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Limits and characteristics of the multifractal behavior of a high-resolution rainfall time series

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Abstract. The multifractal properties of a 2-year time series of 8-min rainfall intensity observations are investigated. The empirical probability distribution function suggests a hyperbolic intermittency with divergence of moments of order greater than 2. The power spectrum *E(f)* of the series obeys a power law form $E(f)=f^{-0.66}$ in the range of scales 8 min to approximately 3 days. The variation of the average statistical moments with scale shows that the series is characterized by a multifractal behavior between 8 min and approximately 11 days. The multifractal parameters associated with universality were estimated to be $\alpha=0.63$ and $C_1=0.44$ by using the Double Trace Moment, DTM, technique. The moment scaling functions obtained from the empirical values and the universal expression are in good agreement in the approximate range $1 \leq q \leq 3$. Outside of this range, however, differences exist which may be related to either limitations of the data or an inexact estimation of the parameters by DTM. The evident multifractal nature of rainfall time series is encouraging since it may lead to new and improved ways of processing rainfall data used in hydrological calculations.

1 Introduction

The applicability of multifractal theory in combination with cascade processes to describe observations from atmospheric processes has been thoroughly investigated during the latest years (e.g., Schertzer and Lovejoy, 1987; Lovejoy and Schertzer, 1990, 1991; Gupta and Waymire, 1993; Hubert et al., 1993; Ladoy et al., 1993; Olsson et al., 1993; Tessier et al., 1993; Davis et al., 1994). These investigations have been performed on data sets of clouds, wind, and rainfall and the aim has been twofold. Firstly to examine the overall scaling behavior to obtain the range of scales over which the multifractal relations hold, i.e., the scaling regime. Secondly, usually on basis of the somewhat contested notion of universality, to estimate parameters characterizing the multifractal behavior.

Regarding the temporal rainfall process, scaling properties have been investigated using time series with resolutions ranging from 1 day down to minutes (e.g., review by Hubert et al., 1993; Fraedrich and Larnder, 1993; Olsson et al., 1993). The overall scaling regime obtained in these analyses is approximately from minutes up to 2 weeks-1 month. However, this range is still associated with some uncertainties. The first one concerns the upper limit which in some studies was estimated to be significantly lower than 2 weeks (Fraedrich and Larnder, 1993; Olsson et al., 1993). The second uncertainty concerns a possible break in the scaling at 2-3 hours, but it has not yet been established whether this break is actually related to the rainfall generating mechanisms (e.g., a characteristic time above which frontal systems are the most influential and below which individual storms dominate the scaling behavior) or if it is an artificial break related to the resolution of the measuring device (Fraedrich and Larnder, 1993). The estimated values of the multifractal parameters associated with the notion of universality (e.g., Lovejoy and Schertzer, 1990) are $\alpha=0.51$ and $C_1=0.44$ with moderate differences between the studies (Hubert et al., 1993).

The aim of this study is to further investigate the temporal multifractal properties of rainfall by analyzing a 2-year series of 8-min rainfall intensity observations. Some multifractal properties of these data have previously been investigated (Olsson et al., 1993), but since that analysis was performed on the raw 1-min data using rather uncertain analyzing techniques, both the limits of the scaling regime and the values of the multifractal parameters obtained in the previous study were associated with uncertainties. In this study the data are analyzed in a more reliable form, the original 1-min values have been aggregated into 8-min values in order to achieve a more correct representation of low rainfall intensities (see Sect.4), but still limitations that affect the results exist and these are thoroughly described. The analysis is done firstly by using standard statistical techniques (empirical probability distribution function, power spectrum), secondly by studying the scaling of average statistical moments, and finally by employing the analyzing technique Double Trace Moments, DTM (Lavallée, 1991), to estimate the multifractal parameters.

