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Percolation on anisotropic media, the Bethe lattice revisited. Application to fracture networks

M. Canals and M. Ayt Ougoudal

CREGU, BP23, 54501 Vandoeuvre-lès-Nancy cedex, France

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Abstract. A bond-percolation model based on the Bethe Lattice is presented. This model handles anisotropic and multiscale situations where, typically, the bond probability is non unique and depends on the sites it connects. The model is governed by a set of non-linear equations which are solved numerically. As a result, the structure of the network is obtained : strenghts of the backbone, dead-end-roads and finite clusters. Percolation thresholds and cluster sizes are also obtained. Application to fissured media is presented and random simulations of 3D distributions of fractures show the good accuracy of the model.

1 Introduction

Percolation theory has been extensively used with success in a wide range of sciences for 25 years (Stauffer, 1985). In most of the problems adressed, only one parameter is used: the bond probability or the site occupancy probability. Percolation theory deals with the size of the clusters when this parameter varies.

In geophysics, conductive properties of rock materials are studied using such method (Gueguen and Dienes, 1989). Permeability of a rock matrix is closely related to the fraction of pores that merely participate to the flow path. The study of the way the pores are interconnected when their density and size vary is typically a percolation problem. As rock materials exhibit anisotropy due to their genesis (sedimentation) or geological story (stresses, tectonics), percolation models must take into account this characteristic. Let us consider the fractured pattern in Fig.1. Two sets of cracks could be distinguished: a set of small dense cracks, and a set of large sparse cracks, each showing a preferential orientation. A model must distinguish two types of sites, S (for small) and L (for large), and four probabilities P_{SS}, P_{SL}, P_{LS} and P_{LL} which are respectively the probabilities for a small (large) crack to intersect a small (large) crack. The number of parameters increases quadratically when other sets of cracks are appended.

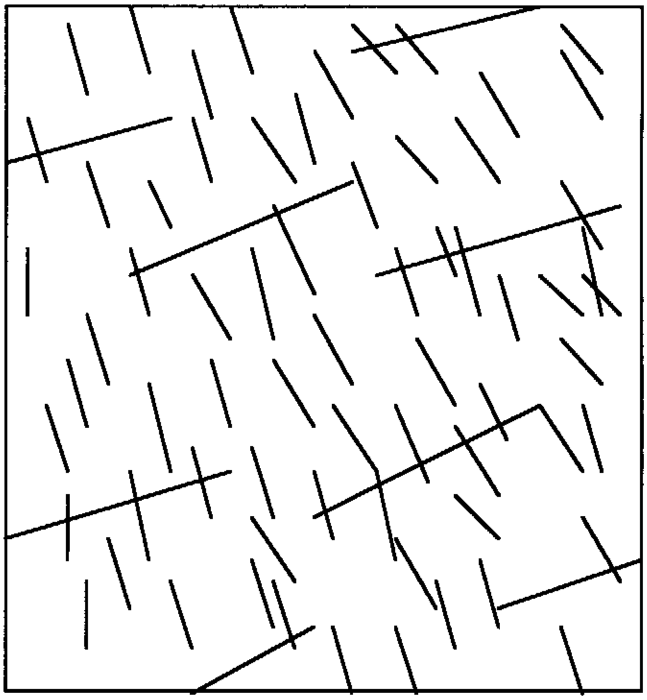

Fig.1. Fractured pattern showing 2 anisotropic sets.(synthetic data)

Blanc et al (1980), considered a square lattice with different vertical and horizontal bond probabilities. Turban (1979) studied the Bethe lattice with a set of bond probabilities but without discrimining the sites.

More recently, Bourget (1990), proposed the Bethe lattice to study percolation on three dimensional anisotropic crack networks but some inconsistencies appear when the crack density increases.

In this paper we shall present a modified Bethe lattice model, handling as many site-types as imposed by the

