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# Time-dependent Long's equation

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**Abstract.** Long's equation describes steady-state two-dimensional stratified flow over terrain. Its numerical solutions under various approximations were investigated by many authors. Special attention was paid to the properties of the gravity waves that are predicted to be generated as a result. In this paper we derive a time-dependent generalization of this equation and investigate analytically its solutions under some simplifications. These results might be useful in the experimental analysis of gravity waves over topography and their impact on atmospheric modeling.

## 1 Introduction

Long's equation (Long, 1953, 1954, 1955, 1959) models the flow of inviscid stratified fluid in two dimensions over terrain. When the base state of the flow (that is, the unperturbed flow field far upstream) is without shear the solutions of this equation are in the form of steady lee waves. Solutions of this equation in various settings and approximations were studied by many authors (Drazin, 1961; Drazin and Moore, 1967; Durran, 1992; Lily and Klemp, 1979; Peltier and Clark, 1983; Smith, 1980, 1989; Yih, 1967). The most common approximation in these studies was to set the Brunt–Väisälä frequency to a constant or a step function over the computational domain. Moreover, the values of the parameters $\beta$ and $\mu$ which appear in this equation were set to zero. In this (singular) limit of the equation the nonlinear terms and one of the leading second-order derivatives in the equation drop out and the equation reduces to that of a linear harmonic oscillator over two-dimensional domain. Careful studies (Lily and Klemp, 1979) showed that these approximations are justified unless wave breaking is present in the solution (Peltier and Clark, 1983; Miglietta and Rotunno, 2014).

Long's equation provides also the theoretical framework for the analysis of experimental data (Fritts and Alexander, 2003; Shutts et al., 1988; Vernin et al., 2007; Jumper et al., 2004) under the assumption of shearless base flow. (An assumption which, in general, is not supported by the data.) An extensive list of references appears in Fritts and Alexander (2003), Baines (1995), Nappo (2012) and Yhi (1980).

An analytic approach to the study of this equation and its solutions was initiated recently by the current author (Humi, 2004). We showed that for a base flow without shear and under rather mild restrictions the nonlinear terms in the equation can be simplified. We also identified the "slow variable" that controls the nonlinear oscillations in this equation and using phase averaging approximation derived a formula for the attenuation of the stream function perturbation with height. This result is generically related to the presence of the nonlinear terms in Long's equation. We explored also different formulations of this equation (Humi, 2007, 2009) and the effect of shear on the solutions of this equation (Humi 2006, 2010).

One of the major obstacles to the application of Long's equation in realistic applications is due to the fact that it is restricted to the description of steady states of the flow. It is therefore our objective in this paper to derive a time-dependent generalization of this equation and study the properties of its solutions. The resulting system contains two equations for the time evolution of the density and the stream function. While the equation for the stream function is rather complicated it can be simplified in two instances. The first corresponds to the classical (steady state) Long's equation while the second is time dependent and new (as far as we know). In this paper we explore the properties of the flow in this second case, which might find some applications in the analysis of experimental data about gravity waves (Vernin et al., 2007; Jumper et al., 2004; Nappo, 2012), and its application to atmospheric modeling (Richter et al., 2010; Geller et al., 2013).

The plan of the paper is as follows: in Sect. 2 we derive the time-dependent Long's equation. In Sect. 3 we consider the time evolution and proper boundary conditions on shearless flow over topography. We end with the summary and conclusion in Sect. 4.

## 2 Derivation of the time-dependent Long's equation

In the paper we consider the flow in two dimensions $(x,\ z)$ of an inviscid, stratified and weakly compressible fluid that is modeled by the following equations:

$$u_x + w_z = 0, \tag{1}$$

$$\rho_t + u\rho_x + w\rho_z = 0, \tag{2}$$

$$\rho\,(u_t + uu_x + wu_z) = -p_x, \tag{3}$$

$$\rho\,(w_t + uw_x + ww_z) = -p_z - \rho g, \tag{4}$$

where subscripts indicate differentiation with respect to the indicated variable, $u = (u,\ w)$ is the fluid velocity, $\rho$ is its density, $p$ is the pressure and $g$ is the acceleration of gravity.

One possible interpretation of Eq. (1), is that the fluid is incompressible while Eq. (2) is an advection equation for a scalar (viz. $\rho$) by the flow. However, since we consider, in the following, derivatives of the density we refer to this formulation as representing a "weakly compressible fluid".