2 Some Multifractal Foundations

The theoretical basis of multifractal atmospheric fields is founded on an assumption that fluxes of water and energy in the atmosphere are governed by multiplicative cascade processes successively transferring these quantities from larger to smaller scales (e.g., Schertzer and Lovejoy, 1987; Lovejoy and Schertzer, 1990; Gupta and Waymire, 1993). This assumption may be justified both on theoretical and empirical grounds. Theoretically, a cascade phenomenology may be deduced from the equations for hydrodynamic turbulence which have been proposed to be approximately valid for the atmosphere (see, e.g., Tessier et al., 1993). Empirically, the often observed hierarchical structure of rainfall fields is directly indicative of a cascade type of behavior (see, e.g., Gupta and Waymire, 1993). A cascade process is generally described as eddies breaking up into smaller sub-eddies, each of which receives a part of the flux of its parent eddy. This way, the main part of the flux is concentrated into smaller and smaller parts of the available space and the resulting field exhibits extreme variability and intermittency.

By considering the statistical properties of a field produced by a cascade process, the following fundamental relationship describing the behavior of the statistical moments at different scales may be deduced (Schertzer and Lovejoy, 1987)

$$\langle \epsilon_\lambda{}^q \rangle \approx \lambda^{K(q)} \tag{1}$$

where $\langle \epsilon_\lambda{}^q \rangle$ is the (ensemble) average qth moment of the normalized intensities at scale ratio λ, the latter here being defined as the ratio of the outer (maximum) scale of the field to the scale of interest, and $K(q)$ is the moment scaling function.

Under a simplifying assumption of universality, i.e., out of infinitely many possible parameters needed to describe a process only a few are relevant due to an inherently converging nature of the process, $K(q)$ may be expressed as (Schertzer and Lovejoy, 1989)

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q) \quad , \; 0 \le \alpha < 1, \; 1 < \alpha \le 2 \tag{2}$$

$$K(q) = C_1 q \log(q) \quad , \; \alpha = 1 \tag{3}$$

where C_1 is the codimension of the mean process and α is the Lévy index. It should be mentioned that this notion of universality has been questioned (Gupta and Waymire, 1993). Here, no attempt to participate in the theoretical debate is made, but the aim is to compare the empirical and the universal moment scaling functions in order to evaluate the applicability of the latter.

According to e.g. Tessier et al. (1993), $K(q)$ is in practice restricted by an upper limit of q related to two different aspects (e.g., Lavallée et al., 1991). The first one is the effect of the limited sample size used in the analysis. The critical value q_S associated with this effect depends on the sample size and by increasing the sample size it is thus possible to increase the range of validity for $K(q)$. The second aspect is that the process may exhibit divergence of moments of order higher than a critical value q_D. This may roughly be interpreted as that at $q < q_D$ the average moments are influenced by all values in the series, whereas at $q > q_D$ the average moments are influenced mainly by the maximum value. According to Schertzer and Lovejoy (1992), the straight-lined behavior of the empirical $K(q)$ resulting from these limiting aspects may be regarded as a multifractal phase transition.

3 Methodology

Before applying any specific multifractal analysis technique to the series, information about the scaling behavior is obtained using two standard statistical descriptions of the data.

The first is the empirical probability distribution function (pdf) which describes the scaling of the intensity fluctuations at a given scale, normally the scale corresponding to the measurement resolution (Fraedrich and Larnder, 1993). If the series is characterized by a hyperbolic intermittency (e.g., Lovejoy and Mandelbrot, 1985) expressed as a power-law form of the tail behavior of the empirical pdf, i.e., for high threshold intensities x

$$Pr(X > x) \propto x^{-q_D} \tag{4}$$

where X is the observed intensity, this is equivalent to the divergence of moments of order greater than or equal to q_D (e.g., Schertzer and Lovejoy, 1987).

