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The quasi-static approximation of the spring-slider motion

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Abstract. The spring-slider is a simple dynamical system consisting in a massive block sliding with friction and pulled through a spring at a given velocity. Understanding the block motion is fundamental for studying more complex phenomena of frictional sliding, such as the seismogenic fault motion. We analyze the dynamical properties of the system, subject to rate- and state-dependent friction laws and forced at a constant load velocity. In particular we study the limits within which the quasi-static model can be used. The latter model approximates the *complete model* of the system without taking into account the inertia effects. The system parameters are here found to be grouped into three characteristic times of the three dynamics present in the complete model. A necessary condition for the quasi-static approximation to hold is that the characteristic time of the inertial equation is much smaller than the other two characteristic times. We have studied a modification of one of the classical forms of the rate- and state-dependent friction laws. Subsequently we have developed a linear analysis in the neighborhood of the equilibrium point of the system. For the quasi-static model we rigorously found, by means of a nonlinear analysis, a supercritical Hopf bifurcation, a dynamical property of the complete model. The classical form of the friction laws can be obtained as a particular case of the one we considered, but fails to preserve the Hopf bifurcation in the quasi-static approximation. We conclude that to have a good quasi-static approximation of the system, even in nonlinear conditions, the form of the friction laws considered is a critical factor.

1 Introduction

In the last three decades sliding with friction has become a theme of increasing interest for empirical and theoretical studies. Friction plays a major role in the relative motion of rock walls along pre-existing faults. At the same time the earthquake instability sequences were often related to stick-slip sequences observed in frictional experiments (e.g. Tse and Rice, 1986; Cao and Aki, 1986; Carlson and Langer, 1989; Okubo, 1989; Miyatake, 1992; Rice, 1993; Cochard and Madariaga, 1994; Dieterich, 1994). The spatially dependent solutions to the fault motion problem when complex friction laws are involved can be better understood knowing the dynamics of a simpler one degree of freedom model: the spring-slider, *i.e.* a massive block subject to friction and pulled through a spring at a given velocity (load). Among the most commonly used friction laws there are the so-called "rate- and state-dependent laws" with several formal expressions. One of the laws first proposed by Ruina (1980) is among the most frequent formalizations of the rate- and state-dependent laws (e.g. Rice and Ruina, 1983; Cao and Aki, 1986; Rice and Tse, 1986; Tse and Rice, 1986; Okubo, 1989; Gu and Wong, 1991; Linker and Dieterich, 1992; Miyatake, 1992, Rice, 1993) and will be referred to as the RR law hereafter. The rate- and state-dependent frictional laws imply two different responses to a sudden velocity variation: an instantaneous (direct) effect and a delayed one (evolving effect). The latter consists in a slow variation of friction where memory of the recent sliding history is kept for a while (fading memory). The evolving effect is associated to the change of the "state" of the sliding surface which cannot be instantaneous since the population of contacts needs a finite sliding distance to vary (e.g. Dieterich and Kilgore, 1994).

The equilibrium state of a spring-slider subject to RR friction laws is accomplished when the block slides at the load velocity. A linear stability analysis in the neighborhood of this state was effected by Rice and Ruina (1983) and Gu et al. (1984) assuming, respectively, a massive (dynamic model) or a massless sliding block (quasi-static model). Further studies considered the proper

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choice of the parameter values on the basis of physical criteria in order to effect the numerical simulation of the dynamical system (Rice and Tse, 1986; Gu and Wong, 1991). Less interest was devoted to the theoretical fundamentals, important for a correct interpretation of the computational results.

