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A centre manifold approach to solitary internal waves in a sheared, stably stratified fluid layer

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Abstract. The centre manifold approach is used to derive an approximate equation for nonlinear waves propagating in a sheared, stably stratified fluid layer. The evolution equation matches limiting forms derived by other methods, including the inviscid, long wave approximation leading to the Korteweg-deVries equation. The model given here allows large modulations of the height of the waveguide. This permits the crude modelling of shear layer instabilities at the upper material surface of the waveguide which excite solitary internal waves in the waveguide. An energy argument is used to support the existence of these waves.

1 Introduction

Much interest has focused on solitary internal waves in stable stratifications ever since the weak nonlinear, long wave analyses of Benney (1966) and Benjamin (1966). The former used a singular perturbation expansion of the infinitely long wave limit with vertical modal structure given by the solution to the Taylor-Goldstein eigenvalue problem. Including a more detailed physical description (viscosity and short wave disturbances) by regular perturbation [Zimmerman and Velarde (1995);(1994a); (1996) and Zimmerman and Rees (1996)] is a cumbersome process for hand calculation. Eventually, further progress in modelling real physical situations will require computational fluid dynamics methods even to produce accurate 1-D wave evolution models. It has been frequently commented that incorporating all the physical mechanisms in a 1-D nonlinear evolution equation will be no easier to solve than the full Navier-Stokes equations. It is true that 1-D models present an enormous simplification in the dimensionality of the problem, but at a cost of analytical complexity that may lead to diminishing returns in incorporating realistic model conditions for meteorological and oceanographic conditions.

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Nevertheless, 1-D models have not yet outlived their utility. For instance, consideration of high shear waveguides leads to the possibility of a propagating chaotic wave [Zimmerman and Velarde (1994a);(1994b)] in vertical velocity stimulated by a passing decoupled temperature wave. This qualitatively describes some observed solitary wave propagation over an Antarctic ice shelf Rees and Rottman (1994). In this paper, an approximate nonlinear evolution equation for solitary internal waves in a viscous fluid layer with stable stratification and shear is derived by a centre manifold technique [Roberts (1994) and references therein]. The centre manifold approach permits large modifications to the waveguide height and thus the possibility of disturbances from other layers exciting waves in the waveguide, providing sufficient energy to overcome dissipative losses. Further, centre manifold approximation does not require the basic disturbance to be inviscid, of small amplitude, or a long wave. The error in the approximation diminishes exponentially with time in a viscous fluid.

2 Model system

2.1 Equations and scaling

Consider a fluid layer bounded above and below by parallel stream surfaces. The height of the layer is nominally h. The upper material surface is constrained to move at velocity U and is held at constant temperature $T = 1$. The lower is held at $T = 0$. The fluid has viscosity μ and thermal diffusivity α. All lengths are scaled by h, velocity by U, and time by h/U. The flow is assumed to be two-dimensional and divergence free, thus able to be expressed in terms of a streamfunction, $(u, v) = (\psi_y, -\psi_x)$. Scalar transport is modelled by this dimensionless convective-diffusion equation

$$T_t + \psi_y T_x - \psi_x T_y = \frac{\delta}{\mathrm{Pr}}\left(T_{xx} + T_{yy}\right) \tag{1}$$

A Boussinesq fluid is assumed, with density linearly related to temperature. The dimensionless density is $\rho = \beta T + 1$, where β is the dimensionless group formed from the coefficient of thermal expansion β^*, the unscaled temperature difference between the upper and lower surfaces ΔT, and ρ_0, the density of the fluid at the reference temperature of the lower surface. Namely, $\beta = \beta^* \Delta T/\rho_0$. $\delta = \mu/\rho_0 U h$ is the inverse Reynold's number. The Prandtl number $Pr = \mu/\rho_0\alpha$ is the ratio of kinematic viscosity to thermal diffusivity.