We can nondimensionalize these equations by introducing

$$\overline{x} = \frac{x}{L},\ \overline{z} = \frac{N_0}{U_0}z,\ \overline{u} = \frac{u}{U_0},\ \overline{w} = \frac{LN_0}{U_0^2}w,$$

$$\bar{\rho} = \frac{\rho}{\rho_0},\ \overline{p} = \frac{N_0}{gU_0\rho_0}p, \tag{5}$$

where $L$, $U_0$, and $\rho_0$ represent respectively characteristic length, velocity, and density. $N_0$ is the characteristic Brunt–Väisälä frequency:

$$N_0^2 = -\frac{g}{\overline{\rho}}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z}, \tag{6}$$

where $\overline{\rho}$ is the ambient density profile of the atmosphere. In the following we let $N_0$ to be a constant.

In these new variables Eqs. (1)–(4) take the following form (for brevity we drop the bars):

$$u_x + w_z = 0, \tag{7}$$

$$\rho_t + u\rho_x + w\rho_z = 0, \tag{8}$$

$$\beta\rho\,(u_t + uu_x + wu_z) = -p_z, \tag{9}$$

$$\beta\rho\,(w_t + uw_x + ww_z) = -\mu^{-2}\,(p_z + \rho)\,, \tag{10}$$

where

$$\beta = \frac{N_0U_0}{g}, \tag{11}$$

$$\mu = \frac{U_0}{N_0L}. \tag{12}$$

$\beta$ is the Boussinesq parameter (Shutts et al., 1988; Baines, 1995) which controls stratification effects (assuming $U_0 \neq 0$) and $\mu$ is the long wave parameter which controls dispersive effects (or the deviation from the hydrostatic approximation). In the limit $\mu = 0$ the hydrostatic approximation is fully satisfied (Baines, 1995; Nappo, 2012). It should be observed that these two parameters $\beta$ and $\mu$ encapsulate the atmospheric conditions which impact the creation of gravity waves over terrain although one of them ($\mu$) can be suppressed by additional scaling.

In view of Eq. (7) we can introduce a stream function $\psi$ so that

$$u = \psi_z,\ w = -\psi_x. \tag{13}$$

Using this stream function we can rewrite Eq. (8) as

$$\rho_t + J\{\rho, \psi\} = 0, \tag{14}$$

where for any two (smooth) functions $f$, $g$,

$$J\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}. \tag{15}$$

Using $\psi$ the momentum equation Eq.( 9), Eq. (10) becomes

$$\beta\rho\,(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{zz}) = -p_x, \tag{16}$$

$$\beta\mu^2\rho\,(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) = -p_z - \rho. \tag{17}$$

We can suppress $\mu$ from the system (Eqs. 14, 16, 17) if we introduce the following normalized independent variables:

$$\overline{t} = \frac{t}{\mu},\ \overline{x} = \frac{x}{\mu},\ \overline{z} = z,\ \mu \neq 0. \tag{18}$$

Equations (14) and (16) remain unchanged and Eq. (17) becomes

$$\beta\rho\,(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) = -p_z - \rho, \tag{19}$$

where we dropped the bars on $t$, $x$, and $z$. However, we observe that in these coordinates $\psi_z = u$ and $\psi_x = -\mu\,w$.

Thus, after all these transformations the system of equations governing the flow is Eqs. (14), (16) and (19).

To eliminate $p$ from Eqs. (16) and (18) we differentiate these equations with respect to $z$ and $x$ respectively and subtract. This leads to

$$\begin{aligned}
&\beta\rho_z\,(\psi_{zt} + \psi_z\psi_{zx} - \psi_x\psi_{zz}) \\
&\quad + \beta\rho\,(\psi_{zzt} + \psi_z\psi_{zzx} - \psi_x\psi_{zzz}) \\
&\quad - \beta\rho_x\,(-\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}) \\
&\quad - \beta\rho\,(-\psi_{xxt} - \psi_z\psi_{xxx} + \psi_x\psi_{xxz}) = \rho_x.
\end{aligned} \tag{20}$$

The sum of the second and fourth terms in this equation can be rewritten as

$$\beta\rho\left[\nabla^2\psi_t + J\left\{\nabla^2\psi, \psi\right\}\right]. \tag{21}$$

(However, observe that when $\mu \neq 1$, $\nabla^2 \psi$ does not represent the flow vorticity due to the transformation Eq. (18) and therefore the sum of the two terms in Eq. (21) is not zero in general.)