The quasi-static model was often used as an approximation of the dynamic model (Rice and Gu, 1983; Gu et al., 1984; Rice and Tse, 1986), also referred here as the *complete model*. The quasi static model assumes the velocity to respond instantaneously, as a dependent variable, to the dynamics of the remaining variables of the system. In fact, a noticeable feature of the complete model is that the dynamic of its components is frequently so strongly differentiated, that the fastest variables, such as velocity, can be considered instantaneous. In these cases the quasi-static approximation can be used. We shall see that the "smallness" of the time constant characterizing the velocity equation with respect to the other time constants is a necessary condition for the quasi-static approximation to hold. The values usually considered for the system parameters fulfil this condition. Moreover, the quasi-static model is not concerned with the computational problems due to the logarithmic, short-term and long-term friction dependence on velocity in the RR law (*overflows* at low velocity values). In fact, the velocity is intrinsically prevented from vanishing by the quasi-static model, as we will show.

On the other hand, the nonlinear analysis of the quasi-static model(Gu et al., 1984) showed that when the RR friction laws are used, "at large" (*i.e.* starting quite far from the equilibrium point) an instability occurs for whatever choice of the parameters and "a stable limit cycle never occurs" (Horowitz and Ruina, 1989). On the contrary, the existence of a stable limit cycle for a particular choice of the dynamic system parameters is pointed out by several authors (e.g. Rice and Ruina, 1983; Horowitz and Ruina, 1989). This difference between the quasi-static model and the dynamic model clearly reduces the validity of the first model as an approximation of the second, at least when the original form of the RR friction laws is used.

In the following paragraphs we will show some dynamic, rigorously deduced, properties of the spring-slider motion using a modified state evolution equation. The RR friction laws can be straightforwardly obtained and studied as a particular case of the laws proposed here. Our results are original in that we propose an original modification of the state equation form previously used in other studies. A different modification of the RR state equation was previously used by Horowitz and Ruina (1989) in the numerical simulation of the quasi-static system, to have a stable limit cycle. We rigorously show in the following paragraphs that the state equation here proposed expands the validity of the quasi-static approximation, both preserving the whole set of dynamic characteristics of the inertial system and keeping the numerical implementation feasible.

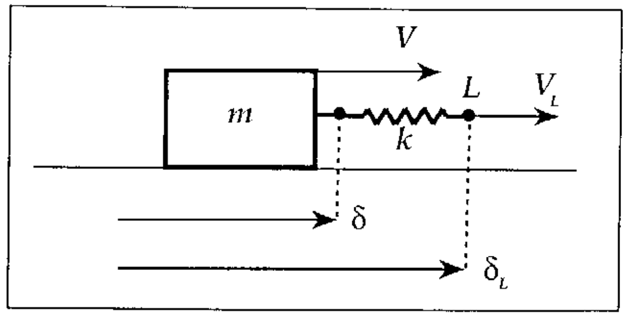

Fig. 1. Schematic representation of a spring-slider.

Finally, a specification is necessary: in the literature the term "state variable" is used implicitly for the variable representing the "state of the sliding surface". We will use the term "state" with this same meaning, and care must be taken to avoid a possible misunderstanding between the surface state variable and the variables defining the "state of the system" from the point of view of the dynamical systems theory. Actually, the surface state variable is only one of the system state variables. In the following paragraphs the three dynamics involved in the spring-slider model are characterized and a linear analysis is presented both for the dynamic complete system and the quasi-static system. A nonlinear analysis of the quasi-static system is finally effected.

2 The spring-slider model

Let us consider (Fig. 1) a block with mass m per unit of surface, sliding with friction τ_a at a velocity V since it is pulled through a spring, the tip of which moves at a velocity V_L. The motion equation is then

$$m\ddot{\delta} = m\dot{V} = \tau_e - \tau_a \tag{1}$$

where the dot means differentiation with respect to time, δ is the slider displacement and τ_e is the elastic traction exerted through the spring on the slider (see also Table 1).