The second is the power spectrum, $E(f)$, which has been used in several investigations to examine the scaling behavior of rainfall time series (e.g., Ladoy et al., 1991, 1993; Fraedrich and Larnder, 1993; Olsson et al., 1993). If the spectrum obeys a power-law form

$$E(f) = f^{-\beta} \tag{5}$$

where f is the frequency, this indicates absence of characteristic time in the range of the power law, i.e., scaling of the fluctuations, and thus a multifractal behavior may be assumed to hold. From the value of the power law exponent β information may be drawn about the stationarity of the data. Stationarity is required in analyses of scaling since they are normally based on some form of averaging over large ranges of scales. It has been argued that if $\beta<1$ the data are stationary, otherwise some processing, e.g. fractional differentiation (Tessier et al., 1993), is needed to produce stationarity (Davis et al., 1994).

To characterize the multifractal behavior, Eq.1 is directly employed to obtain values of the moment scaling function $K(q)$. The data are firstly normalized (and non-dimensionalized) by dividing each value with the average value. Because of this normalization, $\langle\epsilon_\lambda\rangle=1$ for all values of λ. Then the series is averaged over successively doubled time intervals (corresponding to successively halved values of λ) and for each λ the average qth moment $\langle\epsilon_\lambda{}^q\rangle$ is calculated. When this procedure is performed down to $\lambda=1$ the multifractal behavior of the series as expressed by Eq.1 is investigated by plotting $\langle\epsilon_\lambda{}^q\rangle$ as a function of λ in a log-log plot. In the scaling regime the curve will exhibit an approximately straight-lined behavior with a slope that is an estimation of the value of $K(q)$. By performing this procedure for different $q$values, the empirical moment scaling function may be obtained.

To estimate the multifractal parameters α and C_1 producing the best fit of Eq.2 or 3 to the empirical moment scaling function, different ways are possible. One is to directly fit Eq.2 or 3 to the values, but this procedure is associated with large uncertainties because the regression is non-linear and the parameters are correlated. Instead the Double Trace Moment, DTM, technique is employed (Lavallée, 1991; Tessier et al., 1993). By DTM a direct estimation of α is made possible by the introduction of a second ηth moment. The original field ϵ is transformed into ϵ^η, and $K(q)$ into $K(q,\eta)$ which, under the assumption of universality, is related to η by (Lavallée, 1991)

$$K(q,\eta)=\eta^\alpha K(q) \tag{6}$$

through which α may be directly estimated. The procedure is thus implemented by raising all values to η after the normalization and then calculate $\langle\epsilon_\lambda{}^{\eta,q}\rangle$ and $K(q,\eta)$ as described above. By using different values of η and then plot $K(q,\eta)$ as a function of η in a log-log plot, α may be estimated as the slope of the straight-lined part of the curve in accordance with Eq.6. Then C_1 is calculated from Eq.2 or 3 using the relationship $K(q)=K(q,1)$.

4 Rainfall Data

During 1979-81 a detailed observation program of the areal and dynamic properties of short-term rainfall was performed in the city of Lund, Sweden. The rainfall intensity was measured with a time resolution of 1 min by small tipping-bucket gages with an intensity resolution of 0.033 mm/min. The longest continuous measurement period was 2.5 years (January 1979 to July 1981), and the most complete time series was used in the present study. However, the sensitive tipping-bucket gage was unreliable during winter periods where the occurrence of snow introduced errors in the measurements. Therefore, for every day in the 2.5-year period, the tipping-bucket measurements were compared with daily values of precipitation and temperature observed in Lund by the Swedish Meteorological and Hydrological Institute. By this way, winter periods with erroneous registrations could be found and excluded. This modification may affect the analysis results, but since the seasonal variation of rainfall in the region is limited it may be assumed that the influence is small. Because of the removal of winter periods, the length of the analyzed series is 2 years and during this time there were no missing values. For further details about the database and observation area see Niemczynowicz (1986a, b).

