Nonlinear Processes in Geophysics (1996) 3: 41 - 46

Nonlinear Processes in Geophysics

Renormalization group method for Farley-Buneman fluctuations: basic results

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Received 1 June 1995 - Accepted 20 January 1996 - Communicated by E. Marsch

Abstract. *Sudan and Keskinen* in [1979] derived a set of equations governing the nonlinear evolution of density fluctuations in a low-pressure weakly ionized plasma driven unstable by the $\mathbf{E} \times \mathbf{B}$ or gradient-drift instability. This problem is of fundamentAl importance in ionospheric physics. The nonlinear nature of the equations makes it very hard to write a closed form solution. In this paper we propose to use "Dynamical Renormalization Group" methods to study the long-wavelength, long-time behavior of density correlations generated in this ionospheric plasma stirred by a Gaussian random force characterized by a correlation function $\langle f_{\mathbf{k}} f_{\mathbf{k}} \rangle \propto k^m$. The effect of the small scales on the large scale dynamics in the limit $k \to 0$ and infinite "Reynolds" number, can be expressed in the form of renormalized coefficients; in our case renormalized diffusion. If one assumes the power spectra to be given by the kolmogorov argument of cascading of energy, then one can not only derive a subgrid model based on the results of RNG, and this has been done by *Hamza and Sudan* [1995], but one can also extract the skewness of the spectra as we do in this paper.

1 Introduction

Over the past two decades, there has been a large number of observations of organized flow in fluids and plasmas commonly termed turbulent. Historically, turbulent flow has been characterized by extreme incoherence or randomness, the most successful theoretical treatements assuming a quasi-Gaussian probability distribution of the excited fluctuations. It is becoming very clear that new techniques need to be developed in order to sort out the wealth of phenomena that occur in a turbulent system. Renormalization Group Methods (RNG) have tried to fulfill this need, but they come short on many grounds as will be discussed later in this paper. They do, however, provide usefull tools for turbulence analysis as we intend to show explicitly.

In this paper we have chosen a two dimensional problem, namely that of strong turbulence resulting from plasma convection in the E-region of the ionosphere. The choice of this particular case for study is dictated by the fact that it describes fairly accurately the physical situation and the evolution of the plasma in this very specific region. In addition a large number of radar and rocket experiments have been conducted in the past two decades and data has become available to analize and therefore check the validity of the various theories. The calculations, we are presenting, are based on the "Dynamic Renormalization Group" (RNG) methods first developped by *Ma and Mazenko* [1975] for critical phenomena. These techniques have since been successfully applied by a number of authors (see for example *Forster et al.* [1977], *Fournier et al.* [1982], and *Yakhot and Orszag* [1986]) to investigate the problem of randomly stirred fluids. The evolution equation derived in the context of ionospheric physics, is one of the most simple archetypes of nonlinear evolution of inhomogeneous magnetized plasmas (a nonlinear partial differential equation with a quadratic nonlinearity). A great deal of interest has been focussed on applying the strong turbulence theories such as the "Direct Interaction Approximation" of *Kraichnan* [1959] or equivalently the renormalized nonlinear wave interaction theory described by *Kadomtsev* [1965]. The Turbulent spectra for the two dimensional problem we are to investigate have been predicted by the cascade hypothesis (see for example *Sudan and Pfirsch* [1985]), namely that the energy cascades to higher k's with a power law of $(-5/3)$.

WIth all the applications, in fluid turbulence and critical phenomena in mind, here we undertake a detailed description of RNG methods to the problem of strong turbulence arising in a low β, weakly ionized plasma confined in a strong magnetic field subjected to both a gradient in density and a backgroung electric field. Under certain circumstances, the plasma under consideration, was shown by *Simon* [1963] and *Hoh* [1963] to be unstable to electrostatic fluctuations. The $\mathbf{E} \times \mathbf{B}$ drift associated with these very low frequency potential fluctuations is divergence free and convects the electron fluid in two dimensions. The instability is analogous to the Rayleigh-Taylor instability. We should point out that if we were to study the advection of the density by the $\mathbf{E} \times \mathbf{B}$ drift then the natural spectrum that comes out from a kolmogorov type of analysis would be a k^{-1} spectrum as shown by *Batchelor* [1950]. Very interesting results are obtained as we shall see very soon.