In the Rice-Ruina model τ_e is assumed to depend linearly on the spring length variation $\delta_L - \delta$

$$\tau_e = k(\delta_L - \delta) \tag{2}$$

where δ_L is the loading point displacement, and k is the elastic stiffness of the spring. Friction depends on both the velocity and the state variable θ, *i.e.* on the recent slip-rate history according to the following equations

$$\begin{cases} \tau_a(V,\theta) = \tau_* + \theta + A\ln(V/V_*) \\ \dot{\theta} = -\dfrac{1}{L}\dfrac{V}{1+h|V|}[\theta + B\ln(V/V_*)], \end{cases} \tag{3}$$

where the parameters A, B, L, τ_*, h and V_* can be determined on the basis of physical considerations. The

RR laws can be obtained assuming $h = 0$. The factor containing h limits the order of infinity of $\dot\theta$ for $V \to \infty$ so that the choice $h > 0$ seems to be more reasonable than $h = 0$, also from a physical point of view. We recall that bounded time derivatives of the state with respect to V were often employed as a plausible alternative to the RR state evolution equation (see e.g. Linker and Dieterich, 1992). However, this modification of the RR laws tends to be negligible for $V \to 0$. The system variables involved in the set of equations (1-3) are τ_e, V, θ. If we assume a constant load point velocity V_L we have only one equilibrium point

$$\begin{cases} \bar\tau_e = \tau_* + (A - B)\ln(V_L/V_*) \\ \bar V = V_L \\ \bar\theta = -B\ln(V_L/V_*). \end{cases} \quad (4)$$

Assuming the following nondimensional variables in order to have the equilibrium point in the origin (Belardinelli, 1994)

$$\begin{cases} x \equiv \dfrac{V - \bar V}{\bar V} \\ y \equiv \dfrac{\tau_e - \bar\tau_e}{A} \\ z \equiv \dfrac{\theta - \bar\theta}{A}, \end{cases} \quad (5)$$

we obtain the following equations

$$\begin{cases} \dot x = \dfrac{1}{T_1}[y - z - \ln(x+1)] & (6a) \\ \dot y = -\dfrac{1}{T_2}x & (6b) \\ \dot z = -\dfrac{1}{T_3}\dfrac{x+1}{1 + hV_L|x+1|}[z + R\ln(x+1)], & (6c) \end{cases}$$

where

$$R \equiv \frac{B}{A}, \quad T_1 \equiv m\frac{V_L}{A}, \quad T_2 \equiv \frac{A}{kV_L}, \quad T_3 \equiv \frac{L}{V_L}. \quad (7)$$

The parameters T_1, T_2, T_3 have time dimensions and suggest the speed of the three dynamics involved. Smaller values of V_L yield larger values of T_2 and T_3. To have an estimate of the magnitude of the characteristic times T_i, $i = 1,..3$, as far as a fault motion is concerned, we may consider the following mean order of magnitude for the model parameters (Gu and Wong, 1991)

$$A \sim 1\ \text{MPa}, \quad k \sim 10\ \text{MPa/m}, \quad m \sim 10^7\ \text{kg/m}^2. \quad (8)$$

By using two different values for V_L and L (one suitable for fault conditions and the other for experimental conditions) we obtain from definition (7) the results reported in Table 2.

According to Table 2, $T_1 << T_2$, T_3 and the solution of the *complete* set of equations (6) can be approximated with the solution of the following *reduced* one

$$\begin{cases} \dot y = -\dfrac{1}{T_2}(e^{(y-z)} - 1) & (9a) \\ \dot z = \dfrac{1}{T_3}\dfrac{1}{e^{(z-y)} + hV_L}[z + R(y - z)], & (9b) \\ x = e^{(y-z)} - 1 & (9c) \end{cases}$$

obtained from (6) with $T_1\dot x \to 0$. The previous set of equations represents the so-called "quasi-static" problem usually adopted when m is very small. Finally, we may note that Eq. (9c) and the x definition (Eq. 5) imply that for the quasi-static problem the velocity is always nonnegative; in fact $V = V_L e^{(y-z)}$. This avoids the problem of the singularity for $V = -1/h$, $h > 0$ (Eq. 3) in the state evolution.