Correspondance to: M. Canals

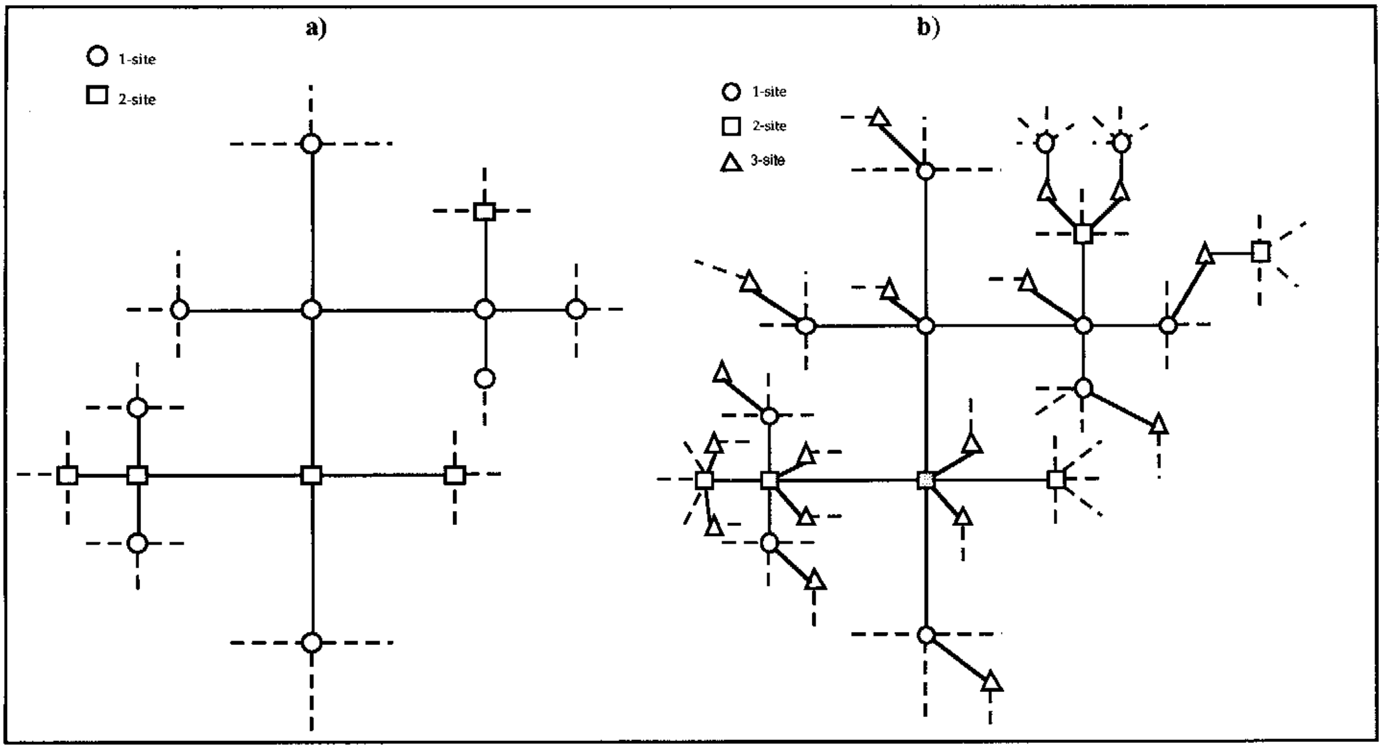

Fig.2. Enhanced Bethe lattice (a) with N=2, z_{11}=3,z_{12}= 1, z_{21}=2,z_{22}=2. (b) with an additionnal family, z_{33}=0, z_{31}=1, z_{32}=1,z_{13}=1 and z_{23}=2.

problem. The outline of the paper is as follows: in Sect.2 we present the lattice and the bond probabilities. In Sect.3 equations governing the model are set and solved numerically. For each site-type three probabilities are calculated. The percolation probability, f_i^p, is the probability for a site from the ith family to belong to the infinite cluster. The flow throught probability, f_i^f, is the probability for a site to be connected by at least two paths to the infinite cluster. The isolation probability, f_i^i, represents sites which belong to finite clusters and we have f_i^i=1-f_i^p. In Sect.4 we show how to apply the model to percolation in fractured media. A sample case is studied with details and results are compared to Monte Carlo simulations.

2 Enhanced Bethe lattice

This lattice is build with the same process than for classical Bethe Lattice (Stauffer, 1985), except that sites are classified by type, and the bond probability depends on the type of the sites.

Let N be the number of site families in the network. A site from the ith familiy is called an i-site. Starting from an i-site as the origin, it has z_{ij} bonds ending in a j-site (j from 1 to N). In turn, each of these j-neighbours has z_{jk} bonds ending in a k-site. z_{ij} is called the ij-coordination of the lattice. This branching process is continued again and again with two simple rules: there are no closed loops: neighbours are always new sites, and z_{ij} is constant in the entire lattice for given i and j. This situation is illustrated in Fig.2a. Figure 2b defines the terminology used. Once the model is built, it can easily accept a new family: new branches are appended to each site (Fig.2b).

In order to define bond probabilities p_{ij} we state that the probablity for no connection between an i-site and a j-site, q_{ij}, may be equal in both the real and the model networks. At this step, we suppose that q_{ij} is well known in the real network (q_{ij} is an input data). In Appendix A, we shall derive q_{ij} for Poissonian distribution of fractures. In the model network an i-site is not connected to a j-site if the z_{ij} links are broken, and this occurs with probability $(1-p_{ij})^{zij}$. We obtain the following relation between p_{ij} and q_{ij}:

$$(1-p_{ij})^{zij}= q_{ij}. \tag{2.1.1}$$

We denote by [i,j[a path starting from an i-site and in which the second site is of j type (Fig.3)

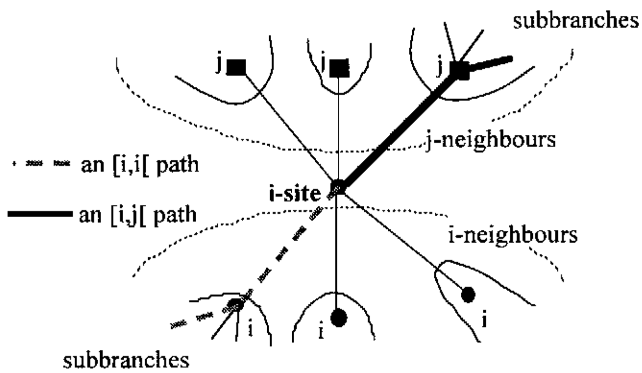

Fig.3. Definition of j-neighbour and [i,j[path.

