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Cosmic ray momentum diffusion in the presence of non-linear Alfvén waves

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Abstract. The relation between the spatial diffusion coefficient along the magnetic field, $\kappa_{\|}$, and the momentum diffusion coefficient, D_p, for relativistic cosmic ray particles is modeled using Monte Carlo simulations. Wave fields with vanishing wave helicity and cross-helicity, constructed by superposing 'Alfvén-like' waves are considered. As the result, particle trajectories in high amplitude wave fields and then - by averaging over these trajectories - the values of transport coefficients are derived. The modeling is performed at various wave amplitudes, from $\delta B/B_0 = 0.15$ to 2.0, and for a number of wave field types. At *our* small amplitudes approximately the quasi-linear theory (QLT) estimates for $\kappa_{\|}$ and D_p are reproduced. However, with growing wave amplitude the simulated results show a small divergence from the QLT ones, with $\kappa_{\|}$ decreasing slower than theoretical prediction and the opposite being true for D_p. The wave field form gives only a slight influence on the wave-particle interactions at large wave amplitudes $\delta B/B_0 \sim 1$. The parameter characterizing the relative efficiency of the second-order to the first-order acceleration at shock waves, $D_p \cdot \kappa_{\|}$, is given in the QLT approximation by the Skilling formula $V_A^2 p^2/9$. In simulations together with increasing δB it increases above this scale in all the cases under our study. Consequences of the present results for the second-order Fermi acceleration at shock waves are briefly addressed.

Keywords : cosmic rays – second-order Fermi acceleration – magnetohydrodynamic turbulence – interstellar medium

1 Introduction

The quasi-linear theory of energetic charged particle transport in weakly perturbed magnetic fields provides a basis for considering cosmic ray propagation in interstellar space and particle acceleration processes at shock waves. It treats the effect of the random field as perturbations of orbits of particles moving in the average background field. Within this approach accounting of interactions of cosmic rays with perturbed magnetic field consists in derivation of a single tensor quantity, the momentum diffusion tensor, including as its components the pitch-angle diffusion coefficient $D_{\mu\mu}$, the momentum diffusion coefficient D_{pp} and the cross-correlation coefficient $D_{\mu p}$. By suitably averaging the coefficients over the pitch angle cosine μ one can derive the spatial diffusion coefficient along the magnetic field, $\kappa_{\|}$ and the *mean over* μ momentum diffusion coefficient D_p . The derivation of these coefficients for perturbations in the form of Alfvén waves was presented by Skilling (1975) and Schlickeiser (1989). In the case of the same wave intensities in the forward and backward waves Skilling obtained a simple relation between the spatial and the momentum diffusion coefficient

$$D_p \, \kappa_{\|} = \frac{1}{9} V_A^2 p^2 \qquad , \qquad (1.1)$$

where V_A is the Alfvén velocity and p is the particle momentum . A more detailed discussion of the problem was presented by Schlickeiser (1989; see also Dung & Schlickeiser 1990a,b, Jaekel & Schlickeiser 1992), who derived transport coefficients in the presence of Alfvén waves with different circular polarizations. By applying the diffusion limit to the full relativistic Fokker-Planck equation, Schlickeiser derived the relevant cosmic-ray transport equation, exact to all orders in V_A/c, for the isotropic part of the distribution function. That precision allows him to determine more general expressions for pitch angle diffusion, momentum diffusion and cross-correlation coefficient for particles, avoiding problems with the 'resonance gap'. The 'application' of the theory in the Schlickeiser paper was to study an influence of the wave polarization state, intensity and propagation direction on the cosmic-ray proton transport in a

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cold medium. In a particular case where *1.)* the right-hand and the left-hand polarized waves stream in both directions with the same intensity, and *2.)* where the power spectrum of right-hand and left-hand polarized magnetic fluctuations at wave number $k_\parallel$ within an interval $dk_\parallel$ is $Q_{xx}(k_0/k_\parallel)^q\, dk_\parallel$ in terms of the spectral density at a reference wave number k_0, for *3.)* $1 \le q < 4$ and *4.)* small V_A/v, one obtains approximately:

$$\kappa_\parallel = \frac{v^{3-q}B_0^2}{4\pi I_0 \Omega} \begin{cases} \frac{2}{(2-q)(4-q)} + \frac{q-1}{q-2}\left(\frac{V_A}{v}\right)^{2-q} & \text{for } q \neq 2 \\ \frac{1}{2} + \ln\left(\frac{V_A}{v}\right)^{-1} & \text{for } q = 2 \end{cases} \tag{1.2}$$

and

$$D_p = \frac{\pi \Omega I_0}{v^{3-q}B_0^2}\, p^2\, V_A^2 \frac{2}{q(q+2)} \quad , \tag{1.3}$$

where Ω is the particle gyrofrequency and I_0 is a constant. For the case of a flat Alfvén wave spectrum, $q = 1$, one obtains[1] the same relation between D_p and $\kappa_\parallel$ as Skilling's one (Eqn. 1.1). In the present considerations we restrict the discussion to the $q = 1$ case.