To reduce the first and third terms in Eq. (20) we use Eq. (14). We obtain

$$\begin{aligned}
&\beta\left[\rho_z\left(\psi_{zt}+\psi_z\psi_{zx}-\psi_x\psi_{zz}\right)\right] \\
&\quad -\beta\left[\rho_x\left(-\psi_{xt}-\psi_z\psi_{xx}\right)+\psi_x\psi_{xz}\right] \\
&\quad =\beta\left[\rho_z\psi_{zt}+\rho_z\psi_z\psi_{zx}-\left(\rho_t+\rho_x\psi_z\right)\psi_{zz}\right. \\
&\quad \left.+\rho_x\psi_{xt}+\left(\psi_x\rho_z-\rho_t\right)\psi_{xx}-\rho_x\psi_x\psi_{xz}\right] \\
&\quad =\beta\left\{\rho_z\psi_{zt}+\rho_x\psi_{xt}-\rho_t\nabla^2\psi+\frac{1}{2}\right. \\
&\quad \left. J\left\{(\psi_x)^2+(\psi_z)^2,\rho\right\}\right\}. \qquad (22)
\end{aligned}$$

Combining the results of Eqs. (21) and (22), Eq. (20) becomes

$$\begin{aligned}
&\rho\left[\left(\nabla^2\psi\right)_t+J\left\{\nabla^2\psi,\psi\right\}\right]+\rho_z\psi_{zt}+\rho_x\psi_{xt} \\
&\quad +\left[-\rho_t\nabla^2\psi+\frac{1}{2}J\left\{(\psi_x)^2+(\psi_z)^2,\rho\right\}\right]=\frac{J\{\rho,z\}}{\beta}. \qquad (23)
\end{aligned}$$

Thus, we have reduced the original four equations (Eqs. 1–4) to two equations (Eqs. 14 and 23). This system of equations can be considered as the generalization of Long's equation to time-dependent flows.

While Eq. (23) is rather complicated in general it can be simplified further in some special cases. The first is when one considers the steady state of the flow. (This simplifies also Eq. 14.) This restriction leads to Long's equation (Long, 1953, 1954, 1955, 1959; Baines, 1995; Yhi, 1980). Furthermore, if the density derivatives associated with the momentum terms are neglected Eq. (23) reduces to the 2-D Boussinesq equation (Tabaei et al.,2005). Another case happens when $\nabla^2 \psi = 0$; i.e., $\psi$ is harmonic. (Note, however, that this does not imply that the physical vorticity $\nabla \times u$ is zero due to the transformation Eq. (18) unless $\mu = 1$.) Equation (23) becomes

$$\rho_z\psi_{zt}+\rho_x\psi_{xt}+\frac{1}{2}J\left\{(\psi_x)^2+(\psi_z)^2,\rho\right\}=\frac{J\{\rho,z\}}{\beta}. \qquad (24)$$

Observe that the derivatives of $\rho$ with respect to time are not present in this equation and this is consistent with Eq. (1).

However, if $\nabla^2 \psi = 0$ we can define $v_1 = \psi_z$ and $v_2 = -\psi_x$. These definitions use the stretched coordinates of Eq. (18) and then

$$(v_1)_z - (v_2)_x = 0.$$

This implies that there exists a function $\eta$ so that

$$\eta_x = v_1,\ \eta_z = v_2.$$

That is,

$$\eta_x = \psi_z,\ \eta_z = -\psi_x.$$

Physically, these relations imply that $\eta_x = u$ and $\eta_z = \mu w$.

Replacing $\psi$ by $\eta$ in Eq. (24) yields

$$J\left\{\eta_t+\frac{1}{2}\left[(\eta_x)^2+(\eta_z)^2\right]+\frac{z}{\beta},\rho\right\}=0. \qquad (25)$$

Hence,

$$\eta_t+\frac{1}{2}\left[(\eta_x)^2+(\eta_z)^2\right]+\frac{z}{\beta}=R(\rho), \qquad (26)$$

where $R(\rho)$ is a parameter function that can be determined from the asymptotic conditions on the flow. This equation is formally similar to the Bernoulli equation with $\eta$ playing the role of the velocity potential. When $\mu = 1$ and the term $\frac{z}{\beta}$ is interpreted as potential energy, $\eta$ represents potential flow.

To summarize, the equations of the flow in this case are

$$\rho_t + \eta_x\rho_x + \eta_z\rho_z = 0 \qquad (27)$$

(which replaces Eq. 14), and Eq. (26).