3 Exact solution for the enhanced Bethe lattice

3.1 Finite paths

For an [i,j[path, we calculate R_{ij}, the probability for this path to be of a finite size. This occurs in the following cases:

- i and j are not connected, this event has the probability: $1-p_{ij}$,
- i and j are connected (p_{ij}) and all the paths starting from the j-site are of finite size. There are $z_{ji}-1$ [j,i[paths and z_{jk} [j,k[paths (k≠i), each with respectively the probability R_{ji} and R_{jk} to be finite. Then, the probability that this fixed [i,j[branch does not lead to infinity is:

$$R_{ij} = \left(1 - p_{ij}\right) + p_{ij} R_{ji}^{z_{ji}-1} \prod_{k \neq i} R_{jk}^{z_{jk}} \qquad (3.1.1)$$

Writing Eq.(3.1.1) for all i and j, we obtain a non-linear system of N^2 equations. The solution of this system is generally non unique. As R_{ij} are probabilities we retain only solutions for which $0 \le R_{ij} \le 1$. The system is solved numerically by a Newton Raphson method with initial values set to 0. More details are presented in Appendix B.

All branches are finite and no percolation occurs when the only solution is $R_{ij}=1$ for all (i,j).

3.2 Finite clusters, dead ends and backbone

In order to study transport and chemical processes in fractured media, Norton and Knapp (1977) introduced three types of porosity: flow porosity, diffusive porosity and residual porosity. Flow porosity represents all pores throught which fluid flow occurs. In these pores the major transport mode is by advection. Diffusive porosity includes all pores for which chemical transport is governed by diffusion effect. Residual porosity consists of pores wich are not connected to the infinite cluster.

These three types of porosity are related to the structure of the crack network. Residual porosity is given by the finite clusters whose sites have only finite paths. As the paths are independent events, the probability for a site to belong to a finite cluster is obtained by the product

$$f_i^i = \prod_{j=1}^{N} R_{ij}^{z_{ij}} \qquad (3.2.1)$$

The sites corresponding to the diffusive porosity are connected to the infinite cluster by only one path and are also called dead-ends. Assuming the infinite path to be an [i,j[one, there are z_{ij} possibilities to choose it, and the $z_{ij}-1$ others as well as the [i,k[ones must be of a finite size. Calculus gives us

$$f_i^d = \sum_{j=1}^{N} z_{ij}(1 - R_{ij}) R_{ij}^{z_{ij}-1} \prod_{k \neq j} R_{ik}^{z_{ik}} \qquad (3.2.2)$$

Finally, the sites which do not belong to these categories pertain to flow porosity with the probability

$$f_i^f = 1 - f_i^d - f_i^i \qquad (3.2.3)$$

3.3 Mean cluster size.

The mean cluster size is calculated according to the method described by Stauffer [1985]. Starting from an i-site, we first calculate the mean size of an [i,j[branch, T_{ij}, which is the average number of sites belonging to this branch. If the first bond is broken (probability $1-p_{ij}$), the branch have a null size. In the opposite case (probability p_{ij}) the size is one plus the mean sizes of each subbranches. There are z_{jk} [j,k[sub-branches for k≠ i, and there are $z_{ji}-1$ [j,i[subbranches for k=i. We obtain

$$T_{ij} = p_{ij}\left(1 + \sum_{k=1}^{N} (z_{jk} - \delta_{ki}) T_{jk}\right) (i,j = 1,N) \qquad (3.3.1)$$

Which we rewrite

$$\sum_{k=1}^{N} p_{ij} z_{jk} T_{jk} - p_{ij} T_{ji} - T_{ij} = -p_{ij} \ (i,j = 1,N) \qquad (3.3.2)$$

As, we have obtained a set of N^2 linear equations involving N^2 unknowns T_{ij}. This system could be formally solved for small N but writing the solution will take more than one page of equations. A Gauss method is very well adapted to solve numerically this system.

An infinite solution significates that system is at the percolation threshold. At this point the system is non inversible and the determinant of the matrix is zero. This gives the critical relation between the probabilities.

A negative solution significates that we are above the percolation threshold.

To obtain the mean size T_i of an i originated cluster we add the mean size of each branch to the size of the origin. Thus

$$T_i = 1 + \sum_{j=1}^{N} z_{ij} T_{ij} \qquad (3.3.3)$$

4 Application to anisotropic fissured media

4.1 The model of anisotropic fissured media

Many transport properties (permeability, diffusivity) of crystalline rocks originates from fractures which exhibits preferential orientations. Fluid flow occurs in cracks networks in which we can distinguish a functionnal hierarchy. As described above, part of the cracks allows transport by advection, others relate to diffusionnal transport while some are isolated . Transport properties are controlled by the geometry of the fractures and the connectivity between them (Gueguen and Dienes, 1989). The question