A major problem arising within the quasi-linear approach comes from the fact that the theory relies on the assumption of small amplitude magnetic field perturbations, $\delta B/B \ll 1$. However, in astrophysical applications the high amplitude MHD waves are common and the assumption can be invalid, or, at most, only of marginal validity. By the method of particle simulations it was shown (Zachary et al. 1989) that even medium amplitude perturbations ($\delta B/B \sim 0.1$) can lead to particle reflections within only a few gyroperiods, a feature not included in the quasi-linear derivations of the transport coefficients (cf., also Carioli 1991). In particular, the wave generation process due to streaming instability acting in the shock wave vicinity can lead to perturbations $\delta B \sim B$ (cf., e.g. Drury 1983, Blandford and Eichler 1987), and thus modify in a quantitative way, but in some cases also qualitatively, the cosmic ray acceleration process. Some aspects of this problem were discussed for non-relativistic shock waves by Decker & Vlahos (1985), Decker (1987), Ostrowski (1988). It is also common to register high amplitude waves by measurements in the solar wind (cf. Goldstein et al. 1995). The discussion of the second-order Fermi acceleration at shock waves presented by Ostrowski & Schlickeiser (1993) is based on the arbitrarily chosen quasi-linear relation between D_p and $\kappa_\parallel$. They demonstrated a possibility of substantial modification of the particle energy spectrum by the second-order process in some conditions, depending however, in a crucial way on the true relation between the momentum diffusion and spatial diffusion in the presence of the high amplitude MHD turbulence. Moreover, the question about the role of the second-order acceleration of energetic particles within the solar magnetosphere and the heliosphere was discussed by numerous authors (e.g. Wibberenz & Beuermann 1971; Morfill & Scholer 1977; Quenby 1984; Moussas et al. 1987; Terasawa 1989; Ryan & Lee 1991; the summary of the last ICRC in Leahy et al. 1994; Baring et al. 1995). These works are often based on the *in situ* measurements in space and thus, it is subject to all reconstruction problems of three dimensional wave field structure. The above mentioned works, as well as some not listed here, suggest a strong need for clarifying the issue of particle transport in media containing high amplitude perturbations of the magnetic field. In the present paper we consider this problem for relativistic particles, with $v \simeq c$, the case relevant to the acceleration processes in large scale interstellar shocks. Our special goal is to understand the influence of the large amplitude waves on the scattering efficiency both in the ordinary space and in the momentum space, as characterized by the respective diffusion coefficients. For particles with lower energies the comparison of scattering coefficients derived from QLT assuming the slab turbulence wave model with the ones determined phenomenologically from spacecraft observations has shown that simple usage of QLT can overestimate the coupling strength between magnetic fluctuations and energetic particles by an order of magnitude. Extensions of QLT which take into account such effects as the dynamical character of the fluctuations, thermal damping of waves, more complicated three-dimensional structure of the turbulence, etc., may overcome the problem (eg., Bieber et al. 1994; Dröge 1994). However, as no one of these modifications is generally accepted we refer our results to 'classical' theory by Schlickeiser (1989). One should also remember that no one QLT approach applies to the case of large amplitude perturbations and can be used only as a reference.

The aim of the present work is to study the relation between the spatial diffusion coefficient and the momentum diffusion coefficient for energetic (relativistic) charged particles propagating in space filled with high amplitude Alfvén waves (cf. also Karimabadi et al. 1992). We do not consider any particular site for application of the present results, the only restrictions are provided by the considered simplified structure of the magnetic field perturbations and relativistic velocities of the considered particles. Monte Carlo simulations involving derivation of particle trajectories in wave fields are applied. In Section 2 we describe them. In Section 3 the actually performed modeling at various wave amplitudes and for a number of wave field types is described and compared to the quasi-linear relations. The QLT estimates for $\kappa_\parallel$ and D_p are approximately reproduced at small amplitude. With growing wave amplitude the simulated results show a small divergence from the QLT

[1] In the Schlickeiser (1989) paper the integral I_1 below the equation (69) reads $I_1 \simeq \frac{2}{3}\left(1 - \frac{1}{5}\frac{V_A^2}{v^2}\right)$ (R. Schlickeiser, private communication).

ones, with $\kappa_{\parallel}$ decreasing slower than the theoretical prediction (outside its range of validity) and the opposite trend for D_p. The wave field form only slightly influences the wave-particle interactions at large wave amplitudes $\delta B/B_0 \sim 1$. The parameter characterizing the relative efficiency of the second-order to the first-order acceleration at shock waves, $D_p \cdot \kappa_{\parallel}$, is given in the QLT approximation by the Skilling formula $V_A^2 p^2/9$. In simulations it increases with increasing δB above this scale in all the cases under our study. In the last section we present a short summary of the results. Consequences of the present results for the second-order Fermi acceleration at shock waves are briefly addressed.

Below, the following notation is used: $\mathbf{E}$ - electric field vector; $\mathbf{B}$ - magnetic induction vector, consisting of the regular component $\mathbf{B}_0$ and the fluctuating component due to waves $\delta\mathbf{B}$(δB is given in the unit of B_0) ; V_A - the Alfvén velocity in the field B_0; q - the wave spectral index. We put the light velocity, the considered particle's mass and the background magnetic field to be the units, $c = 1$, $m = 1$ and $B_0 = 1$, respectively. For cosmic ray particles, we denote as $\mathbf{p}$ - the particle momentum vector with the pitch angle respectively to $\mathbf{B}_0$ denoted with Θ ($\mu \equiv \cos\Theta$), ε - particle energy, $\mathbf{v} = c^2\mathbf{p}/\varepsilon$ - particle velocity vector (here $\approx c$) and the respective Lorentz factor $\gamma \equiv (1 - v^2/c^2)^{-1/2}$.

2 Numerical simulations

The approach applied in the present paper is based on numerical Monte Carlo particle simulations. The general procedure is quite simple: test particles are injected at random positions into a magnetized plasma and their trajectories are followed by integration of the particle equations of motion. Due to the presence of Alfvén waves, particles move diffusively in space and momentum. By averaging over a large number of trajectories one derives the respective diffusion coefficients. Below, we describe a particular simple model chosen for turbulent wave fields. The presentation of technical details of the derivations is given in Appendix A.

2.1 The wave field models

In the case of high amplitude waves, there are no analytic models available reproducing the turbulent field structure. Because of that, approximate models representing such fields are considered. Three of them are described below.The turbulence here is represented as a superposition of Alfvén-like waves. It is a viable model for non-linear turbulence because Alfvénic fluctuations of arbitrary amplitude are exact solutions of dissipationless incompressible MHD (Parker 1979). The constructed perturbed field structures are explicitly divergence-free. In the simulations, for any individual particle a separate set of wave field parameters is selected. As a result all the averages taken over the particles include also averaging over multiple magnetic field realizations.

