u b\rangle, \tag{17}$$

$$J''_u(a,b,c) \equiv \frac{\delta}{\delta u}\langle a, J'_u(b,c)\rangle \dots, \tag{18}$$

where a and b are arbitrary fields. Thus, using the expansions

$$X(u_0 + \delta u) = X_0 + X^{(1)}_0(\delta u) + \frac{1}{2!}X^{(2)}_0(\delta u, \delta u) + \dots \tag{19}$$

and

$$J_{u_0+\delta u} = J_0 + J^{(1)}_0(\delta u) + J^{(2)}_0(\delta u, \delta u), \tag{20}$$

where the subscript 0 represents unperturbed quantities, the different terms in (19) are derived from

$$X^{(1)}_0(a) = J_0 H''_0(a) + J^{(1)}_0(a)H'_0, \tag{21}$$

$$X^{(2)}_0(a,b) = J_0 H'''_0(a,b) + J^{(1)}_0(a)H''_0(b) + J^{(1)}_0(b)H''_0(a) + J^{(2)}_0(a,b)H'_0, \tag{22}$$

$$\left\langle a, J^{(1)}_0(b)c\right\rangle = \left\langle b, J'_0(a,c)\right\rangle, \tag{23}$$

and

$$\left\langle a, J^{(2)}_0(b,c)d\right\rangle = \left\langle b, J''_0(c,a,d)\right\rangle, \tag{24}$$

where again the arguments a, b, c and d are arbitrary fields. Moreover, to investigate the Manley-Rowe symmetries of the coupling coefficients, it turns out to be convenient to express the general result (13) in terms of the conjugate field ξ. Using the linear relations (9) and

$$-i\omega\xi = H_0^{''}(J_0\xi)+J_0^{'}(H_0^{'},\xi), \tag{25}$$

together with (22), (23) and (24) we write (13) as

$$V = V_1+V_2+V_3 \tag{26}$$

with

$$V_1 = \langle J_0\zeta_1, H_0^{'''}(J_0\zeta_2, J_0\zeta_3)\rangle, \tag{27}$$

$$V_2 = i\omega_2\langle J_0\xi_3, J_0^{'}(\xi_1,\xi_2)\rangle + i\omega_3\langle J_0\xi_2, J_0^{'}(\xi_1,\xi_3)\rangle \tag{28}$$

and

$$V_3 = \langle J_0\xi_2, J_0^{'}(\xi_1, J_0^{'}(H_0^{'},\xi_3))\rangle + \langle J_0\xi_3, J_0^{'}(\xi_1, J_0^{'}(H_0^{'},\xi_2))\rangle - \frac{1}{2}\langle J_0\xi_2, J_0^{''}(J_0\xi_3,\xi_1,H_0^{'})\rangle - \frac{1}{2}\langle J_0\xi_3, J_0^{''}(J_0\xi_2,\xi_1,H_0^{'})\rangle. \tag{29}$$

The symmetry of V_1 follows from the equality of mixed partial derivatives, where in this case $\langle a, H_0^{'''}(b,c)\rangle$ is unchanged by permutations of (a,b,c). Furthermore, it can be shown that the second contribution V_2 also has the required symmetry property by using the Jacobi identity, expressed in the form

$$\langle\xi_1, J_0J_0^{'}(\xi_2,\xi_3)\rangle + \langle\xi_3, J_0J_0^{'}(\xi_1,\xi_2)\rangle + \langle\xi_2, J_0J_0^{'}(\xi_3,\xi_1)\rangle = 0 \tag{30}$$

and the resonance condition (8). The last coefficient V_3 requires a little bit more manipulations, but it is straightforward to show that this also is symmetric by using the Jacobi identity and the background condition (5). Thus, we conclude that the coupling coefficient (13) satisfies the Manley-Rowe relations.