Table 1. Symbols

m	slider mass per surface unit
$\tau_a(V, \theta; A, B, L, \tau_*, V_*)$	friction traction
τ_e	elastic traction
δ_L	loading point displacement
$V_L = \dot\delta_L$	loading point velocity
δ	slider displacement
$V = \dot\delta$	slider velocity
$\bar\tau_e$, $\bar V$, $\bar\theta$	equilibrium values
k	spring stiffness
θ	state variable
$R = B/A$	friction parameter
T_1, T_2, T_3	characteristic times
x	nondimensional velocity
y	nondimensional elastic traction
z	nondimensional state
$\lambda_{1,2}$	complex conjugate eigenvalues
T	period at critical conditions

3 Linear stability analysis

Given a set of differential equations like Eq. (6) with an equilibrium point, we define "subcritical" the condition on the system parameters which makes the equilibrium point stable and "supercritical" the condition which makes it unstable. Defining $\phi \equiv T_3'/T_1/(1 + T_1/T_3')$ with $T_3' \equiv T_3(1 + hV_L)$, it can be shown that the complete set of equations (6) for the dynamic system is

Table 2. Time estimates for V_1 = 1 mm/a, L_1 = 10 mm ("fault" values), V_2 = 10 m/a, L_2 = 10 μm ("laboratory" values).

V_L	L	T_1 (s)	T_2 (s)	T_3 (s)
V_1	L_1	3×10^{-10}	3×10^9	3×10^8
"	L_2	"	"	3×10^5
V_2	L_1	3×10^{-6}	3×10^5	3×10^4
"	L_2	"	"	3×10^1

stable in the neighborhood of the equilibrium point if

$$R - 1 < \phi \quad \text{(subcriticality)} \tag{10}$$

and unstable if

$$R - 1 > \phi \quad \text{(supercriticality)}. \tag{11}$$

The following condition will be referred to as the critical condition on the system parameter values

$$R - 1 = \phi \quad \text{(criticality)}. \tag{12}$$

If the latter condition is fulfilled it can be shown that one of the eigenvalues of the linearized system is real and negative, and the other two are imaginary and given by

$$\lambda_{1,2} = \pm i\frac{2\pi}{T},\ T \equiv 2\pi\sqrt{T_2 T_3'}\sqrt{1 + T_1/T_3'}. \tag{13}$$

It is easy to show that these results for $h = 0$ reproduce those found by Rice and Ruina (1983) by using the T_i, $i = 1, 2, 3$ definitions (7), but it is important to stress that the stability of the equilibrium point can change by varying each of the parameters k, A, R, L, h ($h > 0$) and not only k as is usually done.
As shown in the appendix (points *i* and *ii*) the equilibrium point of the linearized quasi-static model (9), can be a node or a focus. The equilibrium point is stable if

$$R - 1 < \psi, \tag{14}$$

where $\psi = T_3'/T_2$ and unstable if

$$R - 1 > \psi. \tag{15}$$

These results are summarized in Fig. 2. The eigenvalues of the system when the critical condition is fulfilled

$$R - 1 = \psi, \tag{16}$$

are imaginary and given by

$$\lambda_{1,2} = \pm i\frac{2\pi}{T},\ T = 2\pi\sqrt{T_2 T_3'}. \tag{17}$$

As expected, these results for the quasi-static system can be obtained from the stability analysis of the dynamical system in the limit $T_1 \to 0$ (cfr. Eqs. 10-13).