It must be emphasized that the gages were used somewhat differently from common practice. Normally the time of each "tip" is recorded and then the bucket volume is evenly distributed over the time interval between this "tip" and the previous "tip". In this investigation, however, for each gage the number of "tips" that occurred during a certain minute was recorded at the beginning of the next minute (provided that at least one "tip" had occurred at at least one gage). Both strategies of measuring are associated with difficulties at low rainfall intensities. In the first case ("time-of-tip" recording), the bucket volume may be distributed in a way that time periods when in reality no rainfall occurred receive a constant low-intensity rainfall. In the second case ("number-of-tips" recording), rainfall intensities lower than the gage resolution are represented as "1-tip" rainfall registrations separated by a number of zero-registrations corresponding to time periods when in reality low-intensity rainfall occurred. When analyzing the series, these errors may affect the resulting scaling behavior at small scales. To overcome or at least reduce the influence of this problem, the original 1-min registrations were aggregated into 8-min values "absorbing" many of the erroneous zero-registrations. Because of the measuring strategy, the high intensity resolution, and the final aggregation, the accuracy of these data should be among the highest available from rain gage observations at this time scale.

5 Results

Figure 1 shows the empirical probability distribution function $\Pr(X>x)$ of the series. The hyperbolic tail behavior is evident with a slope of 2.0, estimated from the fitted regression line shown in Fig. 1. This thus indicates that moments of the series of order greater than or equal

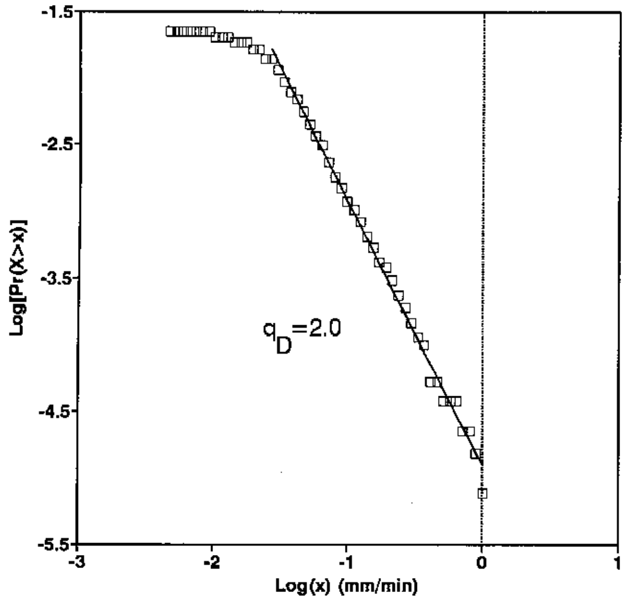

Fig. 1. Empirical probability distribution function $\Pr(X>x)$ of the 8-min rainfall time series. The straight line has been fitted by regression.

to $q_D=2.0$ diverge. These findings agree fairly well with Fraedrich and Larnder (1993) who obtained a similar q_D-value (≈ 1.7) at approximately the same intensity levels. However, at very high intensities (larger than approximately 30 mm/5 min) their empirical pdf breaks and a second straight-lined section is revealed having $q_D \approx 3.0$. In the data used in this study no intensities of that magnitude were observed. This is probably partly due to a less "violent" character of the rainfall generating mechanisms occurring in the present region, partly to the rather short measurement period (2 years).

Figure 2 shows the power spectrum $E(f)$ of the series in the range of scales 16 min (the Nyquist frequency) to 2 years. The spectrum has been averaged over logarithmically spaced frequency intervals. $E(f)$ exhibits a well respected power law form in the range 16 min to approximately 3 days where the spectrum breaks and the slope becomes flatter. The upper limit 3 days is in contradiction with some other studies where it was found to be in the region of 2 weeks (e.g., Ladoy et al., 1993). It must be emphasized that $E(f)$ at these low frequencies is associated with higher uncertainty due to the limited length of the series used here (2 years). However, the 3-day break agrees with the findings in Fraedrich and Larnder (1993) where the scaling regimes of various European rainfall time series are investigated using power spectra. They claim that 3 days is the upper limit of the scaling regime associated with frontal systems, but that this limit may well be higher depending on the climatic characteristics of the region in question. Their results indicate that this regime extends down to 2.4 hours where a break occurs to another regime that they claim is possibly associated with individual meso-scale storms. From Fig.2 no apparent break at 2.4 hours is indicated. The value of the power law exponent β in the range 16 min to 3 days is 0.66 estimated from the solid regression line shown in Fig.2. This thus indicates that the series is scaling with a well respected stationarity in this range of scales.