3 Evaluation of the coupling coefficient for a stratified atmosphere

In section 2 we derived a general expression for the wave coupling coefficient. Our main purpose in the present study is to present a formulation for three-wave interactions between waves in a stratified and ideal conducting atmosphere. We shall here consider a special background magnetic field distribution when both the sound speed and the Alfvén speed are constant (Thomas, 1983). In this case the magneto-acoustic-gravity waves can be described in terms of a global dispersion relation. Thus, in our stratified, isothermal atmosphere at rest, the equilibrium pressure and density are supposed to be exponentially decreasing with altitude according to

$$p_0 = p_{00}\exp(-z/H), \quad \rho_0 = \rho_{00}\exp(-z/H), \tag{31}$$

where p_{00}, ρ_{00} and H are constants. Moreover, in order for the Alfvén speed to be uniform, the background magnetic field will be given by

$$\mathbf{B}_0 = B_{0x}(z)\hat{\mathbf{x}} = B_{00}\exp(-z/2H)\hat{\mathbf{x}}, \tag{32}$$

where B_{00} is constant. The set of equations governing the dynamics of the magneto-acoustic-gravity waves in the atmosphere consists of the usual ideal MHD equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{\rho}\nabla P - \frac{1}{\rho}\mathbf{B}\times(\nabla\times\mathbf{B}) - g\hat{\mathbf{z}}, \tag{33}$$

$$\partial_t S = -\mathbf{u}\cdot\nabla S, \tag{34}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{35}$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{u}), \tag{36}$$

together with the adiabatic pressure model

$$P = S\rho^{\gamma}, \tag{37}$$

where the constant γ is the ratio of specific heats. A constant gravitational force has also been included in the momentum equation. Under the plane-wave assumption, i.e. $\rho \sim \exp(i\mathbf{k}\cdot\mathbf{r} - z/2H)$, the magneto-acoustic-gravity waves are then described by the dispersion relation (Thomas, 1983; Axelsson, 1998)

$$\begin{aligned}\omega^6 &- \left[\left(c_s^2+v_A^2\right)\left(k^2+\frac{1}{4H^2}\right)+v_A^2k_x^2\right]\omega^4 \\ &+\left[v_A^2\left(2c_s^2+v_A^2\right)k_x^2\left(k^2+\frac{1}{4H^2}\right) - g\left(g-\frac{c_s^2}{H}\right)\left(k_x^2+k_y^2\right)+k_y^2 g\frac{v_A^2}{H}\right]\omega^2 \\ &- v_A^2k_x^2\left[c_s^2v_A^2k_x^2\left(k^2+\frac{1}{4H^2}\right)-g\left(g-\frac{c_s^2}{H}\right)\left(k_x^2+k_y^2\right)\right] = 0,\end{aligned} \tag{38}$$

where $k^2 = k_x^2+k_y^2+k_z^2$ and $c_s = (\gamma P_0/\rho_0)^{1/2}$. Now it is useful to describe the linear and nonlinear dynamics in terms of the theory introduced in section 1. The Hamiltonian form of the set of equations (33)-(36) is then derived from the Poisson structure (Morrison, 1982, 1998)

$$J_u = \begin{pmatrix} -\nabla\eta + \frac{1}{\rho}\theta\nabla S + \frac{1}{\rho}\beta\times(\nabla\times\mathbf{u}) - \frac{1}{\rho}\mathbf{B}\times(\nabla\times\alpha) \\ -\frac{1}{\rho}\beta\cdot\nabla S \\ \nabla\times\left(\frac{1}{\rho}\beta\times\mathbf{B}\right) \\ -\nabla\cdot\beta \end{pmatrix}, \tag{39}$$

operating on the Hamiltonian functional

$$H(u) = \int\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho U(\rho,S) + \frac{1}{2}B^2 + \rho g z\right)d\mathbf{r}. \tag{40}$$