2.1.1 Linearly polarized plane waves

In the model we take a superposition of plane Alfvén waves propagating along the z-axis, in the positive (forward) and the negative (backward) direction. Two planar polarizations, along the x and y axes, are considered. In the computations a discrete number of 24 sinusoidal waves, 12 of each polarization, is included. The wave parameters - wave vectors k and wave amplitudes δB_0 - are drawn in a random manner from the flat wave spectrum. Related to the wave 'i' the magnetic field fluctuation vector $\delta\mathbf{B}^{(i)}$ is given in the form:

$$\delta\mathbf{B}^{(i)} = \delta\mathbf{B}_0^{(i)} \sin(k^{(i)}z - \omega^{(i)}t - \Phi^{(i)}) \qquad . \qquad (2.1)$$

The dispersion relation for Alfvén waves, $\omega^2 = V_A^2 k^2$, provides the respective ω parameter for any given wave. The sign of ω is defined by selecting the wave velocity V at, randomly, $\pm V_A$, but a number of waves moving in any direction is kept the same within any selected range of wave vectors (see below). We call the 'isotropic wave field' or 'isotropic turbulence' the wave field with the same number of positive and negative waves in any wave-vector range. The electric field fluctuation related to the particular wave is given as $\delta\mathbf{E}^{(i)} = -\mathbf{V}^{(i)} \wedge \delta\mathbf{B}^{(i)}$.

For selecting any individual set of wave parameters we use the following procedure. Wave vectors, expressed in units of $k_{res} \equiv 2\pi/r_g(< B >, p_0)$ in the *mean* magnetic field $< B >$, are drawn in a random way from the respective ranges: $2.0 < k < 8.0$ for 'short' waves, $0.4 < k < 2.0$ for 'medium' waves and $0.08 < k < 0.4$ for 'long' waves. Four waves are taken from every range for everyone polarization plane. The respective wave amplitudes are drawn in a random manner so as to keep constant

$$[\sum_{i=1}^{24}(\delta\mathbf{B}_0^{(i)})^2]^{1/2} \equiv \delta B \qquad , \qquad (2.2)$$

where δB is a model parameter, and, separately in all wave-vector ranges

$$[\sum_{i=1}^{8}(\delta\mathbf{B}_0^{(i)})^2]^{1/2} = [\sum_{i=9}^{16}(\delta\mathbf{B}_0^{(i)})^2]^{1/2} =$$

$$= [\sum_{i=17}^{24}(\delta\mathbf{B}_0^{(i)})^2]^{1/2} = \frac{\delta B}{\sqrt{3}}. \qquad (2.3)$$

The mean magnetic field for sinusoidal Alfvén waves (with $\delta\mathbf{B} \cdot \mathbf{B} = 0$) is given as $< B >= \sqrt{B_0^2 + \delta B^2/2}$.

2.1.2 Wave packets

The plane wave model is not expected to reproduce the 3-D turbulence in a realistic way. In the same time the 2-dimensional or 3-dimensional character of the turbulence can give a qualitative influence to energetic particle transport properties (cf. Giacalone & Jokipii 1994). Therefore, as a simple extension of the above models to the third dimension we consider waves in the form of wave packets, modulated in one direction perpendicular to the propagation direction. In the present case one can use the formula (3.1) for $\delta \mathbf{B}^{(i)}$, where the phase parameter is subject to modulation on the scale comparable to the corresponding wavelength. Two types of modulation for the x-components in (3.1) are considered: A.) the sinusoidal 'smooth' modulation is given as

$$\Phi_x^{(i)}(y) = \sin(k_y^{(i)} y) \quad , \tag{2.4}$$

and B.) the 'sharp-edged' modulation as

$$\Phi_x^{(i)}(y) = y \; mod \; (1/k_y^{(i)}) \quad . \tag{2.5}$$

The y-components can be obtained from the above formulae by interchanging x and y. Vectors $k_x^{(i)}$ and $k_y^{(i)}$ are drawn in a random manner from the respective wave-vector range for $k^{(i)}$.

3 Results

Simulations were performed for all presented turbulence models. Typically 900 particles were involved in an individual run. At Fig.(1,2,3), on the successive panels we present the derived diffusion coefficients $\kappa_{\|}$, D_p and the product $D_p\,\kappa_{\|}$ versus the wave amplitude δB. However, in order to obtain a realistic estimate of diffusion coefficient for small wave amplitudes requires substantially longer integration time, growing in proportion to roughly the inverse wave amplitude squared, only the results for $\delta B \geq 0.15$ are presented. Also, at small amplitudes the fluctuations ascribed to the long scale phase coherence of the involved waves become visible and continuing simulations below our limiting one would require a substantial increase of the number of the wave mode involved, i.e. a further increase of the computation time. The results are presented in Fig-s (1, 2, 3), where they are compared to the QLT estimates derived from (1.2,3) with $I_0/4\pi = (\delta B^2/8\pi)/[2log(k_{max}/k_{min})]$. The final division by 2 in this expression comes from averaging the squared (sinusoidal) wave amplitude. The expected decrease of the spatial diffusion coefficient and grow of the momentum diffusion with increasing δB is observed. At small amplitudes approximately the QLT estimates for $\kappa_{\|}$ and D_p are reproduced. With growing wave amplitude the simulated results change in a different way with respect to the QLT ones. Diffusion coeficient $\kappa_{\|}$ decreases slower than the theoretical prediction (outside its range of validity) and D_p rises faster then the theoretical prediction. The wave field form gives only slight influence on the wave-particle interactions at large wave amplitudes $\delta B \sim 1$. The parameter characterizing the relative efficiency of the second-order to the first-order acceleration at shock waves, $D_p \cdot \kappa_{\|}$, is given in the QLT approximation by the Skilling formula $V_A^2 p^2/9$. In simulations with increasing δB it increase above this scale in the all cases under our study. The product of two diffusion coefficients shows a systematic trend to increase over the quasi-linear value represented at figures by a horizontal dashed line. One should also note that at larger wave amplitudes there are no qualitative differences between the results presented in Fig.(1,2,3). It may illustrate the non-resonant wave-particle coupling at these amplitudes.