According to Fraedrich and Larnder (1993), another possible explanation for their apparent 2.4-hour break might be an inability of the gages to measure signals below some threshold level (i.e., low rainfall intensities). Thus it would be an artificial break that would be less pronounced the higher the sensitivity of the gage used and that would disappear when the sensitivity is sufficiently high. The present results support this assumption since the resolution of the data used in this study (0.033 mm) is 3 times higher

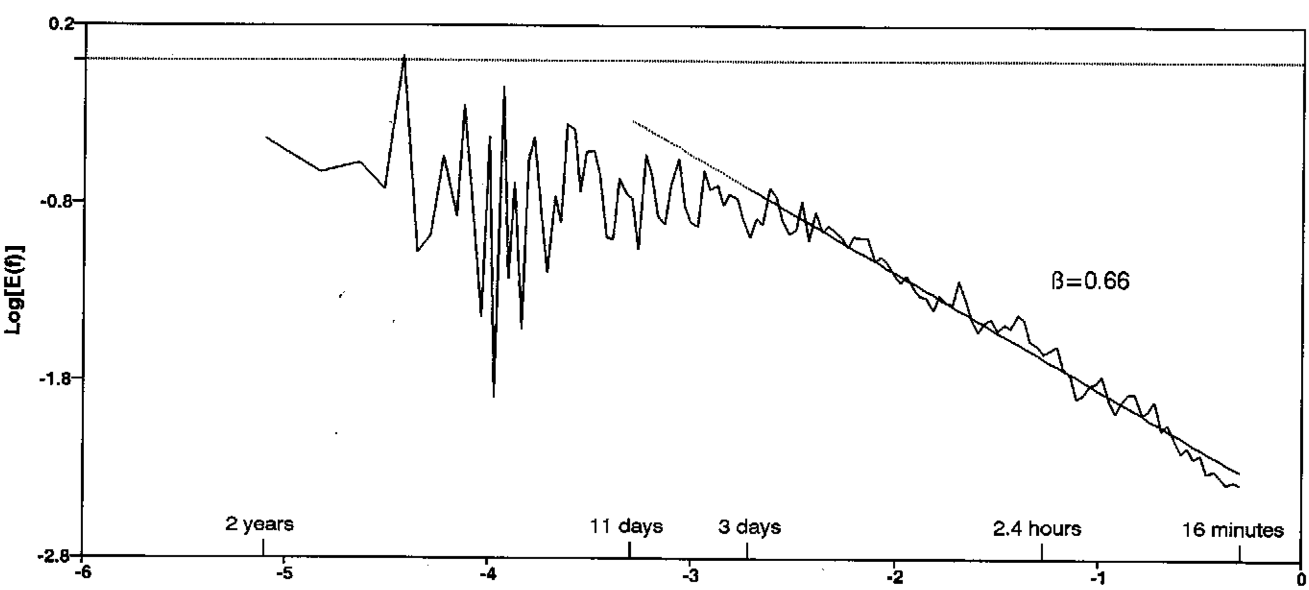

Fig. 2. Power spectrum $E(f)$ of the 8-min rainfall time series averaged over logarithmically spaced frequency intervals. The solid straight line has been fitted by regression. The dotted line has been extrapolated from the solid regression line.