Here we have used the notations $u = (\mathbf{u}, S, \mathbf{B}, \rho)$, $\xi = (\beta, \theta, \alpha, \eta)$, where β, θ, α and η are standard conjugate field variables (Larsson, 1996), and denoted the inner energy by U, noting that $P = \rho^2 U_\rho$. From (39) and (40) we can then calculate explicit expressions for the operators that occur in the different terms in (26). Thus, we immediately find (from now on we will for notational simplicity suppress the space integrations)

$$V_1 = \mathrm{perm}_3\rho_1\mathbf{u}_2\cdot\mathbf{u}_3 + \mathrm{perm}_3 H'_{\rho\rho S0}\rho_1\rho_2 S_3 + H'_{\rho\rho\rho 0}\rho_1\rho_2\rho_3, \tag{41}$$

$$\begin{aligned} V_2 = {} & i\omega_2\Bigg\{\left[\frac{1}{\rho_0}\theta_3\nabla S_0 - \frac{1}{\rho_0}\mathbf{B}_0\times(\nabla\times\alpha_3)\right]\cdot\nabla\times\left(\frac{1}{\rho_0}\beta_1\times\beta_2\right) \\ & - \frac{1}{\rho_0}\beta_3\cdot\nabla S_0\left[\nabla\cdot\left(\frac{1}{\rho_0}\beta_2\theta_1\right) - \nabla\cdot\left(\frac{1}{\rho_0}\beta_1\theta_2\right)\right] \\ & + \nabla\times\left(\frac{1}{\rho_0}\beta_3\times\mathbf{B}_0\right)\cdot\left[\frac{1}{\rho_0}\beta_1\times(\nabla\times\alpha_2) - \frac{1}{\rho_0}\beta_2\times(\nabla\times\alpha_1)\right] \\ & - \nabla\cdot\beta_3\left[\frac{1}{\rho_0^2}\theta_1\beta_2\cdot\nabla S_0 - \frac{1}{\rho_0^2}\theta_2\beta_1\cdot\nabla S_0 \right. \\ & \left. + \frac{1}{\rho_0^2}(\beta_1\times\mathbf{B}_0)\cdot(\nabla\times\alpha_2) - \frac{1}{\rho_0^2}(\beta_2\times\mathbf{B}_0)\cdot(\nabla\times\alpha_1)\right]\Bigg\} \\ & + (2\leftrightarrow 3) \end{aligned} \tag{42}$$

and

$$\begin{aligned} V_3 = {} & \frac{1}{\rho_0}\beta_3\cdot\nabla S_0\nabla\cdot\left[\frac{1}{\rho_0}\beta_1\nabla\cdot\left(\frac{1}{\rho_0}H'_{S0}\beta_2\right)\right] \\ & - \nabla\times\left(\frac{\beta_3}{\rho_0}\times\mathbf{B}_0\right)\cdot\frac{1}{\rho_0}\beta_1\times\left[\nabla\times\left(\frac{1}{\rho_0}\beta_2\times(\nabla\times\mathbf{B}_0)\right)\right] \\ & + \frac{1}{\rho_0^2}\nabla\cdot\beta_3\beta_1\cdot\mathbf{B}_0\times\left[\nabla\times\left(\frac{1}{\rho_0}\beta_2\times(\nabla\times\mathbf{B}_0)\right)\right] \\ & + \frac{1}{\rho_0^2}\nabla\cdot\beta_3\beta_1\cdot\nabla S_0\nabla\cdot\left(\frac{1}{\rho_0}H'_{S0}\beta_2\right) \\ & - \frac{1}{2\rho_0}\beta_2\cdot\nabla S_0\nabla\cdot\left(\frac{1}{\rho_0^2}H'_{S0}\beta_1\nabla\cdot\beta_3\right) \\ & - \frac{1}{2\rho_0^2}\nabla\times\left(\frac{\beta_2}{\rho_0}\times\mathbf{B}_0\right)\cdot\nabla\cdot\beta_3\beta_1\times(\nabla\times\mathbf{B}_0) \\ & + \nabla\cdot\beta_2\left[\frac{1}{\rho_0^3}\nabla\cdot\beta_3\beta_1\cdot\mathbf{B}_0\times(\nabla\times\mathbf{B}_0) \right. \\ & - \frac{1}{2\rho_0^2}\nabla\times\left(\frac{1}{\rho_0}\beta_3\times\mathbf{B}_0\right)\cdot\beta_1\times(\nabla\times\mathbf{B}_0) \\ & + \frac{1}{2\rho_0^2}H'_{S0}\beta_1\cdot\nabla\left(\frac{1}{\rho_0}\beta_3\cdot\nabla S_0\right) \\ & \left. - \frac{1}{\rho_0^3}\nabla\cdot\beta_3 H'_{S0}\beta_1\cdot\nabla S_0\right] + (2\leftrightarrow 3). \end{aligned} \tag{43}$$