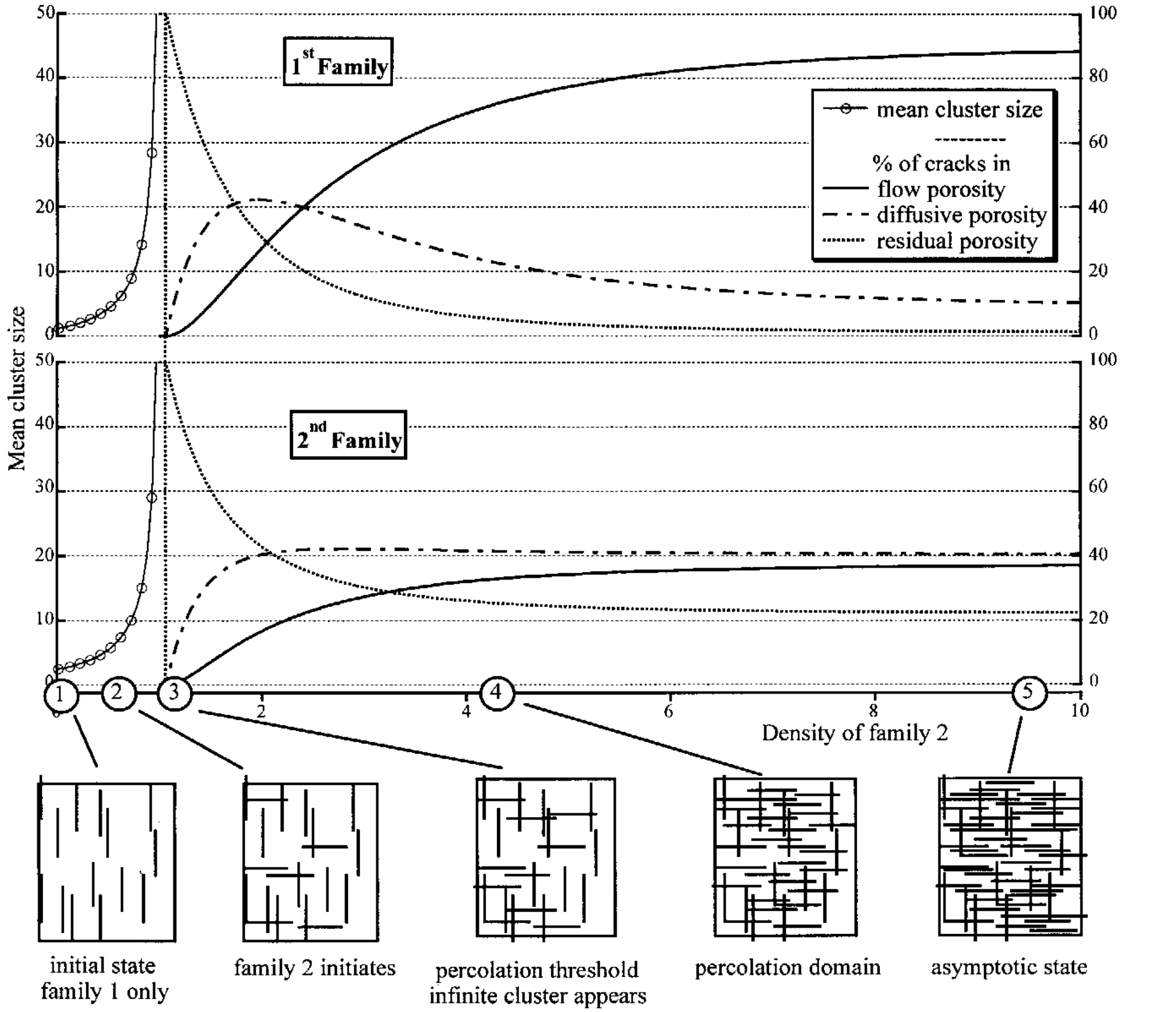

Fig.4. Study of 2 perpendicular families: contribution of both families to each type of porosity as a fraction density (rigth axis) and mean cluster size (left axis). Computation conditions: $D_1=1$, $C_1=C_2=0.5$, $D_2=0$ to 10.

we adress in this section is how to obtain connectivity from structural parameters (density, size and orientation).

As a first approach fractures can be considered as planar disks randomly distributed in space (Long, 1991). When considering their geometrical caracteristics, fractures are splitted in N different sets. Each set is caracterized by the triplet (D_i, C_i, n_i) where D_i is the volumetric density of set i, C_i and n_i the distribution of crack radius and crack normal vector in the i^{th} set. Ayt Ougougdal (1994) describes a procedure to derive the structural parameters of microfracturation from image analysis of rock thin sections.

Assuming Poissonian processes for space-distribution of crack centers, the probability q_{ij} for a crack from ith family to not intersect a crack from jth family is

$$q_{ij} = \exp(-N_{ij}) \qquad (4.1.1)$$

N_{ij} is the average number of j cracks that an i crack intersects.

One way to obtain N_{ij} is to introduce the notion of excluded volume (De Gennes, 1976), V_{ij},. which is the volume of the space around an i-crack in which the center of a j-crack must belong in order to intersect it (see Appendix A). V_{ij} depends on the sizes, frame and orientations of cracks i and j. Then, N_{ij} is the product of the average excluded volume by the density of the j family.

$$N_{ij}=D_j\, V_{ij}. \qquad (4.1.2)$$

Applying Eq.(2.1.1) we obtain the bond probability on the Bethe lattice

$$p_{ij} = 1 - \exp\left(-\frac{D_j V_{ij}}{z_{ij}}\right) \quad (4.1.3)$$

If Dj goes to infinity, pij goes to 1. If Dj goes to 0 pij goes to 0.

4.2 Case study

We consider a simple case of 2 perpendicular crack families defined by the set of 4 parameters (D_1,C_1,D_2,C_2).