Vorticity $\omega = -\nabla^2\psi$ is transported by

$$\begin{aligned}&[\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy}]_y + [\psi_{xt} + \psi_y\psi_{xx} - \psi_x\psi_{xy}]_x \\ &+ RiT_x = \delta\nabla^4\psi\end{aligned} \tag{2}$$

which is the curl of the Boussinesq momentum equation. The Richardson number is a measure of the relative importance of buoyant forces to shear forces, $Ri = \beta g h/U^2$. The assumption of a Boussinesq fluid is that the velocity field is divergence-free, that $\beta \equiv 0$, but the Ri remains finite. Stress-free boundary conditions are imposed:

$$\begin{aligned}&\psi|_{y=0} = T|_{y=0} = \psi_{yy}|_{y=0,\eta} = 0 \\ &\psi|_{y=\eta} = T|_{y=\eta} = 1\end{aligned} \tag{3}$$

Because there is no dynamic boundary condition, pressure was eliminated, resulting in the momentum equation (2). Although the nominal height of the waveguide is h, a centre manifold approach allows the dimensionless height $\eta = \eta^*/h$ to be smoothly changed by even an $O(1)$ amount as long as the change is slow. (see Roberts (1994)).

2.2 The wave evolution equation

Generally, the flow and temperature fields can be arbitrarily decomposed into background fields and disturbance fields. To make the procedure deterministic, the assumption is made that the disturbance field has zero mean in some sense. The sense adopted here is that the background fields are time-averaged and thus steady. Further, the background fields are prescribed and independent of the horizontal x-coordinate.

$$\begin{aligned}&T = \hat{T}(y) + \varepsilon\Theta(x,y,t) \\ &\psi = \hat{\psi}(y) + \varepsilon\Psi(x,y,t)\end{aligned} \tag{4}$$

$\varepsilon = A_0/h$ is the dimensionless characteristic size of the disturbance fields, where A_0 is the maximum value of the initial disturbance. For simplicity, constant buoyancy frequency is assumed, $\hat{T} = y$. Since it is the long time waveform which is required, and it is expected that that the form of solution will be asymptotically steady in a the frame of reference of the phase velocity, the following coordinate transform is made:

$$\begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} \xi = x - ct \\ \tau = t \end{pmatrix}; \qquad \begin{pmatrix} \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi} \end{pmatrix} \tag{5}$$

with $\frac{\partial}{\partial\tau} = 0$ imposed at the outset since only steady waves are considered in this section. Substituting (4) into (1) and (2) yields

$$\begin{aligned}&\Theta_\xi(\hat{u} - c) - \Psi_\xi = \varepsilon(-\Theta_\tau + \Psi_\xi\Theta_y - \Psi_y\Theta_\xi) \\ &+ \frac{\delta}{\mathrm{Pr}}(\Theta_{\xi\xi} + \Theta_{yy}) \\ &[\Psi_{y\xi}(\hat{u} - c) - \hat{u}_y\Psi_\xi]_y + Ri\Theta_\xi \\ &+ [\Psi_{\xi\xi}(\hat{u} - c)]_\xi = \varepsilon(-\Psi_{y\tau} - \Psi_{\xi\tau}) \\ &+ \varepsilon\left([\Psi_\xi\Psi_{yy} - \Psi_y\Psi_{\xi y}]_y + [\Psi_\xi\Psi_{\xi y} - \Psi_y\Psi_{\xi\xi}]_\xi\right) + \\ &\delta(\hat{u}_{yyy} + \Psi_{yyyy} + 2\Psi_{\xi\xi yy} + \Psi_{\xi\xi\xi\xi})\end{aligned} \tag{6}$$

Previous treatments [Benney (1966) Lee and Beardsley (1974) Maslowe and Redekopp (1980) Weidman and Velarde (1992) Zimmerman and Velarde (1995)] begin with the inviscid, vanishing amplitude limit of $\delta = \varepsilon = 0$. This basic disturbance $\Theta^{(0)}, \Psi^{(0)}$ satisfies

$$\begin{aligned}&\Theta^{(0)}_\xi(\hat{u} - c) - \Psi^{(0)}_\xi = 0 \\ &\left[\Psi^{(0)}_{y\xi}(\hat{u} - c) - \hat{u}_y\Psi^{(0)}_\xi\right]_y + Ri\Theta^{(0)}_\xi \\ &+ \left[\Psi^{(0)}_{\xi\xi}(\hat{u} - c)\right]_\xi = 0\end{aligned} \tag{7}$$