4 Summary and final remarks

In the present paper we considered energetic relativistic particle phase-space diffusion due to scattering at the finite amplitude Alfvénic turbulence. In the considered situation of wave backward-forward symmetry and vanishing helicity the diffusion coefficients for a few turbulent field models were derived. At *our* small amplitudes we obtained results close to the quasi-linear theory estimates for $\kappa_{\|}$ and D_p. The decrease of the spatial diffusion coefficient and grow of the momentum diffusion with increasing δB is also reproduced. However, together with growing wave amplitude the simulated results change in a different rate with respect to the QLT ones, with $\kappa_{\|}$ decreasing slower than theoretical prediction and D_p rises faster than theoretical prediction. The wave field form gives only slight influence on the wave-particle interactions at large wave amplitudes $\delta B \sim 1$. The parameter characterizing the relative efficiency of the second-order to the first-order acceleration at shock waves, $D_p \cdot \kappa_{\|}$, is given in the QLT approximation by the Skilling formula $V_A^2 p^2/9$. In simulations with increasing δB it increases above this scale in all the cases under our study. One should also note that at larger wave amplitudes there are no substantial differences between the presented results for the considered turbulence mods. It illustrates the effective non-resonant wave-particle coupling at high wave amplitudes. We do not consider any particular site for application of the present results, as the only restrictions are provided by the assumed simplified structure of the magnetic field perturbations and relativistic velocities of the considered particles.

It was demonstrated by Bell (1978) that shock waves provide conditions for generation of highly non-linear MHD turbulence and our present results have a great importance in such conditions. The possibility to increase the product $D_p \kappa_{\|}$ above the scale $V_A^2 p^2/9$ can give a noticeable impact at the processes of the second-order Fermi acceleration in a number of astrophysical environ-

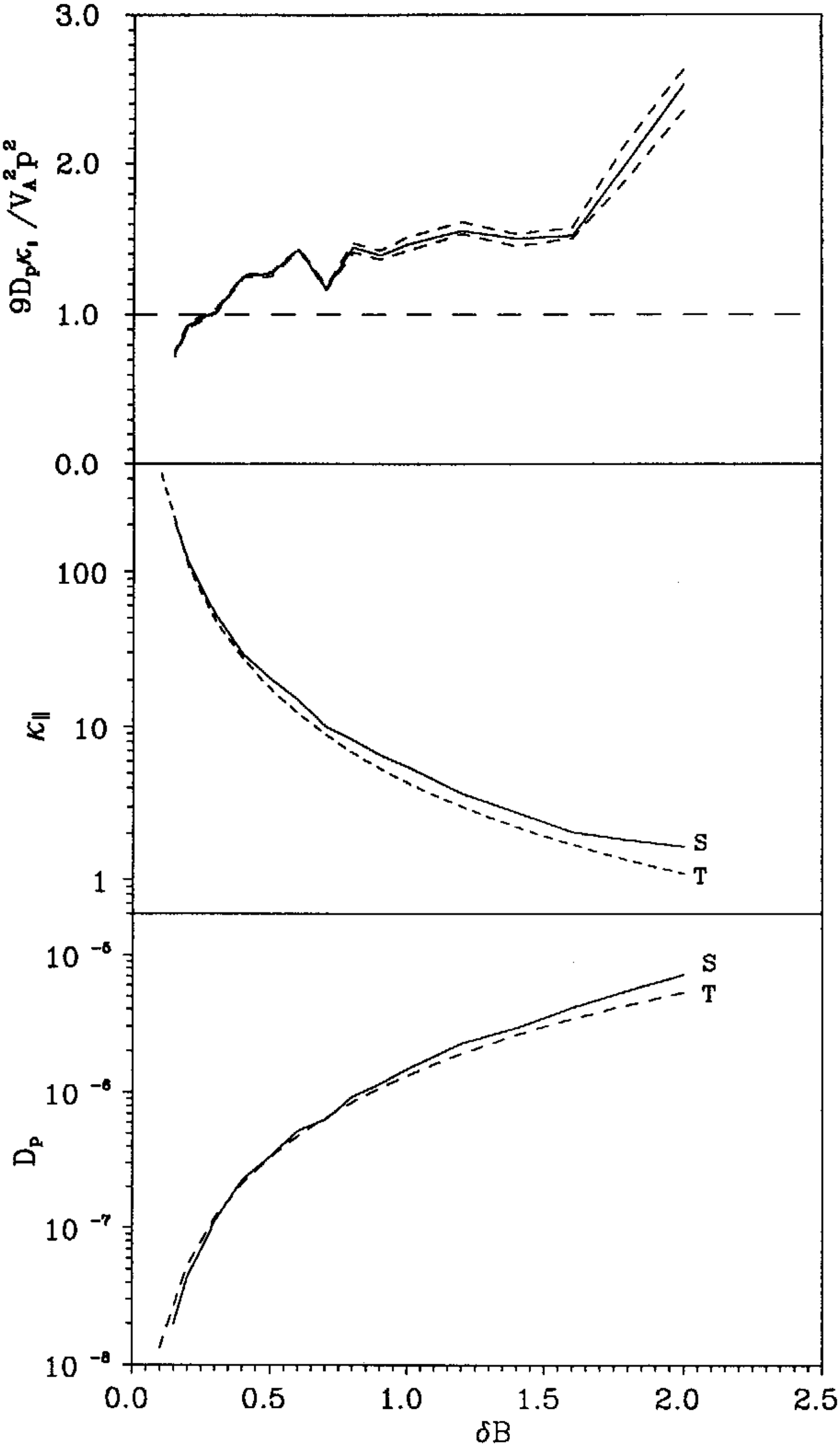

Fig. 1. The simulated values of $\kappa_\parallel$, D_p and $D_p\,\kappa_\parallel$ versus the wave amplitude δB for linearly polarized plane Alfvén waves. Solid lines join the results obtained from simulations (S). Near the results for $\kappa_\parallel$ and D_p the theoretical (T) quasi-linear estimates are presented with dashed lines. The adjacent dashed lines near $D_p\,\kappa_\parallel$, in units of $V_A^2p^2/9$ give the maximun and minimum values of this quantity occurring within the range used for the fitting. The horizontal dashed line is provided in the upper panel for the reference represents the QLT estimate.