Here the notation perm_3 is used to represent the sum of the cyclic permuted terms, the subscripts ρ and S denote functional or partial derivatives, e.g. $H'_\rho = \delta H/\delta\rho$ and $H'_{\rho S} = \partial H'_\rho/\partial S$, and $(2\leftrightarrow 3)$ denotes the terms with index 2 and 3 exchanged. The symmetry property of the coupling coefficient terms (41), (42) and (43) has been confirmed by straightforward, although lengthy manipulations. Thus, using the background relations (31), (32), (37), partial integrations guided by previous calculations (Axelsson et al., 1996a) and the resonance condition (8) we derive the completely symmetric expression

$$\begin{aligned} V = {} & \mathrm{perm}_3\rho_1\mathbf{u}_2\cdot\mathbf{u}_3 + \mathrm{perm}_3\gamma\rho_0^{\gamma-2}\rho_1\rho_2 S_3 + \frac{(\gamma-2)c_s^2}{\rho_0^2}\rho_1\rho_2\rho_3 \\ & + \frac{i}{\rho_0}\mathrm{perm}_6\omega_1\left[\nabla\cdot\left(\frac{1}{\rho_0}\beta_3\beta_1\right)\cdot\mathbf{B}_0\times(\nabla\times\alpha_2) \right. \\ & \left. - \nabla\cdot\left(\frac{1}{\rho_0}\mathbf{B}_0\beta_2\right)\cdot\beta_3\cdot\nabla\alpha_1\right] - \frac{i}{\rho_0^2}\mathrm{perm}_3\omega_2\beta_1\beta_3 : \nabla\nabla\alpha_2\cdot\mathbf{B}_0 \\ & - \frac{1}{\rho_0^2}\mathrm{perm}_3\beta_1\cdot\nabla\mathbf{B}_0\cdot\beta_2\cdot\nabla\mathbf{B}_0\nabla\cdot\left(\frac{1}{\rho_0}\beta_3\right) \\ & + \frac{1}{\rho_0^2}\mathrm{perm}_6\beta_3\cdot\nabla\mathbf{B}_0\cdot\left[\frac{1}{\rho_0}\beta_1\cdot\nabla\mathbf{B}_0\cdot\nabla\beta_2 \right. \\ & \left. + \nabla\cdot\left(\frac{1}{\rho_0}\beta_1\right)\mathbf{B}_0\cdot\nabla\beta_2 - \mathbf{B}_0\cdot\nabla\beta_1\cdot\nabla\left(\frac{1}{\rho_0}\beta_2\right)\right] \\ & + \frac{ic_s^2(\gamma-1)\rho_0^{-(\gamma+2)}}{\gamma}\mathrm{perm}_6\omega_1\beta_3\cdot\nabla\rho_0 \\ & \left[\rho_0\beta_1\cdot\nabla\left(\frac{1}{\rho_0}\theta_2\right) - \nabla\cdot\beta_2\theta_1\right] \end{aligned}$$

$$+\frac{1}{2\rho_0^5\gamma}\left[2c_s^2(1+\gamma-2\gamma^2)+\gamma v_A^2\right]$$
$$(\beta_1\cdot\nabla\rho_0)(\beta_2\cdot\nabla\rho_0)(\beta_3\cdot\nabla\rho_0)$$
$$-\frac{1}{2\gamma\rho_0^3}\left[2c_s^2(\gamma-1)-\gamma v_A^2\right]$$
$$\mathrm{perm}_3(\beta_1\cdot\nabla\rho_0)(\beta_2\cdot\nabla\rho_0)\nabla\cdot\left(\frac{1}{\rho_0}\beta_3\right). \tag{44}$$