The purpose of this paper is to use the renormalization group method to extract turbulent transport properties. We propose to calculate the density-gradient skewness since the application of RNG to numerous problems is available in the literature and to the problem at hand in a publication by *Hamza and Sudan* [1995]. The Skewness calculation presented in this paper has never been published, and constitutes an excellent yard stick for comparison with radar observations related to Farley-Buneman turbulence as it occurs in the auroral E region of the ionosphere. We therefore organize the paper in the following form. In section II we briefly review the results of *Hamza and Sudan* [1995] related to the application of RNG in deriving a sub-grid model for E-region turbulence. In section III we calculte the skewness given a Kolmogorov spectrum, and finally conclude.

2 Dynamical Renormalization Group

In this section we will briefly discuss the results of a recent paper by *Hamza and Sudan* [1995] on the application of RNG to the ionospheric problem in question. We have kept the same notation for convenience, and highlighted a number of equations so that the reading of this manuscript does not really require extensive reference to the mother paper *Hamza and Sudan* [1995].

The nonlinear equation governing the evolution of the plasma density fluctuations in the ionospheric E-region can be written in the following form (for assumptions, derivation and details we strongly recommand the reader to consult the paper by *Hamza and Sudan* [1995]).

$$\left(i\frac{\partial}{\partial t} - \omega_{\mathbf{k}}\right)n(\mathbf{k},t) =$$

$$\int \frac{d\mathbf{q}}{(2\pi)^2}\epsilon^{(2)}(\mathbf{q},\mathbf{k}-\mathbf{q})n(\mathbf{q},t)n(\mathbf{k}-\mathbf{q},t), \qquad (1)$$

where the density fluctuation

$$n(\mathbf{x},t) = \int\int d\omega d\mathbf{k}\frac{1}{(2\pi)^3}n(\mathbf{k},\omega)e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad (2)$$

and $\omega_{\mathbf{k}}$ is given by [see for example, *Fejer et. al.*, 1975]

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}_r} + i\gamma_{\mathbf{k}}$$

$$\gamma_{\mathbf{k}} = \frac{1}{\psi(1+\psi)}\left\{\left(\frac{\Omega_e}{\nu_e}\right)\left[\frac{k_y^2 v_d}{k^2 L(1+\psi)}\right] - \frac{k^2 c_s^2}{\nu_i}\right\}$$

$$\equiv \gamma_0 - k^2 D_0$$

$$\omega_{\mathbf{k}_r} = \frac{\mathbf{k}\cdot\mathbf{v}_d}{1+\psi}$$

$$\psi = \frac{\nu_e\nu_i}{\Omega_e\Omega_i} \qquad (3)$$

where $\mathbf{v}_d = c\mathbf{E}_0\times\mathbf{B}_0/B_0^2 \equiv v_d\hat{y}$ is the electrojet electron drift velocity and $L^{-1} = \frac{1}{n_0}\frac{dn_0}{dz}$ is the electron density scale height in the E-region. Finally the nonlinear coupling matrix element is given by

$$\epsilon^{(2)}(\mathbf{q},\mathbf{k}-\mathbf{q}) =$$

$$\frac{\nu_i}{2\Omega_i}\frac{\mathbf{b}\cdot(\mathbf{q}\times\mathbf{k})}{(1+\psi)^2}\left(\frac{\mathbf{q}\cdot\mathbf{v}_d}{q^2} - \frac{(\mathbf{k}-\mathbf{q})\cdot\mathbf{v}_d}{|\mathbf{k}-\mathbf{q}|^2}\right) \qquad (4)$$

with $\mathbf{b} = \mathbf{B}_0/B_0 \equiv \hat{x}; \mathbf{k} = (k_y, k_z).\mathbf{q} = (q_y, q_z)$.