4 Nonlinear analysis of the quasi-static model

In the previous paragraph we saw that the quasi-static system can be assumed as the limit $T_1\dot{x} \to 0$ of the dynamic system as far as a linear stability analysis is concerned (*i.e.* "at small"). On the basis of the nonlinear analysis for the quasi-static system in the case $h = 0$ (Gu et al., 1984), there are always points that are sufficiently far away from the origin, from which diverging motions develop. For the corresponding dynamical model a supercritical Hopf bifurcation and therefore a stable limit cycle have been computationally found by most authors (e.g. Rice and Tse, 1986). The value of T in (11) is a first approximation estimate of the period on the limit cycle. Accordingly, for $h = 0$, the dynamic and quasi-static models cannot be considered equivalent "at large". On the contrary we will show that a positive value of h deeply modifies the response of the quasi-static model enabling the existence of a stable limit cycle. In this way the quasi-static model becomes a better approximation of the corresponding dynamic model.

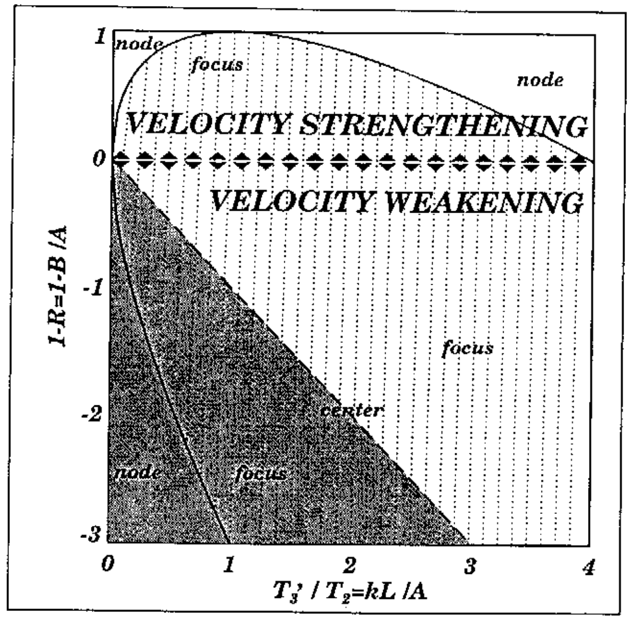

Fig. 2. Parameters plane for the linearized system (9) and instability (dark) and stability (light) domains. $T_3' = T_3(1 + hV_L)$ is the characteristic time of the z evolution in the linearized set of equations. The arrows bound the *velocity weakening* area ($B > A$) from the *velocity strengthening* ($B < A$). The discriminant Δ of the characteristic equation (A1) is negative in the dashed region, and the equilibrium point is a focus. The parabola in solid line is the locus $\Delta = 0$ and the origin is a node in the region complementary to the shaded one. The anti-bisector dashed line is the locus where the origin is a center.

For a Hopf bifurcation to occur in a two-dimensional system, such as (6), with an equilibrium state, we have first to verify that for a given set of parameter values the linearized system has pure imaginary eigenvalues in the equilibrium point ("the critical condition") and that during the monotonic variation of a single parameter through the critical value (determined by the fixed values of the other parameters), the real part of the eigenvalues changes sign. These two conditions enable us to state only the existence of periodic orbits (Farkas, 1994, pp. 417-418) and not the existence of an Hopf bifurcation, as often stated in literature. To completely state the existence and the kind of an Hopf bifurcation it is sufficient (Perko, 1991, p. 317) to prove that in critical conditions the origin is asymptotically stable (supercritical bifurcation) or unstable (subcritical bifurcation).
As shown in the appendix (point *i*) the real part of the

eigenvalues vanishes under condition (16), and if we consider the passage from the subcritical to the supercritical conditions (Eqs. 14-15) with a single parameter varying, it is easy to show that the real part of the eigenvalues changes sign. In order to discuss the asymptotic stability of the origin in critical conditions we can refer (Perko, 1991, pg. 315-317) to the sign of the Liapunov number of the third order approximating system. The latter system can be obtained from Eqs. 9a-b, by expanding the second members up to third order in the Taylor expansion in the neighborhood of the origin. The Liapunov number is given as the following function of the parameter h

$$\sigma(h) = -\frac{3\pi}{4} h V_L \frac{3 + 5hV_L}{(1 + hV_L)^3}. \tag{18}$$

If the Liapunov number of the third order approximating system is greater than 0, then the Hopf bifurcation is subcritical, if it is less than 0 then the bifurcation is supercritical. We cannot state anything on the basis of this criterion if the Liapunov number is vanishing. From (18) we can see that for $h > 0$ σ is always negative; therefore, we have a supercritical bifurcation and, consequently, a stable limit cycle (Fig. 3).