Here we have used perm_6 to denote the sum of the totally permuted terms, and the symbol : to represent tensor operator contractions. Using the linear relations

$$\beta_j=\frac{i\rho_0}{\omega}\mathbf{u}_j, \tag{45}$$

$$\theta_j=\frac{1}{\omega_j^2}P_{S0}\nabla\cdot\mathbf{u}_j \tag{46}$$

and

$$\alpha_j=\frac{1}{\omega_j^2}\mathbf{u}_j\times(\nabla\times\mathbf{B}_0)+\frac{1}{\omega_j}\mathbf{B}_j, \tag{47}$$

the coupling coefficient is finally written in the form

$$V=\mathrm{perm}_3\rho_1\mathbf{u}_2\cdot\mathbf{u}_3+\frac{1}{\rho_0^2}(\gamma-2)c_s^2\rho_1\rho_2\rho_3$$
$$+\frac{i\left[(\gamma-1)^2c_s^2-\gamma v_A^2\right]}{\gamma H^3\omega_1\omega_2\omega_3}\rho_0u_{z1}u_{z2}u_{z3}$$
$$+\frac{\left[2(\gamma-1)(2-\gamma)c_s^2+\gamma v_A^2\right]}{2\gamma H^2}\mathrm{perm}_3\rho_1\frac{u_{z2}}{\omega_2}\frac{u_{z3}}{\omega_3}$$
$$-\frac{i(\gamma-1)c_s^2}{\gamma H\rho_0}\mathrm{perm}_3\left(\gamma-\frac{\omega_1^2}{\omega_2\omega_3}\right)\frac{1}{\omega_1}u_{z1}\rho_2\rho_3$$
$$+\frac{(\gamma-1)c_s^2}{\gamma H}\mathrm{perm}_6\frac{1}{\omega_2\omega_3}\left(\rho_3+\frac{i\rho_0}{H\omega_2}u_{z3}\right)\mathbf{u}_1\cdot\nabla u_{z2}$$
$$+\frac{i\rho_0v_A^2}{\omega_1\omega_2\omega_3}$$
$$\left\{\frac{1}{B_0}\mathrm{perm}_3\left[\mathbf{u}_1\mathbf{u}_3:\nabla\nabla(B_0\nabla\cdot\mathbf{u}_{\perp 2})+\frac{1}{4H^2}B_0u_{z1}u_{z2}\nabla\cdot\mathbf{u}_3\right]\right.$$
$$+\mathrm{perm}_6$$
$$\left[\frac{1}{B_0}\partial_x\mathbf{u}_2\cdot\mathbf{u}_3\cdot\nabla\left(-\frac{1}{2H}B_0u_{x1}\hat{\mathbf{z}}+B_0\nabla\cdot\mathbf{u}_1\hat{\mathbf{x}}-B_0\partial_x\mathbf{u}_1\right)\right.$$
$$+\frac{\omega_1}{\omega_2\rho_0}\nabla\cdot(\rho_0\mathbf{u}_3\mathbf{u}_1)\cdot\left(-\frac{1}{2H}\partial_xu_{x2}\hat{\mathbf{z}}+\partial_x\nabla\cdot\mathbf{u}_2\hat{\mathbf{x}}-\partial_x^2\mathbf{u}_2\right)$$
$$-\frac{\omega_1}{\omega_2\rho_0B_0}\nabla\cdot(\rho_0\mathbf{u}_3\mathbf{u}_1)\cdot\nabla(B_0\nabla\cdot\mathbf{u}_{\perp 2})$$
$$\left.\left.-\frac{1}{2H}u_{z3}\hat{\mathbf{x}}\cdot\left(\partial_x\mathbf{u}_1\cdot\nabla\mathbf{u}_2+\frac{1}{2H}u_{z1}\partial_x\mathbf{u}_2-\partial_x\mathbf{u}_2\nabla\cdot\mathbf{u}_1\right)\right]\right\} \tag{48}$$