Equation (1) describes the nonlinear evolution of density fluctuations around an average density n_0, with a gradient ∇n_0 taken to be constant. This equation has only one quadratic constant of motion $\int d^2k|n_{\mathbf{k}}|^2$ in the limit $D_0 \to 0$. This according to the turbulent cascading theories would lead to one single inertial range. This property plays a signicant role in this problem.

At this stage we are ready to apply RNG and discuss its ramifications. But before, we need to discuss how we handle the unstable system. It will be treated as an externally forced stable system. The external forcing is interpreted as the energy transfer from the unstable part of the spectrum $k_o < k < k_c$ by mode coupling to the range $k > k_c$ by the addition of a forcing function $f(\mathbf{k},\omega)$. Eqn. (1) may be rewritten in $\omega, \mathbf{k}$ space as

$$\epsilon^L(\mathbf{k},\omega)n(\mathbf{k},\omega) = f(\mathbf{k},\omega)$$

$$+\lambda_0\int\frac{d\mathbf{q}d\Omega}{(2\pi)^3}\epsilon^{(2)}(\mathbf{q},\mathbf{k}-\mathbf{q})n(\mathbf{q},\Omega)n(\mathbf{k}-\mathbf{q},\omega-\Omega), \quad (5)$$

where

$$\epsilon^L(\mathbf{k},\omega) = \omega - \omega_{\mathbf{k}} + ik^2 D_0 \qquad (6)$$

and $\omega_{\mathbf{k}}$, D_0 and $\epsilon^{(2)}$ represent the linear eigenfrequency, the diffusion coefficient and the nonlinear coupling coefficient respectively, which were defined earlier. The parameter $\lambda_0(=1)$ is formal; it is introduced to facilitate the perturbation solution of equation (5). The forcing f is random, and chosen to be isotropic and to have Gaussian statistics. It is important to note that no initial and boundary conditions are needed for equation (5), since the plasma described is stirred by the random force such that a statistically steady state can be reached.

The force correlation function is given by:

$$< f(\mathbf{k},\omega)f(\mathbf{k}',\omega') >=$$

$$(2\pi)^3 A_0 F(k)\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}') \qquad (7)$$

$$F(k) = k^m \qquad (8)$$

The results of the renormalization group method as applied to equation (5) will be briefly discussed. But before, we would like to review some of the fundmentals steps and questions raised by this method.

The key question raised by RNG is: Let Λ be the highest wavenumber in the system, the so-called ultraviolet cut off. What then is the effect of the short-wavelength modes $n^{>}(\mathbf{k},\omega)$, from a narrow wavevector band near the ultraviolet cutoff $\Lambda e^{-r} < k < \Lambda$ (where r is a small parameter), on the long-wavelength modes $n^{<}(\mathbf{k},\omega)$ belonging to the interval $0 < k < \Lambda e^{-r}$?

The "Dynamic Renormalization Group" technique answers the question as follows. First eliminate the small scale modes $n^{>}(\mathbf{k},\omega)$ from the equation governing the evolution of the large scales $n^{<}(\mathbf{k},\omega)$. Then average the equation for $n^{<}(\mathbf{k},\omega)$ over the small scale forcing $f^{>}(\mathbf{k},\omega)$ that acts in $\Lambda e^{-r} < k < \Lambda$: this operation redefines the coefficients which enter the reduced equation (r is a measure of the fraction of degrees of freedom eliminated). The next step consists of rescaling space, time, the density $n^{<}(\mathbf{k},\omega)$ and the stirring force $f^{<}(\mathbf{k},\omega)$ and we collect all the terms in the new equation that are *self similar* to the original one. Subsequently, it has to be argued that the remaining terms are "irrelevant", i.e., the higher order terms generated via the perturbation analysis ought to vanish under the rescaling process. Finally one obtains recursion relations (ordinary differential equations (ODE's)) for the different coefficients, such as the diffusion, the coupling constant, etc. If the solutions to ODE's obtained through this procedure have fixed points as $r \to \infty$ we obtain renormalized values for the diffusion.