### 2.1 Other reductions of Eq. (23)

The reduction of Eq. (23) was carried out above under the assumption $\nabla^2 \psi = 0$. However, it can be generalized to case $\nabla^2 \psi = a$, where $a$ is a constant. To this end we define

$$v_1 = \psi_z,\ v_2 = -\psi_x + ax.$$

Therefore,

$$(v_1)_z - (v_2)_x = 0,$$

which implies that there exists a function $\eta$ so that

$$\eta_x = v_1,\ \eta_z = v_2.$$

Hence,

$$\eta_x = \psi_z,\ \eta_z = -\psi_x + ax. \qquad (28)$$

Using these relations to substitute $\eta$ for $\psi$ in Eq. (23) leads to

$$\begin{aligned}
&\rho_z\eta_{xt}-\rho_x(\eta_z-ax)_t \\
&\quad +\left[-a\rho_t+\frac{1}{2}J\left\{(\eta_z-ax)^2+(\eta_x)^2,\rho\right\}\right] \\
&\quad =\frac{J\{\rho,z\}}{\beta}. \qquad (29)
\end{aligned}$$

Therefore,

$$J\{\eta_t,\rho\}-a\rho_t+\frac{1}{2}J\left\{(\eta_z-ax)^2+(\eta_x)^2,\rho\right\}=\frac{J\{\rho,z\}}{\beta}. \qquad (30)$$

Hence,

$$-a\rho_t+J\left\{\eta_t+\frac{1}{2}\left[(\eta_z-ax)^2+(\eta_x)^2\right]+\frac{z}{\beta},\rho\right\}=0. \qquad (31)$$

Using Eq. (14) we have

$$-aJ\{\psi,\rho\}+J\left\{\eta_t+\frac{1}{2}\left[(\eta_z-ax)^2+(\eta_x)^2\right]+\frac{z}{\beta},\rho\right\}=0. \quad (32)$$

It follows then that

$$-a\psi+\eta_t+\frac{1}{2}\left[(\eta_z-ax)^2+(\eta_x)^2\right]+\frac{z}{\beta}=R(\rho). \quad (33)$$

We can eliminate $\psi$ from this equation by differentiating with respect to $z$ and use Eq. (28):

$$-a\eta_x+\left[\eta_t+\frac{1}{2}\left[(\eta_z-ax)^2+(\eta_x)^2\right]\right]_z=-\frac{1}{\beta}+R(\rho)_z. \quad (34)$$

## 3 Time evolution of stratified flow

In this section we shall consider the time evolution of a stratified shearless base flow, viz. a flow which satisfies as $t\to-\infty$,

$$\lim_{x\to-\infty}\rho^0(t,x,z)=\frac{H-z}{H},\ \lim_{x\to-\infty}u=1,\ \lim_{x\to-\infty}v=0; \quad (35)$$

i.e., the far upstream flow is independent of time and satisfies asymptotically $u=1$, $v=0$, and $\rho^0$ is stratified with height ($H$ is a height at which $\rho^0\approx 0$). The conditions on $u$ and $v$ imply that asymptotically $\eta^0=x$. We note that this is the standard setup that has been used to analyze experimental observations of gravity waves (Jumper et al., 2004; Vernin et al., 2007; Shutts et al., 1988). The solutions of Eqs. (26) and (27) which we discuss below represent therefore gravity waves which are generated by low lying topography.

In these limits Eq. (27) is satisfied. Substituting these limiting values in Eq. (26) we obtain that

$$R(\rho)=\frac{z}{\beta}+\frac{1}{2}=\frac{H(1-\rho)}{\beta}+\frac{1}{2}. \quad (36)$$

However, it is obvious that different profiles of the base flow will yield a different $R(\rho)$.

We now consider perturbations from the (shearless) base flow described by Eq. (35) due to shape of the topography, viz.

$$\eta=\eta^0+\epsilon\phi,\ \rho=\rho^0+\epsilon\zeta. \quad (37)$$

From Eqs. (26) and (27) we obtain to first-order in $\epsilon$ the following equations for $\phi$ and $\zeta$:

$$\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}+\frac{H\zeta}{\beta}=0, \quad (38)$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}-\frac{1}{H}\frac{\partial\phi}{\partial z}=0. \quad (39)$$

To find the general form of the solution of these equations we use Eq. (38) to express $\zeta$ in terms of $\phi$ and substitute in Eq. (39). This yields the following equation for $\phi$:

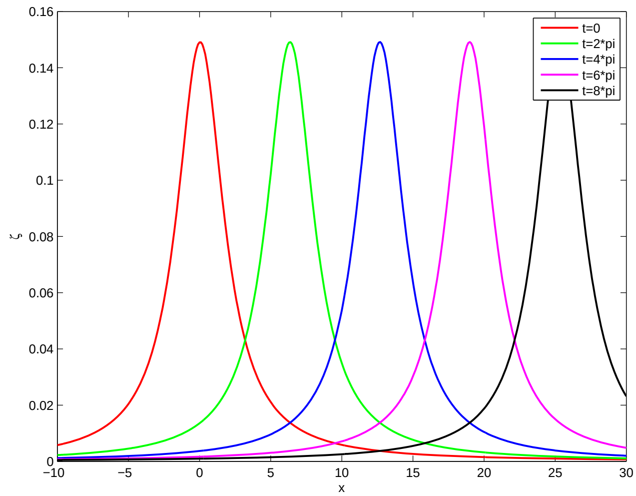


**Figure 1.** A cross section of the perturbation in $\rho$ at $z=2$.

$$\frac{\partial^2\phi}{\partial t^2}+2\frac{\partial^2\phi}{\partial t\partial x}+\frac{\partial^2\phi}{\partial x^2}+\frac{1}{\beta}\frac{\partial\phi}{\partial z}=0. \quad (40)$$

It is possible to find "elementary solutions" to this equation by separation of variables if we let

$$\phi=p(t,x)F(z),$$

where $c$ is an arbitrary positive constant so that $\phi$ represents a perturbation moving forward in time. This leads to

$$\frac{\frac{\partial^2 p}{\partial t^2}+2\frac{\partial^2 p}{\partial t\partial x}+\frac{\partial^2 p}{\partial x^2}}{p}=-\frac{1}{\beta}\frac{F(z)'}{F(z)}=-\omega^2, \quad (41)$$

where $\omega^2$ is the separation of variables constant. Primes denote differentiation with respect to the appropriate variable.

Solving Eq. (41) we obtain the following elementary solution for $\phi$:

$$\phi_\omega=C_\omega\exp\left[\beta\omega^2 z\right][G(x-t)\cos\omega t+K(x-t)\sin\omega t], \quad (42)$$

where $G(x-t)$, $K(x-t)$ are arbitrary smooth functions and $C_\omega$ is a constant.

The corresponding solution for $\zeta$ can be obtained by substituting this result in Eq. (38):

$$\zeta=C_\omega\frac{\beta\omega}{H}\exp\left[\beta\omega^2 z\right][G(x-t)\cos\omega t-K(x-t)\sin\omega t]. \quad (43)$$

Hence, the general solution for $\phi$ can be written as

$$\phi=\int_0^\infty\exp\left[\beta\omega^2 z\right][G_\omega(x-t)\cos\omega t+K_\omega(x-t)\sin\omega t]\,\mathrm{d}\omega \quad (44)$$

with a similar expression for $\zeta$.

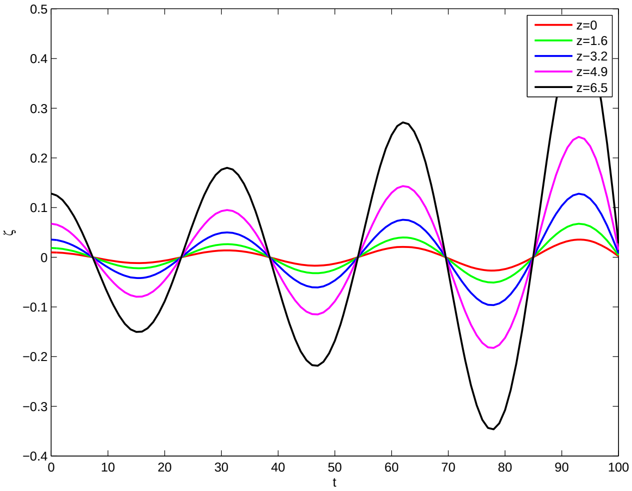


**Figure 2.** A cross section of the perturbation in $\rho$ at $x = 20$.

### 3.1 Boundary conditions

We consider a flow in an unbounded domain over topography with shape $f(x)$ and maximum height $h$ and impose the following boundary conditions on $\rho$ and $\psi$ in the limits $x = -\infty$ and $t = -\infty$:

$$\psi(-\infty, -\infty, z) = z, \ \rho(-\infty, -\infty, z) = \rho^0(z). \tag{45}$$

(This implies that in these limits $\eta = x$.)