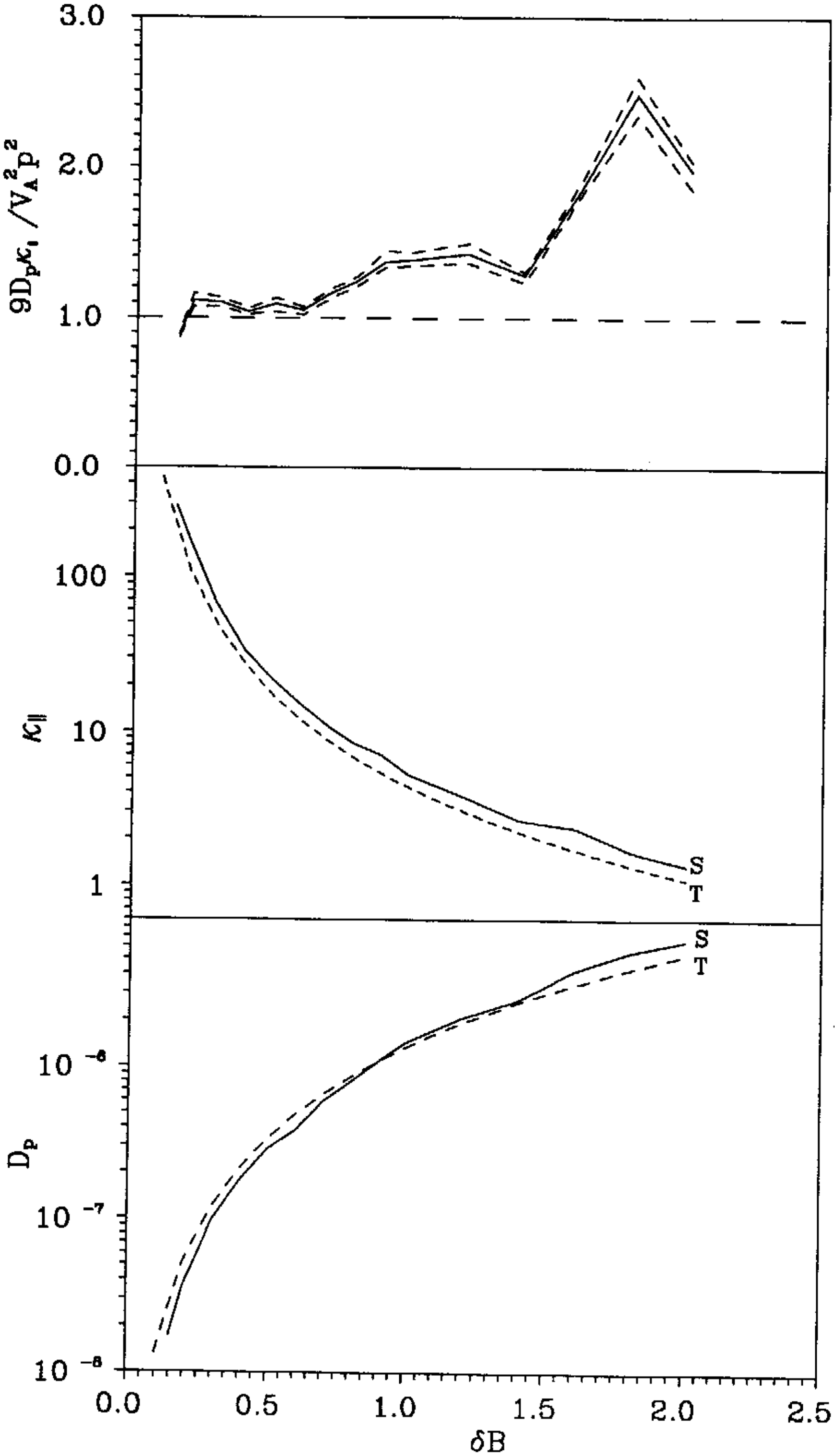

Fig. 2. The simulated values of $\kappa_\parallel$, D_p and $D_p\,\kappa_\parallel$ versus the wave amplitude δB for sinusoidal modulated Alfvén waves. The meaning of curves and symbols is the same as in Fig. 1.