If now one defines the large and small scale densities $n_{\hat{k}}^{<}$ and $n_{\hat{k}}^{>}$ for $\{|\mathbf{k}| < \Lambda e^{-r}\}$ and for $\{\Lambda e^{-r} < |\mathbf{k}| < \Lambda\}$ respectively, then one can write two coupled equations, one for $n_{\hat{k}}^{<}$ and another for $n_{\hat{k}}^{>}$ respectively. This leads, after substituting the ">" quantities and averaging over the high k shell with the proper statistics for the forcing term f, to the renormalized evolution equation

$$\epsilon^{R}(\mathbf{k},\omega)n(\mathbf{k},\omega)^{<} =$$
$$-\lambda_0 \int_{q<\Lambda e^{-r}} \frac{d\mathbf{q}d\Omega}{(2\pi)^3}\epsilon^{(2)(\mathbf{q},\mathbf{k}-\mathbf{q})}n(\mathbf{q},\Omega)^{<}n(\mathbf{k}-\mathbf{q},\omega-\Omega)^{<}$$
$$+f^{<}(\mathbf{k},\omega) - \Delta f^{<}(\mathbf{k},\omega) + 0\Big(\lambda_0^2,(n(\mathbf{k},\omega)^{<})^3\Big) \quad (9)$$

where the renormalized dielectric function of the lhs of Eqn. (9) is

$$\epsilon^{R}(\mathbf{k},\omega) = \epsilon^{L}(\mathbf{k},\omega)\Big(1-$$
$$4A\lambda_0^2 \int_{\Lambda e^{-r}<q<\Lambda} \frac{d\mathbf{q}d\Omega}{(2\pi)^3}\frac{\epsilon^{(2)}(\mathbf{q},\mathbf{k}-\mathbf{q})}{\epsilon^{L}(\mathbf{q},\Omega)\epsilon^{L}(\mathbf{k}-\mathbf{q},\omega-\Omega)}$$
$$\times\frac{\epsilon^{(2)}(\mathbf{k},-\mathbf{q})}{\epsilon^{L}(\mathbf{k},\omega)\epsilon^{L}(-\mathbf{q},-\Omega)}F(q)\Big) \quad (10)$$

and where the correction to the forcing, $\Delta f^{<}(\mathbf{k},\omega)$, is shown to vanish because of the structure of the nonlinear coupling coefficient in the evolution equation, i.e., the bare vertex $\epsilon^{(2)}$ vanishes (see appendix A of paper *Hamza and Sudan* [1995]).

One can at this stage evaluate the renormalized coefficients as done in *Hamza and Sudan* [1995] to obtain

$$\epsilon_{\mathbf{k}}^{R} = (\omega - \omega_{\mathbf{k}}) + ik^2 D_r \quad (11)$$

where

$$D_r = D_0\left(1 + \frac{3}{64\pi}\left(\frac{\nu_i}{\Omega_i}\right)^2 \frac{v_d^2 \sin^2\theta}{(1+\psi)^4}\right) \quad (12)$$

where the dimensionless coupling constant $\bar{\lambda}_0$ is defined by

$$\bar{\lambda}_0^2 = \frac{A_0\lambda_0^2}{D_0^3\Lambda^{2-m}} \quad (13)$$

The renormalized Eqn. (9) is defined on the domain $0 < k < \Lambda e^{-r}$, unlike the original evolution equation, which is defined on the larger interval $0 < k < \Lambda$.