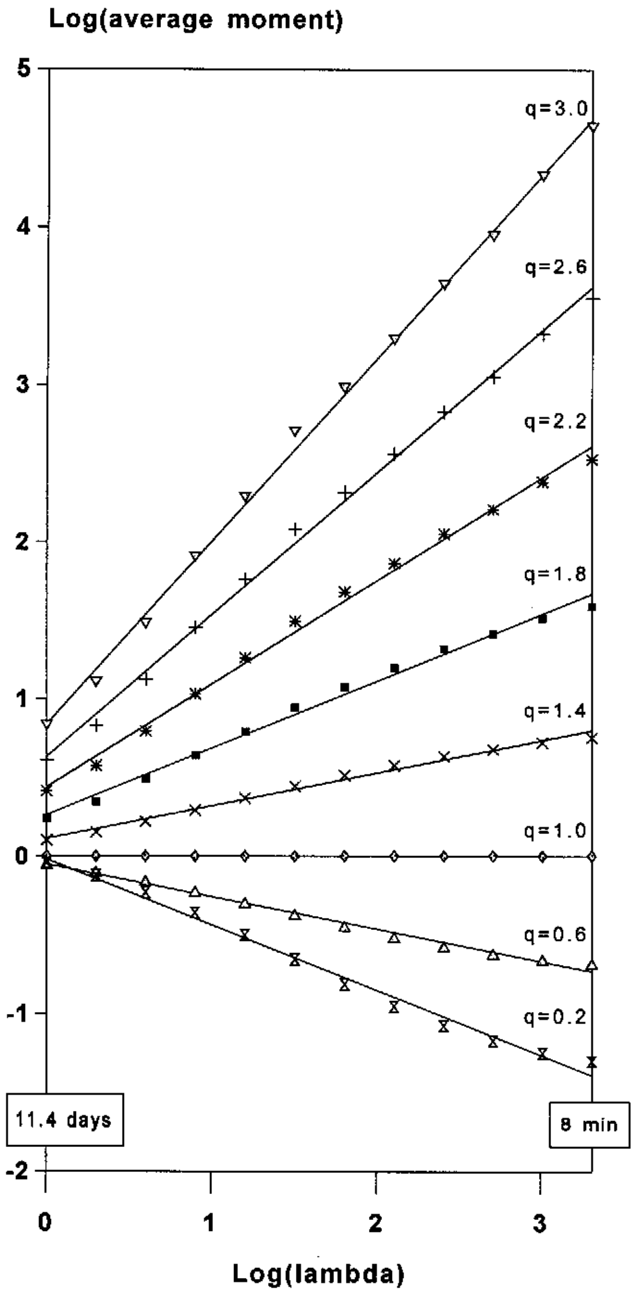

Fig. 3. The average moment $\langle \epsilon_\lambda{}^q \rangle$ as a function of λ (lambda) in the range 8 min to 11.4 days for values of q between 0.2 and 3.0. The straight lines have been fitted by regression.

than the 0.1-mm resolution used in the investigation by Fraedrich and Larnder (1993) (Fraedrich, 1994). Another observation in favor of the assumption is that the power spectrum obtained from the raw 1-min data (intensity resolution 0.033 mm/min) used in this study exhibited a pronounced break at about 45 min (Olsson et al., 1993), whereas in the spectrum from the more sensitive 8-min data (intensity resolution 0.033 mm/8 min, approximately 0.004 mm/min) this break has disappeared.

From the power spectrum, the scaling regime was thus estimated to be from 8 min up to approximately 3 days.

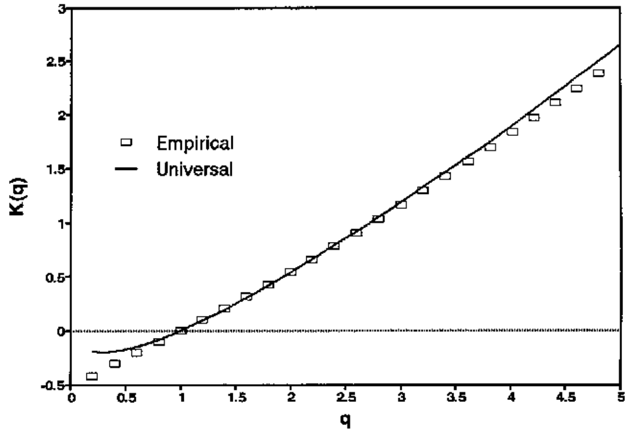

Fig. 4. The moment scaling function $K(q)$ for the 8-min rainfall time series; empirical (point values) and universal using $\alpha=0.63$ and $C_1=0.44$ (the solid line).