$\hat{u} = \hat{\psi}_y$ has been introduced to agree with prior treatments including Davey and Reid (1977). The expectation that the disturbance should be a stationary horizontal wave in the reference frame moving with the phase velocity c at any given height y leads one to seek a solution in the form of the separated variables

$$\begin{aligned}&\Theta^{(0)}(\xi, y) = A(\xi)\theta(y) \\ &\Psi^{(0)}(\xi, y) = A(\xi)\phi(y)\end{aligned} \tag{8}$$

The system (7) then simplifies to

$$\begin{aligned}&A_\xi(\theta(\hat{u} - c) - \phi) = 0 \\ &A_\xi[\phi_y(\hat{u} - c) - \hat{u}_y\phi]_y + Ri\theta A_\xi + A_{\xi\xi\xi}(\hat{u} - c)\phi = 0\end{aligned} \tag{9}$$

The system (9) only has solutions if a separation condition on A is satisfied, namely

$$A_{\xi\xi\xi} + k^2 A_\xi = 0 \tag{10}$$

The separation constant was chosen to clarify that A must be a harmonic wave to leading order. If (10) holds, then the system (9) reduces to

$$\begin{aligned}&\theta = \frac{\phi}{\hat{u} - c} \\ &[\phi_y(\hat{u} - c) - \hat{u}_y\phi]_y - k^2(\hat{u} - c)\phi + Ri\frac{\phi}{\hat{u} - c} = 0\end{aligned} \tag{11}$$

The second equation above is the well known Taylor-Goldstein equation [Taylor (1931); Goldstein (1931)] for the particular case of a constant buoyancy frequency. For prescribed $\hat{u}$ and Ri, (11) is a two point boundary value problem ($\phi|_{y=0} = \phi|_{y=\eta} = 0$) for the eigenvalue c. In the long wave limit and with $\hat{u} = y$, the regular spectrum [Davey and Reid (1977)] of c_n, $n = \pm 1, \pm 2, \ldots$, and the singular spectrum [Maslowe and Redekopp (1980)] ($c_n \in [0,1]$) are known when $Ri > 1/4$. For $0 < Ri < 1/4$, c_n are complex conjugates, implying that one harmonic wave grows without bound and the other decays.

If the solvability condition is developed to $O(\varepsilon)$, Zimmerman and Velarde (1996) found the following modifications to (10):

$$A_{\xi\xi\xi} + k^2 A_\xi = \varepsilon\gamma_1 A A_\xi + \delta\left(-|\gamma_2|A + |\gamma_3|A_{\xi\xi} - |\gamma_4|A_{\xi\xi\xi\xi} + \gamma_0\right) \quad (12)$$

Explicit quadratures are given for the γ_i, depending solely on the eigenfunction ϕ and Ri and Pr. (12) simplifies to the steady Korteweg-de Vries equation in the inviscid case $\delta = 0$. This steady wave equation is valid in a slightly viscous fluid. In the next section, the transient approach to the centre manifold is computed in a fully viscous fluid. The resulting nonlinear evolution equation has the same steady limiting form as (12).