With further use of the linear relations, the wave activities (12), related to the wave energies, are written as

$$W_j=2\int\rho_0\mathbf{u}_j\cdot\overline{\mathbf{u}}_j d\mathbf{r}. \tag{49}$$

Thus there is equipartition between the kinetic- and poten-tial energies of the waves.

4 Conclusions

In this paper we have derived a coupling coefficient for waves in a stratified and magnetized ideal atmosphere. This is a significant generalization of the theory for three-wave interactions between acoustic-gravity waves. Thus, in the limit of vanishing magnetic field, the result reduces to eq. (9) in Axelsson et al. (1996a). Another interesting limit is the case with wave coupling of MHD waves in a homogeneous plasma. Thus, dropping all derivatives of the unperturbed density and magnetic field, we reconfirm the results in the paper by Brodin and Stenflo (1988). For example, using the linear relations for the usual magnetosonic wave, the coupling coefficient reduces to (21) in Brodin and Stenflo (1988) for interactions between three magnetosonic waves.

In section 3 we considered a particular background state where the Alfvén velocity was constant. For more general background atmospheres it is still straightforward to find an explicit expression for the wave coupling coefficient, because the equations (41)-(43) are valid also in the general case and we may use the linear relations (45)-(47) to obtain the coupling coefficient. We know from general theory (as discussed in section 2) that the resulting coupling coefficient is symmetric. However, the explicit symmetric form is presented only for the above mentioned special case.

In presence of strong magnetic fields and when the collision rates are sufficiently small, the pressures can be significantly different parallel and perpendicular to the magnetic field direction. This occurs for instance in the subsolar magnetosheath plasma (Hau et al., 1993). It is then necessary to replace the isotropic pressure model with the Chew-Goldberger-Low (CGL) pressure model or the generalised CGL model used by Duhau and Gratton (1975). It is straightforward to write down explicit expressions for the coupling coefficient for wave interactions also for this anisotropic and inhomogeneous atmosphere, thus generalising the work of Brodin and Stenflo (1989). This is possible because the MHD equations including the CGL pressure model is within the class of Hamiltonian models (Larsson, 1996).

Acknowledgements. The author thanks Prof. Lennart Stenflo and Dr. Jonas Larsson for valuable comments.

References

Axelsson, P., Larsson, J., and Stenflo, L., Nonlinear interaction between acoustic gravity waves, *Ann. Geophysicae, 14*, 304-308, 1996a

Axelsson, P., Larsson, J., and Stenflo, L., Nonlinear interaction between acoustic gravity waves in an rotating atmosphere, *Nonlin. Proc. in Geophysics, 3*, 216-220, 1996b

Axelsson, P., Dispersion relations for waves in anisotropic and stratified magnetoplasmas, *Physica Scripta, 57*, 242-245, 1998

Brodin, G. and Stenflo, L., Three-wave coupling coefficients for MHD plasmas, *J. Plasma Phys., 39*, 277-284, 1988

Brodin, G. and Stenflo, L., Three-wave coupling coefficients for magnetized plasmas with pressure anisotropy, *J. Plasma Phys., 41*, 199-208, 1989

Dong, B. and Yeh, K. C., Resonant and nonresonant wave-wave interactions in an isothermal atmosphere, *J. Geophys. Res., 93*, 3729-3744, 1988

Dong, B. and Yeh, K. C., On resonant interactions of atmospheric waves in a rotating Earth, *Physica Scripta, 43*, 534-544, 1991