However, when performing the moment analysis it was found that a multifractal behavior as expressed by Eq.1 was well respected up to more than 10 days (Fig.3). By studying the power spectrum between 3 and 11 days (Fig.2), it may be observed that it does not deviate too strongly from the straight-lined behavior in the range 16 min to 3 days (the dotted line in Fig.2, extrapolated from the solid regression line). Thus a multifractal behavior is expected to hold approximately also for this range.

Figure 3 shows $\langle \epsilon_\lambda{}^q \rangle$ as a function of λ using values of q between 0.2 and 3.0. The curves exhibit a well respected straight-lined behavior in the range 8 min ($\lambda=2048$) to 11.4 days ($\lambda=1$). To the curves, straight lines have been fitted by regression (the straight lines in Fig.3) having an average R^2 of 0.994 with a standard deviation of 0.004.

From the slopes of the lines in Fig.3, the values of the empirical moment scaling function may be estimated. In Fig.4, this function is shown for values of q between 0.2 and 4.8 (the point values).

As previously mentioned, to estimate the multifractal parameters α and C_1 producing the best fit of Eq.2 or 3 to the empirical moment scaling function, DTM is employed. Figure 5 shows $K(q,\eta)$ as a function of η for $q=1.3$, 1.4, and 1.5. The curves exhibit a straight-lined behavior for a range of η values in which straight lines have been fitted by regression. At both ends the curves deviate from the straight-lined behavior and the range of η values where the straight-lined behavior is respected is in fact rather limited. The upper limit corresponds to $q\eta \approx 3.4$ which seems reasonable considering that it is related to the divergence of high-order moments which makes $K(q,\eta)$ approach a constant value. The lower limit ($q\eta \approx 1.7$) is higher than what has been found in other DTM-analyses of rainfall time series (e.g., Tessier et al., 1993). Generally, when raising the values of the series to small exponents the importance of the extreme values is decreased whereas the importance of the low values are increased. Thus inaccuracies of the low values may affect the analysis

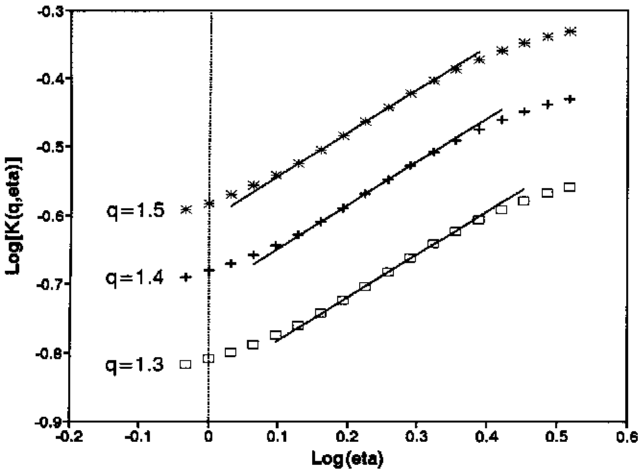

Fig. 5. DTM-analysis: $K(q,\eta)$ as a function of η (eta) for $q=1.3$, $q=1.4$, and $q=1.5$. The straight lines have been fitted by regression.

results to such an extent that the scaling behavior cannot be accurately determined at low values of q and η. Slight inaccuracies may exist in the present data depending both on the low-intensity problem previously described (see Sect.4) which may remain to some extent also in the aggregated 8-min values, and on possible noise in the data.

From the slopes of the straight lines in Fig.5, the estimated values of α are 0.624 ($q=1.3$), 0.631 ($q=1.4$), and 0.633 ($q=1.5$). The average value of α obtained from $1.1 \leq q \leq 1.6$ is 0.626 with a standard deviation of 0.009. Using these values, the average value of C_1 becomes 0.439 with a standard deviation of 0.003.