In the case $h = 0$ of the RR laws the Liapunov number is vanishing and the critical condition has to be studied in another way. The origin is a center as shown in the appendix (point *iii*) and therefore the third condition of the Hopf bifurcation theorem is not fulfilled.

If the $h = 0$ quasi-static system had a supercritical Hopf bifurcation, it would exhibit a stable limit cycle when the origin is unstable (Eq. 15). But Gu et al. (1984) demonstrated that in this case the quasi-static model is globally unstable. Thus a stable limit cycle does not exist; therefore a supercritical Hopf bifurcation does not exist.

Finally, another difference between the quasi-static and dynamic model for $h = 0$ can be stated. The dynamic system motion is, at least asymptotically, independent of the initial position; the quasi-static system in critical conditions, on the contrary, exhibits a strong dependence on the initial conditions, as hinted also by Horowitz and Ruina (1989). In fact, by definition, the trajectories characterizing the center depend on the initial condition. By the way, for the latter reason, even if they are periodic, the trajectories of the quasi-static system in critical conditions cannot be associated to a limit cycle as they could easily be misinterpreted.

5 Conclusive remarks

According to the numerical simulation (e.g. Rice and Tse, 1986; Gu and Wong, 1991) the dynamical (or complete) model (6) of a spring-slider motion under rate- and state-dependent friction exhibits a supercritical Hopf bifurcation. Therefore a stable limit cycle necessarily occurs for specific parameter values. A smaller order model can approximate the behaviour of the third order, complete system, only if it preserves this fundamental property.

In the previous paragraphs we dealt with the properties of such a dynamical system. We introduced nondimensional variables that allows for the definition of three characteristic times relative to the three independent dynamics of this system: sliding velocity, state of the surface, elastic traction. A linear analysis in the neighborhood of the equilibrium point shows that a Hopf transition is likely for variation of each of the parameters across the critical condition and not only the stiffness of the spring k, as is usually reported. In the second paragraph we state a general necessary condition for the use of a second order model as an approximation of the complete model. The approximating model is the so-called quasi-static model and is obtained by supposing the inertial dynamics to be instantaneous.

As shown by previously made numerical simulations, if the RR frictional laws are not modified, the quasi-static model does not preserve the property of the existence of a stable limit cycle. In particular, we proved that in critical conditions the equilibrium point is a center and thus is not asymptotically stable and shows a strong dependence on the initial conditions. In the literature, on the contrary, a Hopf bifurcation was often incorrectly attributed to the quasi-static model with RR laws. Since the quasi-static model lacks inertia effects, as far as the RR frictional laws are concerned, the existence of a stable limit cycle for the complete model can be explained with the dynamic effect of the mass according to the arguments just given. Therefore the effect of the mass can be regarded as strictly stabilizing, the stable limit cycle being a system attractor by definition. In particular it is worth stressing that in a global analysis of the system it is not correct to extrapolate a result of the linear analysis such as "mass is always destabilizing" (Rice and Ruina, 1983).

However, the quasi-static model has noticeable computational advantages with respect to the complete system, besides that of being of a lower order. We proved, in fact, in the second paragraph that in the quasi-static model the sliding velocity does not vanish or change sign, and thus overflows linked to the logarithmic dependence on velocity of the rate- and state- dependent laws are prevented. Moreover, we saw that the parameter values of geophysical interest fulfil the general condition necessary for the quasi-static model to be a good approximation of the complete model.