Calculation of excluded volume gives (see Appendix A)

$$V_{11}=V_{22}=0$$

$$V_{12} = V_{21} = V = 2\pi C_1 C_2 (C_1 + C_2) \quad (4.2.1)$$

Applying Eq.(4.2.1) we obtain the bond probabilities:

$$p_{11}=p_{22}=0$$

$$p_{12} = 1 - exp\left(-\frac{V.D_2}{z_1}\right) = p_1 \qquad p_{21} = 1 - exp\left(-\frac{V.D_1}{z_2}\right) = p_2 \quad (4.2.2)$$

The consequence of $p_{11}=p_{22}=0$ is that $z_{11}=z_{22}=0$ and $R_{11}=R_{22}=1$. For simplicity we have replaced indexes $_{12}$ and $_{21}$ by $_1$ and $_2$.

The non linear system Eq.(3.1.1) rewrites

$$\begin{cases} R_1 = 1 - p_1 + p_1 R_2^{z_2-1} \\ R_2 = 1 - p_2 + p_2 R_1^{z_1-1} \end{cases} \quad (4.2.3)$$

The linear system governing the mean cluster size rewrites:

$$\begin{bmatrix} p_2(z_1-1) & -1 \\ -1 & p_1(z_2-1) \end{bmatrix}\begin{bmatrix} T_{12} \\ T_{21} \end{bmatrix} = \begin{bmatrix} -p_1 \\ -p_2 \end{bmatrix} \quad (4.2.4)$$

whose solutions are

$$T_{12} = \frac{p_1 + p_1 p_2 (z_2-1)}{1 - p_1 p_2 (z_1-1)(z_2-1)} \qquad T_{21} = \frac{p_2 + p_1 p_2 (z_1-1)}{1 - p_1 p_2 (z_1-1)(z_2-1)} \quad (4.2.5)$$

Under percolation threshold T_{12} and T_{21} must be positive and finite. So we obtain the critical condition for percolation :

$$p_1 p_2 > (p_1 p_2)_c = \frac{1}{(z_1-1)(z_2-1)} \quad (4.2.6)$$

which we may compare to the percolation threshold in the classical Bethe lattice:

$$p_c = \frac{1}{z-1} \quad (4.2.7)$$

The model is ran with the following settings: an initial family is set (D_1 =1, C_1 =2) and then a second one appears with increasing density (C_2 =0.5, D_2 varies from 0 to 10). At each step Eq.(4.2.4) is solved and T_i, f_i^i,f_i^f and f_i^d are calculated (Fig.4). We have pointed up five points:

1) Initial state. Family 2 is empty and no percolation occurs. The mean cluster size is 1.

2) Family 2 initiates. Mean cluster size is growing but insufficiently to create an infinite cluster. All cracks belong to the non connected porosity.

3) Percolation threshold is reached. Mean cluster size goes to infinity with the formation of an infinite cluster. The infinite cluster is dominated by dead-end-roads (diffusive porosity).

4) The infinite cluster is growing, aggregating finite clusters and creating new flow paths. After a maximum, the fraction of sites belonging to dead end roads decreases.

5) Asymptotic state. When D_2 is very high, connection attains its maximum. It must be noted that a fraction of cracks remains isolated: the cracks which are too far from others will never be connected (Fig.5).

The value of $\mathcal{D}_2$ = $D_2C_1^3$ (*italic* is used for non dimensional parameters) for which the percolation threshold is attained depends on the two adimensional parameters $\mathcal{D}_1=D_1C_1^3$ and $\mathcal{C}_2=C_2/C_1$. This value is defined by combining Eq.(4.2.5) and Eq.(4.2.2) :

$$\left(1-\exp\left(-\frac{2\pi \mathcal{C}_2(1+\mathcal{C}_2)}{z_1}\mathcal{D}_2\right)\right) x \left(1-\exp\left(-\frac{2\pi \mathcal{C}_2(1+\mathcal{C}_2)}{z_2}\mathcal{D}_1\right)\right) = \frac{1}{(z_1-1)(z_2-1)} \quad (4.2.8)$$

Figure 6 explorates this relation for $\mathcal{D}_1$ taking the values 10^{-3},10^{-1}, 1, and 5. The critical value of $\mathcal{D}_2$ is plotted versus $\mathcal{C}_2$. For $\mathcal{C}_2$ under $\mathcal{C}_{2crit}$, there is no solution to Eq.(4.2.6).

Effectively, each term of the product in Eq.(4.2.6) is smaller than 1, a necessary (but not sufficient) condition for percolation is

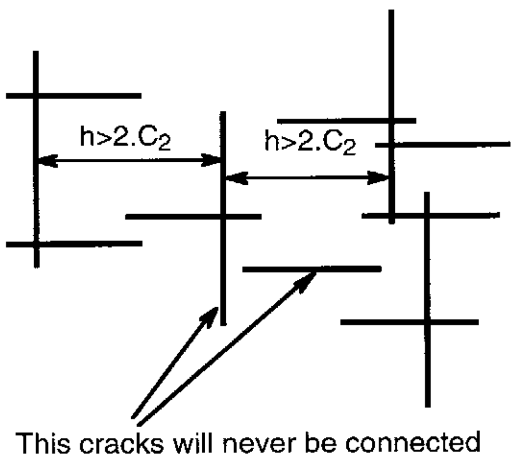

Fig.5. Study of 2 perpendicular families: diagram explaining why some craks of the 1st family remain isolated.