2.3 Centre manifold approach

In the development of the Taylor-Goldstein equation above, there are only small amplitude harmonic waves propagating horizontally if the vertical boundary value problem has non-trivial solutions. Specifically, there exist nontrivial vertical stationary modes θ and ϕ with a given phase velocity c which depends on the wavenumber k and Ri. Perturbation expansions have been developed leading to nonlinear evolution equation for the amplitude $A(\xi,\tau)$ for the steady wave in a weakly viscous fluid [Zimmerman and Velarde (1996)] and for the weakly transient case in and inviscid fluid [Zimmerman and Rees (1996)] with arbitrary wavenumber. In this paper, a centre manifold approach is put forth for the nonlinear evolution. It is based on the normal mode expansion of $\phi(y)$,

$$\phi(y) = \sin(\zeta) + a_2 \sin(2\zeta) + a_3 \sin(3\zeta) + \ldots \quad (13)$$

where $\zeta = \frac{\pi y}{\eta}$. Because of viscosity, these normal modes have temporal eigenvalues: $\frac{-\delta\pi^2}{\eta^2}, \frac{-4\delta\pi^2}{\eta^2}, \frac{-9\delta\pi^2}{\eta^2}, \ldots$ and thus decay with $exp(\frac{-\delta\pi^2\tau}{\eta^2}), exp(\frac{-4\delta\pi^2\tau}{\eta^2}), exp(\frac{-9\delta\pi^2\tau}{\eta^2}), \ldots$. Clearly, very rapidly all but the first normal mode becomes unimportant. There is also a zero eigenvalue associated with variation of the layer height η. This zero eigenvalue is due to the fact that there is a one parameter family of solutions with any layer height η and thus η can be smoothly varied in space and time [Roberts (1994)]. The centre manifold analysis leaves η as a free parameter and fixes contributions from only the slowest decaying vertical modes

$$\phi(y) \approx \sin\zeta \qquad \theta(y) \approx \frac{\sin\zeta}{\hat{u} - c} \quad (14)$$

The momentum equation (2) is then required to be satisfied on weighted average over the layer. The weighting function that gives most satisfactory numerical results is $\sin\zeta$ (private communication with A.J. Roberts). For instance, in the $\delta = \varepsilon = 0$ limit, the momentum equation becomes (10) with

$$k^2 I = \frac{\pi^2 c}{2\eta^2} + Ri\int_0^\eta \frac{\sin^3\zeta}{\hat{u} - c}dy - \int_0^\eta \sin^2\zeta\left(\frac{\pi^2\hat{u}}{\eta^2} + \hat{u}_{yy}\right)dy \quad (15)$$

$$I = \int_0^\eta \sin^2\zeta\,(\hat{u} - c)\,dy$$

(15) is the one mode Rayleigh-Ritz approximation to the horizontal eigenvalue k^2 given the phase velocity c and the Richardson number Ri. Alternatively, once $k(c_n, Ri)$ is known, along any branch c_n the relation can be inverted to find $c_n(k, Ri)$.

The above harmonic, inviscid eigenvalue relation (15) can then be used to guide the approximate viscous, transient evolution of the approach to the centre manifold. Allowing the full $A(x,t)$ dependence in (8), then solving the heat equation (1) for T_x, substituting T_x in the momentum equation (2), and taking the $\sin\zeta$ weighted average over the depth of the layer, developing the equation $O(\varepsilon^2)$, results in a nonlinear evolution equation:

$$\begin{aligned}&\lambda_{0,1}A_t + \lambda_{1,0}A_x + \lambda_{2,1}A_{xxt} + \lambda_{3,0}A_{xxx} = \\ &-\varepsilon\left(\delta r_0 A^2 + r_1 A A_t + r_2 A A_x + \delta r_3 A A_{xx}\right) \\ &+\delta(\lambda_0 A + \lambda_{2,0}A_{xx} + \lambda_{4,0}A_{xxxx} + f)\end{aligned} \quad (16)$$

with the coefficients $\lambda_{i,j}$ and r_i given by

$$\lambda_0 = -\frac{\pi^4}{2\eta^4} - \frac{Ri}{Pr}\int_0^\eta \Big(\frac{\pi^2\sin^3\zeta}{\eta^2\hat{u}(\hat{u}-c)} + \frac{2\pi\hat{u}_y\sin^2\zeta}{\eta\hat{u}(\hat{u}-c)^2} - \frac{2\hat{u}_y^2\sin^3\zeta}{\hat{u}(\hat{u}-c)^3} - \frac{\hat{u}_{yy}\sin^3\zeta}{\hat{u}(\hat{u}-c)^2}\Big)dy \quad (17)$$