$C_1=0.44$ agrees very well with the results from previous analyses of rainfall time series. In the review by Hubert et al. (1993), C_1 is found to be 0.44 ± 0.16 based on five different sets of rainfall data. The estimated values of α show some difference, 0.63 in this study compared to 0.51 ± 0.05 in Hubert et al. (1993). One reason for the difference may be the differences in the upper limit of the scaling regime, it was at least 2 weeks in Hubert et al. (1993) compared to 11 days in this study.

In Fig.4, a comparison between the universal moment scaling function obtained from Eq.2 using $\alpha=0.63$ and $C_1=0.44$ (the solid line) and the empirical function (the point values), is shown. In the approximate range $0.8 \leq q \leq 3.2$ the agreement is good, but outside of this range the functions differ somewhat. From $q \approx 3.0$ and up the empirical function is straight-lined. This behavior has been found also in other analyses of rainfall time series (e.g., Ladoy et al., 1993) and it is most likely associated with the restrictions of $K(q)$ imposed by the two limiting aspects, undersampling and divergence of moments, described in Sect.2. The empirical probability distribution function, however, indicated divergence of moments (and consequently a straight-lined $K(q)$) from $q_D=2.0$ (Fig.1). This discrepancy may be related to difficulties in estimating the true tail behavior of the probability distribution due to an insufficient amount of data, as mentioned above. The almost straight-lined behavior of the empirical function from $q \approx 2.0$ and down, which produces the most striking difference from the universal function at $q<0.8$, has not been found in other studies. This part of the empirical function may be affected by the low-intensity problem (see above and Sect.4).

Another possible reason for the difference between the empirical and the universal moment scaling function may be that the values of α and C_1 obtained by DTM are not ideal and that other values may provide better fits. The almost straight-lined behavior of the empirical function at low values of q actually indicates that a universal function with $\alpha=0$ (i.e., implying a monofractal behavior) would produce a better fit in this range. This is, however, in striking contrast to the apparent multifractality of the data found in a previous investigation (Olsson et al., 1993) and therefore this finding is left merely as an observation.

6 Summary and Discussion

A 2-year time series of 8-min rainfall intensity observations, aggregated from 1-min data, was analyzed in order to investigate whether it was characterized by a multifractal behavior. The empirical probability distribution function indicated a hyperbolic intermittency with divergence of moments of order higher than or equal to 2. The power spectrum indicated that the series was scaling within the range 8 min to 3 days with a well respected stationarity. By studying the average statistical moments at different scales, a multifractal behavior was well respected from 8 min up to 11.4 days. By using Double Trace Moments, DTM, the multifractal parameters associated with the notion of universality was estimated to be $\alpha=0.63$ and $C_1=0.44$. The agreement between the universal and the empirical moment scaling function is satisfactory although at high and low values of q the curves differ. At $q>3$ the difference is likely to originate from either undersampling or divergence of higher-order moments. At $q<1$ the difference may be related to an improper representation of low-intensity rainfall in the analyzed time series. Another possibility is that the parameter values obtained by DTM are imprecise and that other values would produce better fits.

This study constitutes yet a contribution to the growing number of analyses where the hypothesis of a multifractal behavior of rainfall fields is supported. The multifractal behavior is accurately respected from the 8-min resolution of the series to approximately 11 days. From a hydrological point of view it is interesting to note that the interval 1 day to 1 hour is well within the limits of the scaling regime. Present sophisticated hydrological modelling often requires rainfall data input at a time resolution of at least 1 hour, sometimes even minutes. Such data are very seldom available, instead daily or at best 6-hour values observed by meteorological services must be used. Thus new ways of connecting the rainfall properties at these different scales are of great importance.

The estimated values of the multifractal parameters $\alpha=0.63$ and $C_1=0.44$ agree fairly well with previous studies. The number of investigations is, however, still low and more analyses of data observed at different time and space scales in different climatic regions are needed to further validate the parameter values. However, in parallel with the continuing rainfall data analyses the development of methods for using the multifractal approach in practical rainfall applications should be commenced. For example, to extract robust and useful information about the small-scale rainfall properties from larger-scale data, something that would increase the accuracy of the output from hydrological models.

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