The particular form of the state equation according to the RR laws is likely to be a destabilizing factor for the quasi-static model. A suitable modification of the state equation with respect to the RR laws gives rise to a supercritical Hopf bifurcation and in particular a stable limit cycle. A similar modification can be made

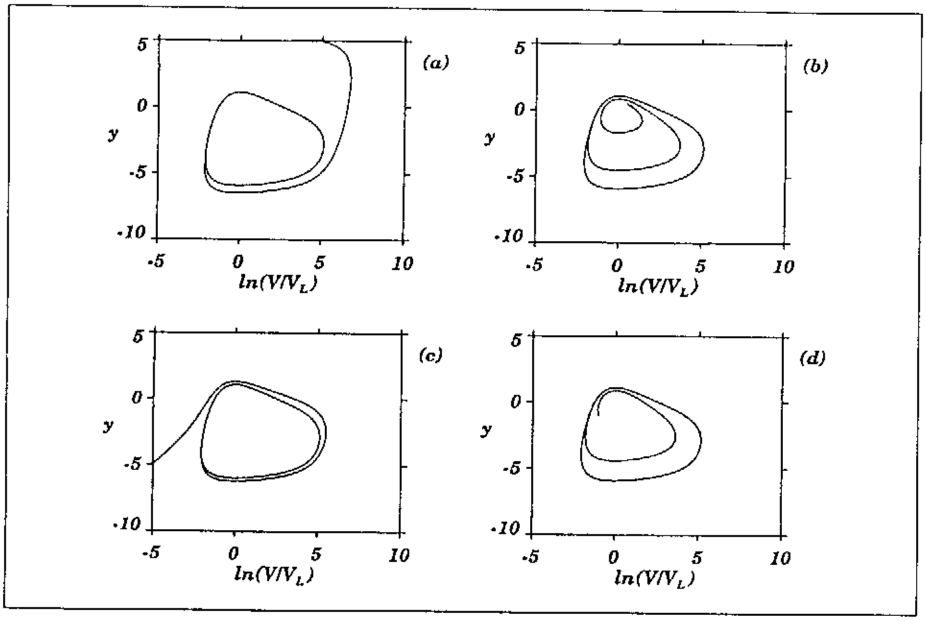

Fig. 3. Trajectories in the phase plane y versus $\ln(V/V_L)$ of the quasi-static system for $R = 2.04$, $hV_L = 0.01$ and $T_2 = T_3 = 1.1 \times 10^3$s. A limit cycle stable both externally (a, c) and internally (b, d) is evident. Greater values of the ratio T_2/T_3 reduce the transient part of trajectories tending to the limit cycle.

on the complete system without changing the essential properties (limit cycle) of the system.
In summary, the analytical results we showed in this paper indicate that: 1) the quasi-static model with unchanged RR laws is equivalent to the complete model only as far the LINEAR dynamical properties are concerned, 2) by modification of the frictional laws it is possible to obtain a quasi-static model equivalent to the complete model in a more general sense (*i.e.* "at small" and "at large").

Appendix

In this appendix *i*) we obtain the stability condition for the origin in the quasi-static system (9), *ii*) classify the origin as a critical point (Fig. 2), *iii*) prove that the origin is a center in critical conditions for $h = 0$.

i)
The characteristic polynomial of the linearized set of equations obtained from (9a) and (9b) is

$$p(\lambda) = \lambda^2 + a\lambda + b$$

$$\text{where } a \equiv 1/T_2 + (1 - R)/[(1 + hV_L)T_3], \qquad (A1)$$

$$b \equiv R/[T_2T_3(1 + hV_L)].$$

Since the eigenvalues are solutions of $p(\lambda) = 0$, we have

$$\lambda_{1,\,2} = (-a \pm \Delta^{1/2})/2, \quad \Delta = a^2 - 4b \qquad (A2)$$

and $a > 0$ is the necessary and sufficient condition for the origin stability, *i.e.* (14) in the text; analogously (15) is the necessary and sufficient condition for the origin instability.