The next step of the RNG technique is to test the self-similarity of the equation under the transformations:

$$K = ke^{r}\ ,\Omega = \omega e^{\alpha(r)}\ ,n(\hat{k})^{<} = \xi(r)\psi(\hat{K}). \quad (14)$$

Therefore the new variable $\mathbf{K}$ is defined on the same interval $0 < K < \Lambda$ as the vector $\mathbf{k}$ in the original evolution equation. In terms of the new variables, the renormalized equation is self-similar with the following new parameters:

$$\epsilon^{R}(\mathbf{K}) = \Omega - e^{\alpha-r}\omega_{\mathbf{K}_r} + iD(r)K^2,$$
$$f(\mathbf{K}) = f(\mathbf{k})^{<}e^{\alpha(r)}\xi^{-1}(r),$$
$$\lambda(r) = \lambda_0\xi(r)e^{-3r},$$
$$D(r) = D_r e^{\alpha(r)-2r}. \quad (15)$$

On the other hand the correlation function characterizing the force $f_{\mathbf{K}}$, given by the expression (7) can be constructed easily using the original definition and the rescaled variables

$$< f(\mathbf{k},\omega)f(\mathbf{k}',\omega') >= \frac{e^{(3\alpha(r)+(2-m)r)}}{\xi^2}$$
$$\times(2\pi)^3 AK^m\delta(\mathbf{K}+\mathbf{K}')\delta(\Omega+\Omega'). \quad (16)$$

We choose the function $\xi(r)$ such that the amplitude A_0 of the forcing remains unchanged; therefore

$$\xi(r) = e^{\frac{3}{2}\alpha(r)+(\frac{2-m}{2})r}. \quad (17)$$

The different steps so far described are formally exact in the limit $r \to 0$. In order to eliminate a finite band of $\mathbf{k}$ space, one can iterate and therefore eliminate infinitesimally narrow wavenumber bands. The different coupling constants generated depend on r and satisfy the following ordinary differential equations:

$$\frac{dD}{dr} = D(r)(z - 2 + C\bar{\lambda}^2), \tag{18}$$

$$\frac{dA_0}{dr} = 0, \tag{19}$$

$$\frac{d\lambda}{dr} = \lambda(r)\left(\frac{3}{2}z - \frac{m+4}{2}\right), \tag{20}$$

where $d\alpha/dr \equiv z, C \equiv C_0 \sin^2\theta = \frac{3}{32\pi}\left(\frac{\nu_i}{\Omega_i}\right)^2 \frac{v_d^2 \sin^2\theta}{(1+\psi)^4}$, and the dimensionless parameter $\bar{\lambda}$ is defined as

$$\bar{\lambda}^2 = \frac{\lambda^2 A_0}{D^3 \Lambda^{2-m}}. \tag{21}$$

The equation satisfied by $\bar{\lambda}$ is then

$$\frac{d\bar{\lambda}}{dr} = \frac{\bar{\lambda}}{2}(2 - m - 3C\bar{\lambda}^2). \tag{22}$$

The solution to this Eqn. (22) tends to zero when $r \to \infty$ if $m > 2$. On the other hand when $m < 2$, the solution tends to a fixed point $\bar{\lambda}^*$

$$\bar{\lambda} \to \bar{\lambda}^* = \sqrt{\frac{(2-m)}{3C}}, \quad as \quad r \to \infty. \tag{23}$$

The exact solution to Eqn. (22) can be written as follows:

$$\bar{\lambda}(r) = \bar{\lambda}_0 e^{(\frac{2-m}{2})r}\left(1 + \frac{3C}{(2-m)}\bar{\lambda}_0^2(e^{(2-m)r} - 1)\right)^{-1/2}, \tag{24}$$

and therefore the diffusion coefficient can be written as:

$$D(r) = D_0 e^{(z-2)r}\left(1 + \frac{3C}{2-m}\bar{\lambda}_0^2(e^{(2-m)r} - 1)\right)^{1/3}. \tag{25}$$

At the fixed point the diffusion coefficient $D(r)$ becomes r-independent if

$$z = \frac{m+4}{3} \tag{26}$$

from Eqns. (18) and (20).