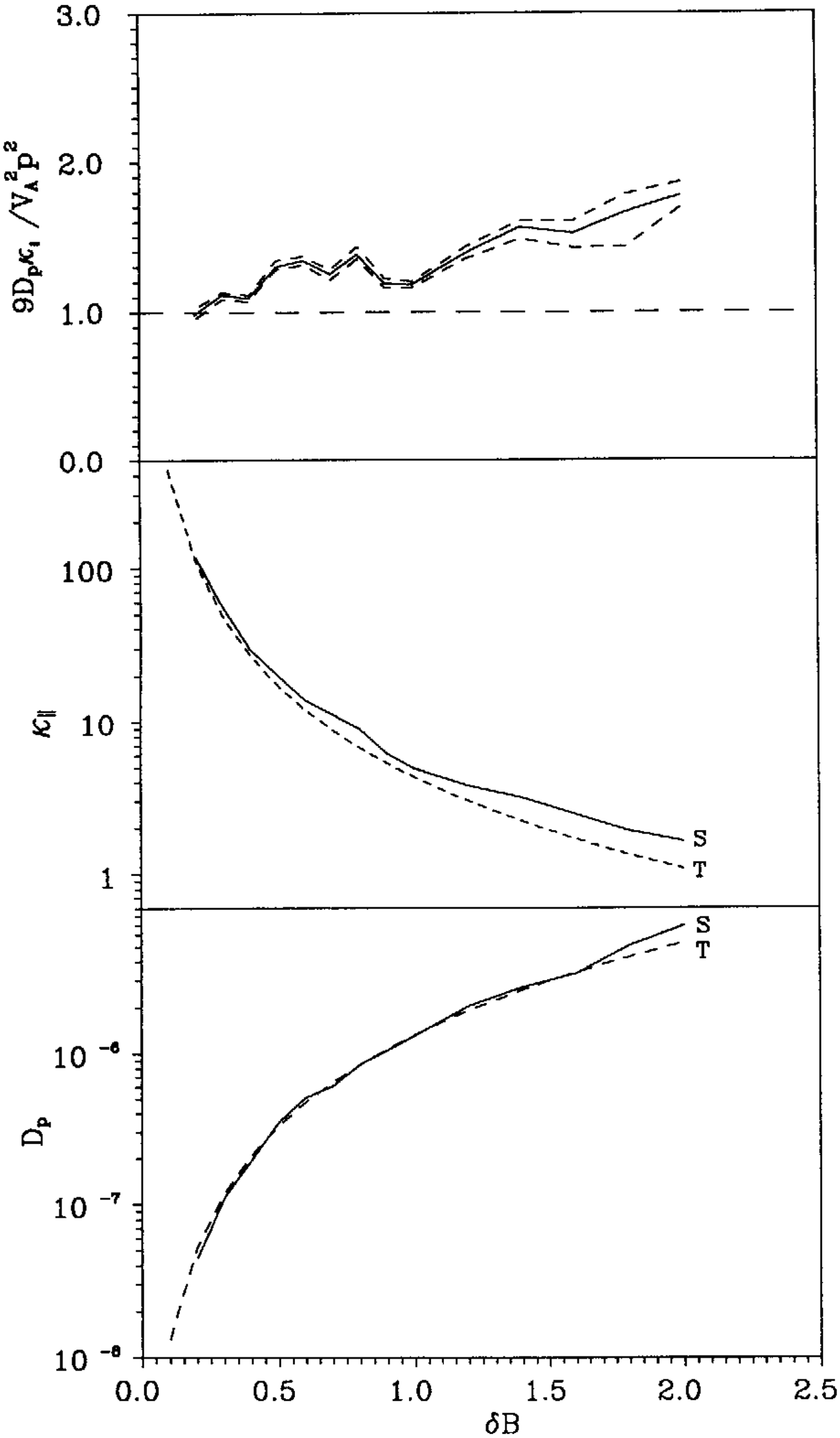

Fig. 3. The simulated values of $\kappa_\|$, D_p and $D_p\kappa_\|$ versus the wave amplitude δB for sharp-cut modulation. The meaning of curves and symbols is the same as in Fig. 1.

ments. As it was demonstrated by Ostrowski & Schlickeiser (1993) and Ostrowski (1994) in some shock conditions the second-order accelerati on can give a substantial influence to the accelerated particle spectrum. To illustrate this fact when varying the ratio $\alpha \equiv D_p \kappa_\| / 9 V_A^2 p^2$ one can derive the respective spectral indices according to formulae given in Appendix A of Ostrowski & Schlickeiser (1993; in A7 one should read U_1^2 instead of U_1). The results with $\alpha > 1$ – occurring in our simulations for large δB – suggest an interesting possibility to obtain a noticeable flattening of particle spectra at shocks, which create non-linear waves or propagate in highly turbulent medium.

Among other simplifications and limitations introduced in the present paper the most important ones are due to our simplified turbulence model. In actual wave fields modifications may occur because of a few reasons. In general, the wave spectral index is $q \neq 1$ and the values of q expected to be larger than unity will lead to somewhat greater QLT values of α. The processes generating waves in the shock vicinity may be balanced by the non-linear wave damping (cf. Lagage & Cesarsky 1983) or the generating process can become saturated (cf. Völk 1984) with $\delta B \leq B_0$. Then our derivations at higher amplitudes would be of only 'academic' importance, perhaps with the exception of the perturbations appearing immediately behind the strong shock. Moreover, the general perturbation field in the considered conditions can involve a number of magnetosonic wave modes complicating the reality with respect to the considered model. Finally, in non-linear situation the phase velocities of perturbations will be governed by the actual magnetic field B instead of the smaller field B_0 used by us. It may lead to larger particle momentum diffusion.

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Appendix A. The method of simulations

Let us consider an infinite region of tenuous plasma with the uniform *mean* magnetic field. As the reference frame we introduce a plasma rest frame with the z-axis directed along the mean magnetic field. The Alfvén waves, described in Section 3, propagate along this axis. Particles used in the simulations are injected with the same initial momentum p_0, randomly at initial positions (x_0, y_0, z_0) within the cube with dimension much larger than the longest wave. Integration of the Lorentz equation, $\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$ $(\equiv \mathbf{F})$ and the energy equation $\dot{\varepsilon} = \mathbf{F} \cdot \mathbf{v}$ is performed for the time much longer [1] than the mean scattering time. In a chosen sequence of time instants, t_i, particle spatial positions and momentum values are recorded. Basing on the information collected for numerous particles, at any instant 'i' one derives the values of the 'partial' diffusion coefficients by averaging the respective squared dispersion over all particles. For the spatial diffusion along the mean magnetic field, $\kappa_\| \equiv \kappa_z$, one obtains