ii)
The analysis of the discriminant Δ of the characteristic equation enables us to classify the character of the origin as in Fig. 2. Since in (A1) $b > 0$ for $h > 0$, we have a node (a focus) in the origin if $\Delta > 0$ ($\Delta < 0$). The locus $\Delta = 0$ is a parabola in the plane $\alpha \equiv T_3(1 + hV_L)/T_2$, $\beta \equiv 1 - R$

$$\alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha = 0. \qquad (A3)$$

iii)
The solution $y(t)$, $z(t)$ of the set of equations (9a) and (9b) for $h = 0$ represents a point of the family of trajectories with locus $F(y, z) =$ cost, where (Minorsky, 1962, pg. 33) F is such that

$$Y\frac{\partial F}{\partial y} + Z\frac{\partial F}{\partial z} = 0 \qquad (A4)$$

$$\text{with} \quad Y(y,z) \equiv -\frac{1}{T_1}(e^{(y-z)} - 1)), \qquad (A5)$$

$$Z(y,z) \equiv \frac{1}{T_3}\frac{z + R(y - z)}{e^{(z-y)} + hV_L}. \qquad (A6)$$

We solve Eq. (A4) by subsequent approximations, expanding Y, Z and F in a series of powers of y and z: $Y = \sum_{i=1}^{\infty} Y_i(y,z)$, $Z = \sum_{i=1}^{\infty} Z_i(y,z)$, $F = \sum_{i=1}^{\infty} F_i(y,z)$, where Y_i, Z_i, F_i, $i = 1, 2, \ldots$ are homogeneous polynomials of order i in y and z. At the first order in the power expansion we may write Eq.(A4) in the form

$$Y_1\gamma + Z_1\epsilon = 0 \qquad (A7)$$

where we put $F_1 \equiv \gamma y + \epsilon z$. Equation (A5) owing to the polynomials identity principle is equivalent to a homogeneous set of linear equations for α, β, with not vanishing determinant $D = (T_2T_3)^{-1}$, and then with only the vanishing solution. Thus we have $F_1 = 0$. We now consider Eq. (A4) at the second order

$$Y_1(\chi y + \xi z) + Z_1(\xi y + \mu z) = 0 \qquad (A8)$$

where we put $F_2 = \chi y^2 + 2\xi yz + \mu z^2$. Owing to the polynomials identity principle Eq. (A8) is equivalent to a homogeneous set of linear equations for χ, ξ, μ with a vanishing determinant if the critical condition (16) holds. Accordingly, its solution is a one-dimensional subspace of $\Re^3$ parameterized by the value of ξ, for instance: $\chi = -RT_2/T_3\xi$, $\mu = -\xi$. We finally have

$$F_2 = -\xi T_2 \left(\frac{R}{T_3} y^2 - \frac{2}{T_2} yz + \frac{1}{T_2} z^2 \right). \qquad (A9)$$

The previous one, whereby equated to an arbitrary constant, and owing to the condition (16), is the equation of an ellipses family surrounding the origin, *i.e.* a family of closed curves. Thus we proved that for each point in the neighborhood of the origin, we have a closed trajectory surrounding it; this is sufficient for asserting that the origin is a center for the quasi-static system (9) for $h = 0$ and in critical conditions. This result was obtained considering the trajectories embedded in such a small neighborhood of the origin that we can neglect terms of order greater than the second order in the expansion of the trajectory locus F =cost. Nevertheless, the former analysis is nonlinear since we took into account the original form of the system equations (Eq. 9) and not its linearized form. This is important since the existence of imaginary eigenvalues in a linear analysis enables us only to assert that the corresponding equilibrium state is a center or a focus, stable or unstable (Perko, 1991, p. 142). Only nonlinear analysis can decide among these alternatives.

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