This now lays down the ground for the calculation of the skewness. Most of the results presented here will be used to obtain the final expression for the skewness. We will need the expression for the fixed point as well as the renormalized expressions for both the renormalized coupling coefficient and the diffusion coefficient.

3 The Skewness

Now that we have prepared the neccessary tools, we shall compute the density-gradient skewness S_n, defined as

$$S_n = \frac{\langle(\frac{\partial n}{\partial x})^3\rangle}{\langle|\frac{\partial n}{\partial x}|^2\rangle^{3/2}} = \frac{A}{B^{3/2}} \tag{27}$$

where

$$A = \langle(\frac{\partial n}{\partial x})^3\rangle =$$

$$-i\int\int \frac{d\hat{q}}{(2\pi)^3}\frac{d\hat{t}}{(2\pi)^3} q_x t_x (k_x - q_x - t_x) n(\hat{q}) n(\hat{t}) n(\hat{k} - \hat{q} - \hat{t}) \tag{28}$$

This expression can now be decomposed according to the RNG scheme as follows

$$A = A^<$$

$$-i\int\int \frac{d\hat{q}}{(2\pi)^3}\frac{d\hat{t}}{(2\pi)^3} q_x t_x (k_x - q_x - t_x)(a+b+c+d+e+f+g) \tag{29}$$

where

$$\begin{aligned}
a &= n(\hat{q})^> n(\hat{t})^< n(\hat{k} - \hat{q} - \hat{t})^> \\
b &= n(\hat{q})^> n(\hat{t})^> n(\hat{k} - \hat{q} - \hat{t})^< \\
c &= n(\hat{q})^< n(\hat{t})^> n(\hat{k} - \hat{q} - \hat{t})^> \\
d &= n(\hat{q})^> n(\hat{t})^< n(\hat{k} - \hat{q} - \hat{t})^< \\
e &= n(\hat{q})^< n(\hat{t})^> n(\hat{k} - \hat{q} - \hat{t})^< \\
f &= n(\hat{q})^< n(\hat{t})^< n(\hat{k} - \hat{q} - \hat{t})^> \\
g &= n(\hat{q})^> n(\hat{t})^> n(\hat{k} - \hat{q} - \hat{t})^>
\end{aligned} \tag{30}$$

At this stage we need to evaluate A explicitly, eliminating the small scale structures from the problem following the RNG procedure. It is tedious, however straightforward to show that one can write the contribution from term (a) in expression (30) in the following form (again some of the details can be found in appendix (A) of the paper by *Hamza and Sudan* [1995] regarding the eleimination of small scales and their effects on the large scales)

$$A_a = -4i\lambda_0 A_0 \int\int \frac{d\hat{q}}{(2\pi)^3}\frac{d\hat{t}}{(2\pi)^3} q^m q_x t_x (k_x - q_x - t_x)$$

$$\Big\{\epsilon^{(2)}(\mathbf{k} - \mathbf{q} - \mathbf{t}, \mathbf{k} - \mathbf{q}) G^{(0)}(\hat{q}) G^{(0)}(-\hat{q}) G^{(0)}(\hat{k} - \hat{q} - \hat{t})$$

$$+ \frac{|\mathbf{q} - \mathbf{t}|^m}{q^m}\epsilon^{(2)}(\mathbf{q}, \mathbf{t}) G^{(0)}(\hat{q}) G^{(0)}(\hat{q} - \hat{t}) G^{(0)}(-\hat{q} - \hat{t})\Big\}$$

$$n(\hat{t})^< n(\hat{k} - \hat{t})^< \tag{31}$$

where

$$G^{(0)}(\hat{k}) = \frac{1}{\epsilon^L(\mathbf{k}, \omega)} = (\omega - \omega_{\mathbf{k}} + i\nu_0 k^2)^{-1}$$