At the topography we impose the following boundary condition on $\rho$ at $t = 0$:

$$\rho(0, x, \epsilon f(x)) = \rho^0(\epsilon f(x)) = \frac{H - \epsilon f(x)}{H} \tag{46}$$

but

$$\rho(0, x, \epsilon f(x)) \approx \rho^0(0, x, 0) + \epsilon\zeta(0, x, z).$$

Hence, at the topography

$$\zeta(0, x, \epsilon f(x)) = -\frac{f(x)}{H}. \tag{47}$$

To derive the corresponding boundary condition for $\eta$ we first consider the appropriate boundary condition on the stream function $\psi$ along the topography. To this end we assume that the topography is a line on which the stream function is constant and this constant can be chosen to be zero. For the base flow described in Eq. (45), $\psi_0 = z$ and $\psi = \psi_0 + \epsilon\, \psi_1$ where $\psi_1$ is the perturbation due to to the topography. Hence, along the topography

$$\begin{aligned} 0 &= \psi_0 + \epsilon\psi_1 = z + \epsilon\psi_1(0, x, \epsilon f(x)) \\ &= \epsilon f(x) + \epsilon\psi_1(0, x, \epsilon f(x)). \end{aligned} \tag{48}$$

Therefore, along the topography we let $\psi_1(0, x, \epsilon\, f(x)) = -f(x)$. We now observe that by definition $\psi_x = \eta_z$. But $\psi_x = \epsilon\, \psi^1_x = -\epsilon\, f'(x)$, (where primes denote differentiation with respect to $x$) and $\eta = -x + \epsilon\,\phi$. Therefore, we infer that the boundary condition on $\eta$ along the topography is

$$\phi_z(0, x, \epsilon f(x)) = -f'(x) \tag{49}$$

(which is consistent with Eq. 39).

As to the boundary condition on $\eta(t, \infty, z)$, we observe that the system of Eq. (26) and Eq. (27) contains no dissipation terms and therefore only radiation boundary conditions can be imposed in this limit. (Physically, this means that the horizontal group velocity is positive and energy is radiated outward.) Similarly, at $z = \infty$ it is customary to impose (following Peltier and Clark, 1983) radiation boundary conditions. However, in view of Eqs. (42) and (43) it is obvious that the perturbation described by these equations is propagating forward in time and this condition is satisfied. A formal verification of this constraint is possible by expressing $F$, $G$, and $K$ in these equations using Fourier transform representation.

For low lying topography (viz $\epsilon \ll 1$) it is customary to replace the boundary conditions Eqs. (46) and (47) by

$$\zeta(0, x, 0) = -\frac{f(x)}{H}, \ \phi_z(0, x, 0) = -f'(x). \tag{50}$$

*Example*: if $f(x)$ is given by a "witch of Agnesi" curve, then

$$f(x) = \frac{a^2}{(a^2 + x^2)}, \ f'(x) = -\frac{2a^2 x}{(x^2 + a^2)^2}. \tag{51}$$

Let the initial perturbation in $\rho$ be

$$\zeta(0, x, z) = e^{\beta\lambda^2 z},$$

where $\lambda$ is a constant. From Eq. (50) we infer that the general expression for $\zeta$ is given by Eq. (43) with $\omega = \lambda$. Hence, at $t = 0$ we must have

$$G(x) = -\frac{f(x)}{\beta\lambda}.$$

Similarly, the boundary condition on $\phi$ yields

$$K(x) = -\frac{f'(x)}{\beta\lambda^2}.$$

Figures 1 and 2 exhibit cross sections of the perturbation at $z = 2$ and $x = 20$ at different times with $C_\omega = 0.1$, $a = 2$, and $\lambda = 1$.

## 4 Summary and conclusions

Steady-state solutions of Long's equation model the vertical structure of plane parallel gravity waves. These solutions are useful, for example, in the parameterizations of unresolved gravity wave drag where the WKBJ (Wentzel–Kramers–Brillouin–Jeffreys) approximation is invoked to describe the time-dependent amplitude spectrum of a packet of gravity waves propagating in a slowly varying background.

The present paper presents an alternative analytical approach to solve (under several restrictions) this and similar time-dependent problems without having to invoke the WKBJ approximation. The analytical insights derived from this approach might be used to complement and verify the numerical results obtained from the WKBJ method.

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