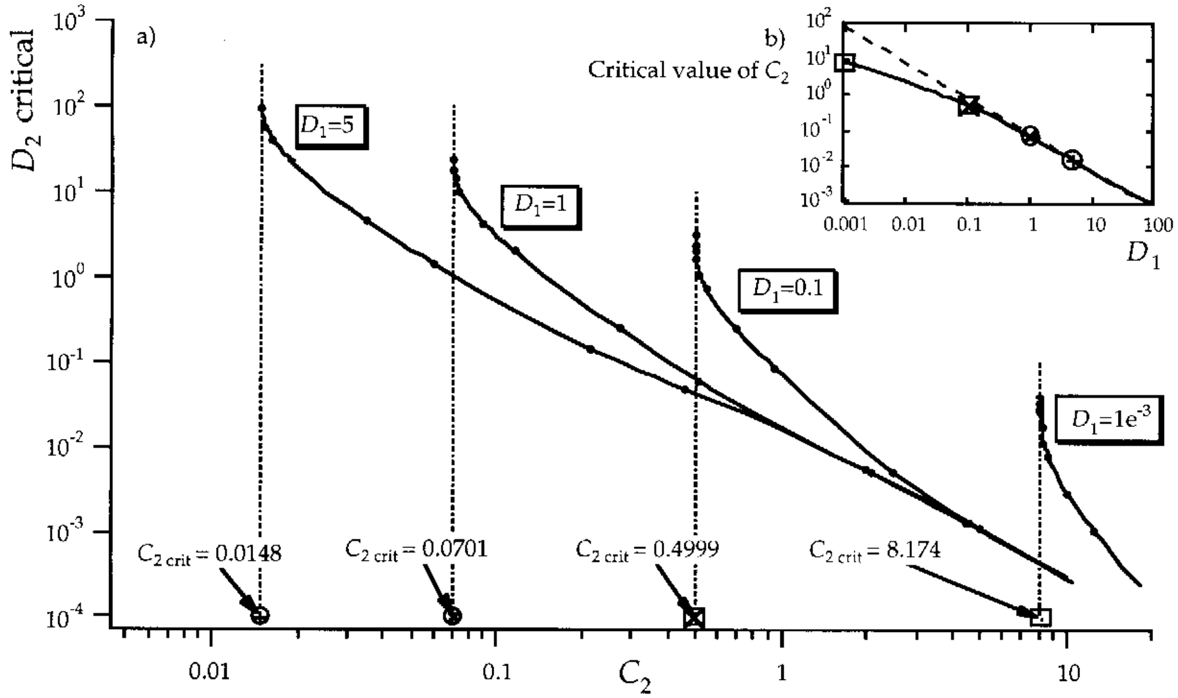

Fig.6. Critical curves for a network of two perpendicular families. C_1 = 1 and D_1 lies from 1e^{-3} to 5. Small graph: critical radius of second set as a function of D_1, comparison with mean distance between cracks in family 1.

$$1-\exp\left(-\frac{2\pi C_2(1+C_2)}{z_2}D_1\right)>\frac{1}{(z_1-1)(z_2-1)} \qquad (4.2.9)$$

This expresses that the crack size of second family must be sufficiently large to connect the cracks of the first family. C_{2crit} depends on the relative density (D_l) of the first family, not on the density of the second. Figure 6b traces C_{2crit} versus D_1. C_{2crit} has the same asymptotic behaviour than a particular measure of distance between cracks in family 1: the average bi-closeness (Appendix A2).

$$C_{2crit} \sim \frac{1}{4\pi D_1} \qquad (4.2.10)$$

4.3 Monte Carlo simulations

In order to test the accuracy of the model, Monte Carlo simulations have been conducted on the previous case. Large numbers of cracks are generated in a finite cubic volume. This volume depends on the desired density. As described above, each crack is a disk and is caracterized by the coordinates of its center (randomly set), its radius and normal. So it is easy to calculate the intersection between two cracks. Cluster counting is made by scanning the whole set of cracks and giving the same cluster label to cracks which intersect. A cluster joining two opposite sides of the cube is considered infinite. The strength of the percolating cluster is the number of cracks belonging to an infinite cluster divided by the total number of cracks.

We set D_1=1 and C_1=0.5 for all numerical experiences. We study the percentage of each family belonging to the infinite cluster when C_2 is increasing and for various values of D_2 (D_2=0.5,1 ,5).

Each run is made on the same set of 10000 cracks. Only size and orientation varies according to D_2 and C_2. We set the coordination to 4 in each site for Bethe lattice calculations.

Figure 7 shows the result of Monte carlo simulations (markers only) compared to Bethe lattice results (line+markers). Behaviors are very similar and thresholds are closed, smallest for Bethe lattice due to the approximation of no closed loops. In this case, Bethe lattice is a good approximation of the three dimensionnal network.

5 Comparison with previous works

Robinson (1983) studied connectivity of two-dimensionnal fracture systems by a Monte Carlo method. In his work, lines of specified length and orientation distributions are uniformly generated in a square domain. Percolation occurs when there is a path joining two opposite sides of the square. The important parameter is (line density) x (length scale)^2 called N. The length scale used is a half of the average line length l_{av}.

For two perpendicular sets of same density d/2 and same average length lav, he found a critical value $Nc \approx 1.62$.