The frequency integrals can be performed readily to give

$$A_a = 2\frac{\lambda_0 A_0}{\nu_0^2}\int \frac{d\hat{t}}{(2\pi)^3} n(\hat{t})^< n(\hat{k} - \hat{t})^<$$

$$\times \int \frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{q^2 + |\mathbf{q} - \mathbf{t}|^2}\left\{q_x(t_x - k_x)(q_x - t_x)|\mathbf{q} - \mathbf{t}|^{m-2}\right.$$

$$\times\epsilon^{(2)}(\mathbf{q},\mathbf{t}) + q_x t_x(k_x - q_x - t_x)q^{m-2}\epsilon^{(2)}(-\mathbf{q}+\mathbf{t},\mathbf{t})\Big\} \quad (32)$$

expressing explicitly the coupling coefficients one can easily verify that the expression (32) goes to zero when $q \to 0$. Expanding in powers of q/t and retaining the first nonvanishing terms in the expansion leads to

$$A_a = \frac{\Lambda^m}{16\pi}\left(\frac{\lambda_0 A_0}{D_0^2}\right)\left(\frac{\nu_i}{\Omega_i}\right)\frac{1-e^{-mr}}{m}\frac{V_d \sin\theta_v}{(1+\psi)^2}$$

$$\int \frac{d\hat{t}}{(2\pi)^3} p_x^2 n(\hat{t})^< n(-\hat{t})^< \quad (33)$$

where θ_v represents the flowangle.
It can easily be shown, using symmetry arguments, that the contributions from the terms a, b, and c to the integral (29) are all equal to A_a, while that of d, e, f, and g vanish to the order kept. Therefore

$$A = A^< + 3A_a$$

The next consists of using the energy conservation law for the system and this enables us to eliminate the spectral dependence of the skewnes. In order to achieve our goal we use the following relation derived by *Hamza and Sudan* [1995].

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle |n|^2\right\rangle = -D_0\int\frac{d\mathbf{k}}{(2\pi)^2}k^2\left\langle |n_{\mathbf{k}}(t)|^2\right\rangle$$

$$+Im\int\frac{d\mathbf{k}}{(2\pi)^2}\left\langle f_{\mathbf{k}} n^*_{\mathbf{k}}\right\rangle \quad (34)$$

where $\left\langle |n|^2\right\rangle = \int \frac{d\mathbf{k}}{(2\pi)^2}\left\langle |n_{\mathbf{k}}|^2\right\rangle$. In the steady state the lhs of Eqn. (34) vanishes and we have a balance between the energy dissipated per unit time per unit volume $\bar{\epsilon}$ and the power provided by the forcing agent. Thus

$$\bar{\epsilon} = D_0\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2}k^2\left\langle |n_{\hat{k}}|^2\right\rangle =$$

$$Im\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2}\left\langle f_{\hat{k}} n^*_{\hat{k}}\right\rangle. \quad (35)$$

If we were now to assume an isotropic spectrum, we can then define an energy rate transfer in the x direction

$$\bar{\epsilon}_x = D_0\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2}k_x^2\left\langle |n_{\hat{k}}|^2\right\rangle = \frac{\bar{\epsilon}}{3} \quad (36)$$

and consequently be able to write expression (33) in a closed form for $m = -2$, that is

$$A = A^< + \frac{1}{16\pi}\left(\frac{\lambda_0 A_0}{\Lambda^2 D_0^2}\right)\bar{\epsilon}C_1\frac{e^{2r}-1}{2} \quad (37)$$

where

$$C_1 = \frac{\nu_i}{\Omega_i}\frac{V_d\sin\theta_v}{(1+\psi)^2} = \sqrt{\frac{32\pi C}{3}}$$

where C was defined earlier.