$$\kappa_{\|,i} \equiv \frac{<(z_i - z_0)^2>}{2t_i} \qquad , \qquad (A1)$$

and, for diffusion perpendicular to the mean field, κ_x and κ_y are obtained by substitution in Equ. (A1) of x and y instead of z, respectively. Analogously, the momentum diffusion coefficient can be *estimated* as

$$D_{p,i} \equiv \frac{<(p_i - p_0)^2>}{2t_i} \qquad . \qquad (A2)$$

For *sufficiently* long integration times (but with V_A small enough to keep $<(p_i - p_0)^2> \ll p_0^2$) these coefficients tend to the limiting values corresponding to the final coefficients $\kappa_\|$, $\kappa_\perp$ and D_p. The estimates (A1-2) have a noticeable weakness for limited particle numbers in the simulations: because of the square dependence of the estimated coefficients on particle shift along the respective co-ordinate the few particles with highest shifts determine the result. That feature leads to substantial fluctuations of the derived diffusion coefficients. Therefore, in our approach, we applied a procedure of fitting the obtained limiting distribution to the particle distribution function derived from the respective diffusion equation, with the diffusion coefficient being the fitting parameter. For the spatial distributions

$$f(\rho) = \frac{1}{2\sqrt{\pi \kappa_\rho t}} \exp\left[-\frac{(\rho - \rho_0)^2}{4\kappa_\rho t}\right] \qquad , \qquad (A3)$$

where ρ stands for space co-ordinate x, y, or z and the respective spatial diffusion coefficient is κ_ρ. The momentum diffusion coefficient with the considered flat wave spectrum ($q = 1$ in equation (1.3)) is proportional to momentum for relativistic particles, $D_p \equiv D_0 \cdot p$. Then, the solution of the momentum diffusion equation can be obtained with the Bessel function technique, to yield (cf. Toptygin 1985):

$$f(p,t) = \frac{1}{4\pi D_0 t p p_0} \exp\left[-\frac{p + p_0}{D_0 t}\right] I_2\left(\frac{2\sqrt{p p_0}}{D_0 t}\right) \qquad . \qquad (A4)$$

[1] A small digression is now required. The above energy equation can be derived from the momentum equation and it seems to be sufficient to solve only the later one. However errors introduced into particle energy by an iterating numerical code are of the order of $\delta B / \delta E \sim c / V_A$ larger for the momentum equation than the errors arising from integration of the energy equation. Due to this fact we solved equations for momentum components, as well as the energy equation, and, after any step of iteration procedure, momentum length was scaled to preserve $\varepsilon^2 = 1 + p^2$.

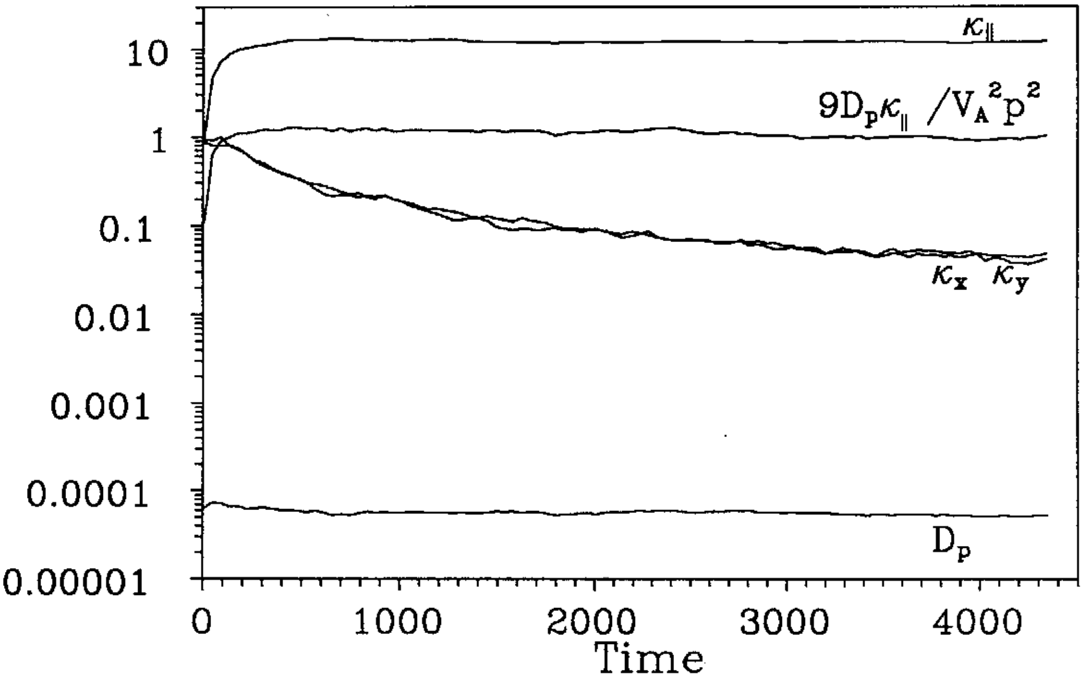

Fig. 4. An example of the derived instantaneous diffusion coefficients (A1,2) versus integration time are presented for the case of $\delta B = 0.6$ and the plane Alfvén wave model.

where the injection have the form $\propto \delta(t)\delta(p-p_0)$ and I_2 is the modified Bessel function (Abramowitz & Stegun 1973). In the expressions (A3,4) t is the diffusion time ($\equiv$ integration time) and the index 0 is for the particle initial co-ordinates. The fitting was performed by looking for the highest probability according to the Kolmogorov-Smirnov test (see, e.g. Press et al. 1989) of the theoretical distribution to be the real one. Examples of the results for a given value of δB are presented at Fig. 4. Variations of the recorded data can be easily accounted for on the qualitative level. The first, quickly changing section represents transition from the initial free particle flow to that dominated by wave scattering. Then the curves for the spatial diffusion coefficient and the momentum diffusion coefficient get flat. However, in deriving the diffusion coefficient perpendicular to the magnetic field the moment of reaching the stabilized flat range is delayed, until the scatter of particle position due to diffusion is greater than the one related to probing the regular particle gyration (Michalek & Ostrowski, in preparation). The effect is stronger for smaller wave amplitudes and, so, it requires long integration times to become negligible. Various tests of the simulation procedure are presented in Appendix B.