Duhau, S. and Gratton, J., Kelvin-Helmholtz instability of anisotropic plasma in a magnetic field, *J. Plasma Phys., 13*, 451-479, 1975

Gryanik, V. M. and Dobritsyn, A. A., Localized vortices with finite cores in the equatorial atmosphere: structure and dynamics, *Izv. Atmos. Ocean Phys., 26*, 260-266, 1990

Hau, L.-N., Phan, T.-D., Sonnerup, B. U. Ö., and Paschmann, G., Double-polytropic closure in the magnetosheath, *Geophys. Res. Lett., 20*, 2255-2258, 1993

Larsson, J., A new Hamiltonian formulation for fluids and plasmas. part 2. MHD models, *J. Plasma Phys., 55*, 261-278, 1996

Larsson, J., The Hamiltonian adjoint equation and perturbation theory, *Physica Scripta, 58*, 97-101, 1998a

Larsson, J., Hamiltonian theory for waves and the resonant three-wave interaction process, *Physica Scripta, T75*, 173-175, 1998b

Lindgren, T., Wave coupling in nonuniform plasmas, *Physica Scripta, 25*, 568-572, 1982

Morrison, P. J. and Green, J. M., Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics, *Phys. Rev. Lett., 45*, 790-794, 1980

Morrison, P. J., Mathematical methods in hydrodynamics and integrability in dynamical systems, *AIP conference proceedings*, 1982

Morrison, P. J. and Pfirsch, D., Dielectric energy versus plasma energy, and Hamiltonian action-angle variables for the Vlasov equation, *Phys. Fluids B, 4*, 3038-3057, 1992

Morrison, P. J., Hamiltonian description of the ideal fluid, *Rev. Mod. Phys., 70*, 467-521, 1998

Olver, P. J., *Applications of lie groups to differential equations*, Springer-Verlag, 1993

Pavlenko, V. P. and Stenflo, L., The acoustic-gravity vortex turbulence spectrum, *Europhys. Lett., 20*, 693-695, 1992

Shukla, P. K. and Shaikh, A. A., Dust-acoustic gravity vortices in a nonuniform dusty atmosphere, *Physica Scripta, T-75*, In press, 1998

Shukla, P. K. and Stenflo, L., Comment on "two dimensional solitary kinetic Alfvén waves and dipole vortex structures" [Phys. Plasmas 4, 611 (1997)], *Phys. Plasmas, 4*, 3445-3446, 1997

Stenflo, L., Nonlinear equations for acoustic gravity waves, *Physica Scripta, 33*, 156-158, 1986

Stenflo, L., A nonlinear equation for low-frequency acoustic gravity waves, *Physica Scripta, 35*, 488-489, 1987a

Stenflo, L., Acoustic solitary vortices, *Phys. Fluids, 30*, 3297-3299, 1987b

Stenflo, L., Resonant three-wave interactions in plasmas, *Physica Scripta, T50*, 15-19, 1994

Stepanyants, Y. A., Cylindrical eddies with a horizontal axis in the equatorial atmosphere, *Izv. Atmos. Ocean. Phys., 25*, 729-731, 1989

Stepanyants, Y. A., Cylindrical vortices in an unstably stratified fluid, *Soviet Phys. Dokl., 36*, 127-129, 1991

Thomas, J. H., Magneto-atmospheric waves, *Ann. Rev. Fluid Mech., 15*, 321-343, 1983

Yeh, K. C. and Liu, C. H., The instability of atmospheric gravity waves through wave-wave interactions, *J. Geophys. Res., 86*, 9722-9728, 1981

Zhou, C. T., Lai, C. H., and Yu, M. Y., Chaos, bifurcations and periodic orbits of the Lorenz-Stenflo system, *Physica Scripta, 55*, 394-402, 1997a

Zhou, C. T., Lai, C. H., and Yu, M. Y., Bifurcation behaviour of the generalized Lorenz equations at large rotation numbers, *J. Math. Phys., 38*, 5225-5239, 1997b