This allows us to write and use the rescaling ODE's for the fixed point to obtain a ODE describing the evolution of the triple correllation under the group transformation, which we calculate to be

$$A = -\frac{\bar{\epsilon}C_1}{32\pi}\frac{\lambda_0 A_0}{D^3\Lambda^2} \quad (38)$$

On the other hand the calculation of B is straightforward and leads to

$$B = \int\frac{d\hat{t}}{(2\pi)^3}t_x^2 n(\hat{t})n(\hat{k}-\hat{t}) \quad (39)$$

In the limit of $k \to 0$ one can follow the same procedure as for the calculation of A to obtain

$$B = B^< - A_0\int\frac{d\hat{t}}{(2\pi)^3}\frac{t_x^2}{t^2}|G^{(0)}(\hat{t})|^2 \quad (39)$$

performing the frequency and wavevector integrations leads to

$$B = B^< - \frac{A_0}{8\pi D_0}r$$

Using the rescaling arguments one can show that the fixed solution can be written in the following form

$$B = \frac{3}{4}\frac{A_0}{8\pi D} \quad (40)$$

This finally allows us to evaluate the skewness

$$S_n = \frac{A}{B^{3/2}} = -\frac{\bar{\epsilon}C_1}{32\pi}\frac{\lambda_0 A_0}{\Lambda^2 D^3}\left(\frac{32\pi}{3}\right)^{3/2}\frac{D^{3/2}}{A_0^{3/2}} \quad (41)$$

using the results of *Hamza and Sudan* [1995] for the expression of the fixed point (20)-(23), one can eliminate $\bar{\epsilon}$ for the expression for the skewness and express it as a constant

$$S_n = -\frac{16\pi^2}{3\sqrt{3}}g \quad (42)$$

where $g < 1$. For $g = 0.1$ we obtain a skewness $S_n \approx -3$.

4 Summary and Conclusions

In this paper we have presented a calculation of the density-gradient skewness for a turbulence problem, that occurs in the ionospheric E and F regions, using the "Renormalization Group Methods" (RNG). This calculation is subject to many of the same reservations mentioned by *Kraichnan* [1987], for example, concerning the renormalization group methods in general. One has to question the validity of RNG methods near the ultraviolet cutoff. For more details on the limitations of RNG techniques the reader is refered to *Hamza and Sudan* [1995]. As described above, RNG is another perturbation technique with even stronger constraints than the classical perturbation methods dealing with similar problems; indeed the constraint of selfsimiliraty is very robust and may be the exception rather than the rule in most turbulent systems. Moreover, RNG treats only half the problem, namely that of the effects of the small scales on the large scales. It does not address, not at all, the effects of large scales on small scales, and would not be able to do so using a perturbative approach since the large scales contain more energy than the small ones. This removes the possibility of cascading, and it can easily be seen in the predictions of RNG; it predicts a family of power laws linked to the forcing which tries to provide the missed Kolmogorov power law. Since the effects of the small scale turbulence is, to lowest order, independent of whether that turbulence was generated by self-consistent cascade or external forcing so is the renormalized diffusion. Thus RNG provide only half a turbulence theory by modeling the effects of the small scales on the large ones. This is the only half that matters for sub-grid modeling as emphasized by *Hamza and Sudan* [1995].

In this paper we have been able to show that given a set of assumptions and limitations, one can still predict certain transport properties such the skewness. One can try to verify whether the predicted value for the skewness , by RNG, matches that predicted by other turbulence closure techniques. In this specific case, we have been able to show (see *Hamza et al.* [1995]), that one can obtain an expression for the skewness, with the same order of magnitude along with the proper sign, using an "Eddy Damped Quasi-Normal Markovian" closure scheme. This gives us more confidence about the usefulness of RNG methods, given their limitations. On the other hand the value quoted above, and obtained via RNG is very close to what radar data seems to suggest (see for example *Schlegel, Thomas and Ridge* [1986]). The present calculation suggests that the model used, though severe assumptions were imposed, retains the physics relevant to Farley-Buneman fluctuations and their evolution.

Acknowledgements. This work was supported by the Canadian Space Agency.

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