Appendix B. Testing the simulation scheme.

Simulations with the use of Monte Carlo method require very careful checking against possible systematic numerical effects. In the considered by us wave fields constructed by superposition of a finite number of sinusoidal waves an additional danger of occurring some resonant phenomena arises. In order to avoid or control these problems a number of tests of the code, including the ones described below were performed.

1.) The particle trajectories in the phase space were checked to be smooth and 'reasonable' ones. In the magnetic field with static perturbations imposed the particle energy was conserved to the high degree of accuracy (of course, without using the above mentioned momentum scaling). With the randomly oriented uniform magnetic field the particle pitch angle was analogously conserved.

2.) In the presence of resonant phenomena one expects the final particle distributions to diverge from the purely stochastic behaviour. For example, one could find particles that diffused much further in space or in momentum in comparison to what could be deduced from the expected Gaussian-like distributions. As presented at Fig. 5, we compared the typical simulated particle distributions with the theoretical (A3,4) ones for the diffusion coefficients derived in the simulations involving the expressions (A1,2). One should note that the distributions for larger wave amplitudes are always well

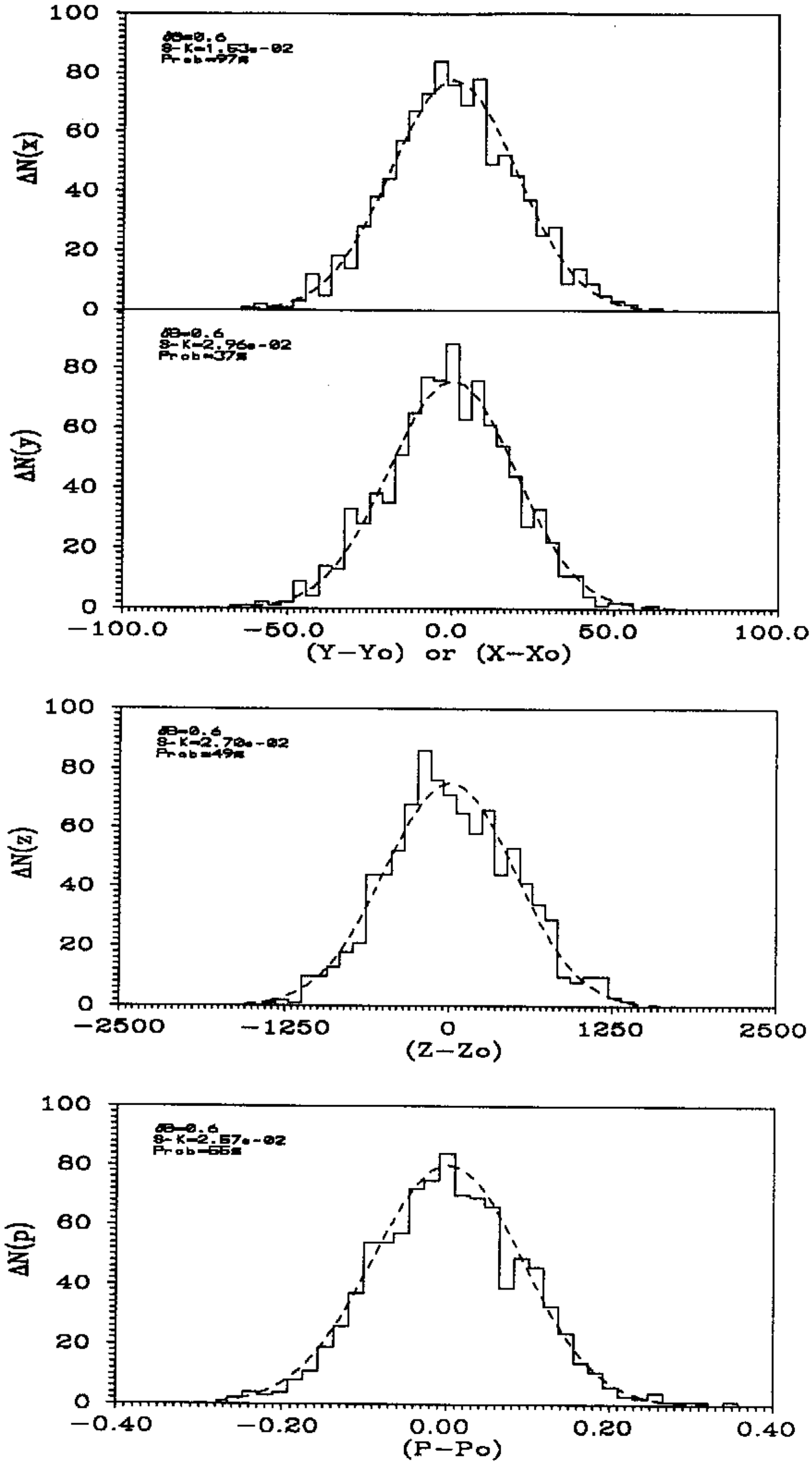

Fig. 5. Histograms of the generated particle distributions (solid lines) along the indicated phase space co-ordinate compared to the respective theoretical curves (A3,4; dashed lines). The results are presented for linearly polarized plane Alfvén waves .

fitted by the diffusive distributions. Additional checks whether the numerical code preserves the momentum distribution isotropy were performed for all momentum components p_x, p_y and p_z, for all considered δB.
3.) The curves presenting time evolution of the transport coefficient were inspected by eye and checked against any abrupt change or systematic variation at advanced simulation times. The additional consistency verification for the considered axial symmetry of the perturbation field is due to good agreement of the derived values of the perpendicular diffusion coefficients along axes x and y.

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