$$\lambda_{0,1} = -\frac{\pi^2}{2\eta^2} - Ri\int_0^\eta \frac{\sin^3\zeta}{\hat{u}(\hat{u}-c)}dy$$

$$\lambda_{1,0} = Ri\int_0^\eta \frac{\sin^3\zeta}{\hat{u}}dy - \int_0^\eta \sin^2\zeta\left(\frac{\hat{u}\pi^2}{\eta^2} + \hat{u}_{yy}\right)dy$$

$$\lambda_{2,0} = \frac{\pi^2}{\eta^2} + \frac{Ri}{Pr}\int_0^\eta \frac{\sin^3\zeta}{(\hat{u}-c)\hat{u}}dy$$

$$\lambda_{2,1} = \frac{1}{2}$$

$$\lambda_{3,0} = \int_0^\eta \hat{u}\sin^2\zeta\,dy$$

$$\lambda_{4,0} = \frac{1}{2}$$

$$f = \varepsilon^{-1} \int_0^{\eta} \sin\zeta \hat{u}_{yyy} dy$$

$$r_0 = \frac{Ri}{Pr} \int_0^{\eta} (\hat{u}^{-2} \cos\zeta \sin^3\zeta (\frac{\pi^3}{\eta^3 (\hat{u} - c)} - \frac{2\pi \hat{u}_y^2}{\eta (\hat{u} - c)^3} + \frac{\pi \hat{u}_{yy}}{\eta (\hat{u} - c)^2}) + \cos^2\zeta \sin^2\zeta \frac{2\pi^2}{\eta^2 (\hat{u} - c)^2}) dy \quad (18)$$

$$r_1 = \frac{Ri\pi}{\eta} \int_0^{\eta} \frac{\cos\zeta \sin^3\zeta}{\hat{u}^2 (\hat{u} - c)} dy$$

$$r_2 = r_1 + Ri \int_0^{\eta} \left(\frac{\pi \cos\zeta \sin^3\zeta}{\eta \hat{u} (\hat{u} - c)} - \frac{\sin^4 \hat{u}_y}{\hat{u} (\hat{u} - c)^2} \right) dy$$

$$r_3 = \frac{Ri\pi}{Pr\eta} \int_0^{\eta} \frac{\cos\zeta \sin^3\zeta}{\hat{u}^2 (\hat{u} - c)} dy$$

For any background shear flow $\hat{u}$, phase velocity c, and Ri, the above coefficients of the nonlinear wave equation (16) can be determined. The form of the nonlinear wave equation is similar to that found earlier in the steady-wave limit [Zimmerman and Velarde (1996)] and in the inviscid limit [Zimmerman and Rees (1996)], with the exception that the centre manifold technique easily gives the dependence of the coefficients on modulation of the height of the waveguide. The drawback of the centre manifold approach is that the order of the approximation is unclear, although it improves with time in a viscous fluid.

Strictly, (16), is valid only in the transient approach to the centre manifold, not the full range of t in an initial value problem. It follows that the steady wave equation, using (5) is given by

$$k^2 I A_\xi + I A_{\xi\xi} = - \varepsilon \left(\delta r_0 A^2 + (r_2 - c r_1) A A_\xi + \delta r_3 A A_{\xi\xi} \right) + \delta(\lambda_0 A + \lambda_{2,0} A_{\xi\xi} + \lambda_{4,0} A_{\xi\xi\xi\xi} + f) \quad (19)$$

If terms of $O(\varepsilon\delta)$ are neglected, the the form of (19) is identical to that derived by slightly viscous perturbation theory (12).

3 Solutions to the forced nonlinear wave equation

There are two types of modes for the boundary value problem (11). Regular modes have $c_n > \hat{u}$ for $y \in [0,1]$. Singular modes have $c_n = \hat{u}(y_{crit})$ for a critical height y_{crit}. The analysis given here tacitly assumes that c_n is positive, i.e. the waves move rightward. For regular modes, it is clear that the denominator I in the coefficients λ_i is always negative, regardless of k, Ri, and Pr. Singular modes are inadmissible to this study as they violate the assumption of regularity of $\phi(y)$, permitting the vertical Fourier expansion. This is a common bane of Orr-Sommerfield type analysis.