Application of the model described in this paper to the same case gives the excluded volume $V = l_{av}^2$. Then, the average number of intersection is

$$N = \frac{d}{2} l_{av}^2 = 2N$$

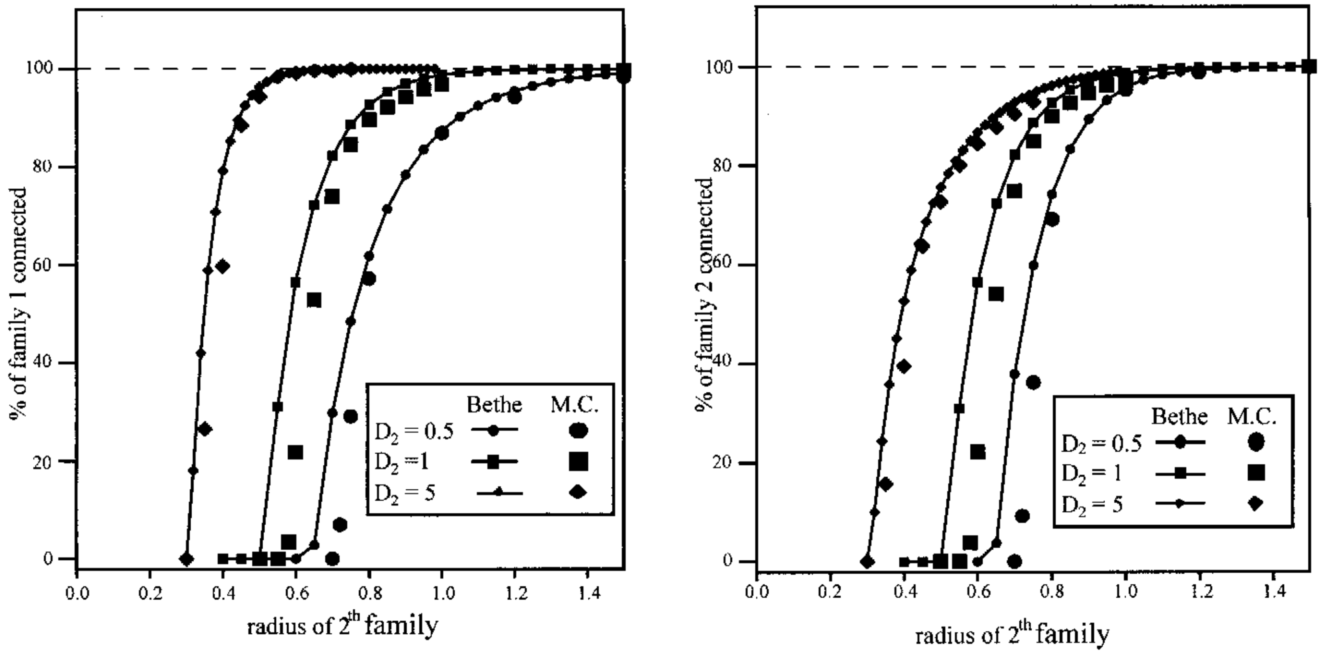

Fig.7. Comparison of enhanced Bethe lattice results and Monte Carlo simulations for 3D fracture networks. Study of 2 perpendicular families: (a) percentage of first family belonging to the infinite cluster (b) same for second family. Computational conditions: $D_1 = 1$; $C_1=0.5$; D_2 takes the values 0.5, 1, 5; C_2 varies from 0 to 1.5.

and the bond probability in the Bethe lattice with z=4 is

$$p = 1 - e^{\frac{-N}{2}}.$$

Setting $p_c = \frac{1}{3}$ we obtain $Nc \approx 0.81$.

This value is too low and we can conclude that Bethe lattice is irrelevent for 2D studies, especially in that case where the hypothesis of no closed loops is too strong.

6 Conclusion

The enhanced Bethe lattice model is a reliable tool to study anisotropic problems. It handles as many site types as imposed by the problem in a common way. It is easily and quickly solved by simple numerical analysis. The model has been applied to fracture networks. In this case, the bond probability could be determined from the structural parameters. This permit to establish a percolation criterion involving the structural parameters. For two perpendicular set of cracks, the minimal crack size of one set varies as the average bi-closeness of the other set. Comparison with Monte Carlo simulation shows the good accuracy and behaviour of the model for 3D fracture networks. In the 2D case, the model appears less accurate. This is due to the strength of the no closed loop hypothesis wich is no more realistic in this case.

Appendix A

A1 Derivation of excluded volume

Excluded volume between two objects is the measure of the region of space where their barycenters must belong to allow intersection. It depends on the sizes and orientations of the two objects. Considering a crack as a finite disk defined by its center, radius C and normal to the plan n, the excluded volume of 2 cracks (C_1,n_1), (C_2,n_2) is

$$v_{12} = 2\pi\, C_1C_2\,(C_1+C_2)\,\sin(n_i, n_j) \qquad (A1.1)$$

By averaging over all cracks in the sets i and j we obtain the exluded volume of two families:

$$V_{ij} = 2\pi\left[C_iC_j(C_i + C_j) + C_i\sigma_j^2 + C_j\sigma_i^2\right]\int_0^{\frac{\pi}{2}} \sin\theta\, f(\theta)d\theta \qquad (A1.2)$$

C_i,σ_i,C_j and σ_j are respectively the mean and standard deviation of radius for families i and j. θ is the angle between n_i and n_j and f its probability density function. Two cases are investigated for f:

1) one of the families is isotropic, θ is equally distributed on the hemisphere and has density $f(\theta)=\sin(\theta)$. The integral term becomes:

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi}{4}$$

and the excluded volume is:

$$V_{ij} = \left[C_iC_j(C_i + C_j) + C_i\sigma_j^2 + C_j\sigma_i^2\right]\frac{\pi^2}{2} \qquad (A1.3)$$

2) The two families are anisotropic, in each case the normal takes a single value. The excluded volume is:

$$V_{ij} = 2\pi\left[C_iC_j(C_i + C_j) + C_i\sigma_j^2 + C_j\sigma_i^2\right]\sin(n_1, n_2) \quad (A1.4)$$

A2 Derivation of average bi-closeness

We say that 2 cracks are close when a perpendicular straight line can join them (Fig.A1). Cracks close to A are in a virtual tube of radius 2C (ligth grey). They can be ordered according to their distance to A. We define the bi-closeness, L, as the distance to the second crack close to A. The bi-closeness is a measure of the proximity between the cracks. Considering a Poissonian distribution of cracks with density D, the probability that L is smaller than ℓ is

$$p(L < \ell) = 1 - \exp\left[-8\pi DC^2\ell\right](1 + 8\pi DC^2\ell) \quad (A2.1)$$

and the mean value of L is:

$$\bar{L} = \frac{1}{4\pi DC^2} \quad (A2.2)$$

Dividing by C to obtain the nondimentional parameter

$$\bar{L} = \frac{1}{4\pi DC^3} = \frac{1}{4\pi D} \quad (A2.3)$$

Appendix B

B1 Numerical solution of the system

The system is rewrited

F (**R**) = 0

with

F = $(F_{ij})_{i,j=1,N}$ and **R** = $(R_{kl})_{k,l=1,N}$

The Newton Raphson algorithm writes

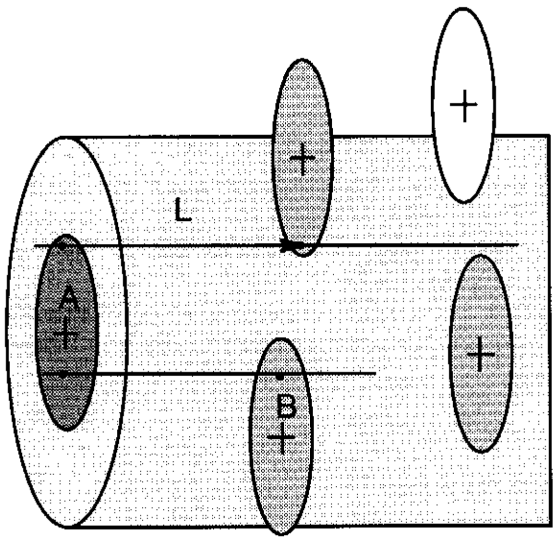

Fig.A1. Definition of bi-closeness

1) R0 = 0

2) r = |F(Rn)| if r < e then stop

3)Rn+1 = Rn - $\frac{\partial F}{\partial R}$ F(Rn)

4) go to step 2

Step 3 involves the evaluation of the Jacobian matrix of the system:

$$\frac{\partial F_{ij}}{\partial R_{mn}} = -\delta_{im}\delta_{jn} + \delta_{jm}\, p_{im} \times$$
$$\left(\delta_{in}(z_{mn} - 1)R_{mn}^{z_{mn}-2}\prod_{l\neq n} R_{ml}^{z_{ml}} + (1 - \delta_{in})z_{mn}R_{mn}^{z_{mn}-1}R_{mi}^{z_{mi}-1}\prod_{l\neq i,\,n} R_{ml}^{z_{ml}}\right) \quad (B1.1)$$

where δ is the Kronecker symbol $\delta_{ij}=1$ if i=j, 0 elsewhere.

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References

Ayt Ougougdal, M., Controles magmatiques, structuraux et hydrothermaux de la formation des épisyénites de la Marche Occidentale, *INPL Thesis*, 245p, 1994.

Blanc, R., Mitescu, C.D., and Thévenot, G., Percolation anisotrope: conductivité d'un réseau carré de liens aléatoires, *J. Physique*, 41, 387-391, 1980.

Bourget, E., *Rapport de DEA*, EOPG Strasbourg, 1989.

De Gennes, P.G., *The physics of liquid crystals*, Oxford University Press, 1976.

Gueguen, Y. and Dienes, J., Transport properties of rocks from statistics and percolation, *Mathematical Geology, 21*, 1-13, 1989.

Long, J.C.S., Hestir, K., Karasaki, K., Davey, A., Peterson, J., Kemen, J. and Landsfeld, M., Fluid flow in fractured rock: theory and application, *Transport Processes in Porous Media*, Kluwer Academic Publishers, 203-241, 1991.

Norton D., and Knapp R., Transport phenomena in hydrothermal systems: the nature of porosity, *American Journal of Science*, 913-936, 1977.

Robinson, P.C., Connectivity of fracture systems- a percolation theory approach, *J. Phys. A: Math. Gen.*, *16*, 605-614, 1983.

Stauffer, D., *Introduction to percolation theory*, Taylor & Francis, 1985

Turban, L., Anisotropic percolation on the Bethe lattice, *J. Phys. C: Solid State Phys.*, *12*, 1479-90, 1979.