If transients, viscous and thermal diffusive effects were ignored and only weak dispersion retained, only the accumulation term A_ξ, the dispersive term $A_{\xi\xi\xi}$ and the nonlinear term AA_ξ survive in (16). Regardless of the values of the coefficients (unless zero), this equation is equivalent to the KdV equation and always has localised $sech^2$ soliton solutions and periodic cnoidal wave solutions. The addition of dissipative effects leaves open the question of whether or not there are nontrivial asymptotic attractors to the transient dynamics. This question can be addressed in terms of the energy integral formed by multiplying (16) by the amplitude A and integrating over all ξ. Since it is required that solutions be localised, at $\pm\infty$, $A = A_\xi = A_{\xi\xi} = \ldots = 0$, so it is readily shown that the energy integral (accumulation) of the KdV terms in (16) vanish. The surviving equation (ignoring $O(\varepsilon\delta)$ terms) is

$$\lambda_{0,1} \frac{d}{d\tau} \int_{-\infty}^{\infty} A^2 d\xi = \delta(-|\lambda_{4,0}| \int_{-\infty}^{\infty} A_{\xi\xi}^2 d\xi - |\lambda_{2,0}| \int_{-\infty}^{\infty} A_\xi^2 d\xi - |\lambda_0| \int_{-\infty}^{\infty} A^2 d\xi + f \int_{-\infty}^{\infty} A d\xi) \quad (20)$$

Clearly, $A \equiv 0$ can satisfy this constraint only if $f \equiv 0$. The LHS represents the pseudo-energy present in the wave. The λ_0,$\lambda_{2,0}$, and $\lambda_{4,0}$ terms dissipate wave energy. The f term pumps energy into the disturbance from the viscous decay of a nonuniform background vorticity profile in the waveguide. It has been proposed [Zimmerman and Velarde (1996)] that localised-in-ξ disturbances of the boundary or at critical layers due to either shear layer instability or critical layer instability can be crudely modelled as a localised-in-ξ effective f term. In the analysis here, the level of the waveguide can be smoothly varied to model the gross effects of a Kelvin-Helmholtz vortex being shed from a shear layer instability. If the level of the waveguide η were to change abruptly as a roll is shed and then abruptly return to normal, then η is unity plus a pulse. The shear layer instability is bound to create a boundary layer where the shear profile has $\hat{u}_{yyy} \neq 0$ with thickness $\delta^{1/2}$. It follows that the propagating Kelvin-Helmholtz roll provides a localised-in-ξ energy source to a solitary internal wave. The dissipative terms will asymptotically match this energy input, resulting in a non-trivial solitary internal wave of permanent form. Since the phase velocity of the Kelvin-Helmholtz roll sets c and the Richardson number Ri is a property of the background conditions, (15) approximates the wavenumber of the internal solitary wave and (17–18) approximately gives the coefficients of the nonlinear wave equation it satisfies.

4 Conclusions

A centre manifold approach was used to derive an approximate nonlinear wave equation satisfied by 1-D disturbances (16). The waveguide is a stable stratification of viscous fluid under shear. The equation simplifies to the well known time-reduced Korteweg-deVries equation for inviscid long waves, so that it must have solitary internal wave solutions similar to the famous $sech^2$ waveform under some conditions. Because a centre manifold approach readily permits modulation of the height of the waveguide, a crude model for shear layer instability at the upper material surface of the waveguide is proposed. This model predicts that Kelvin-Helmholtz rolls that propagate at the shear layer interface pump energy into internal wave disturbances that propagate in phase with the roll. These solitary internal waves perforce must be localised. An interesting question is their waveform, which can only be derived by numerical solution of the nonlinear evolution equation.

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