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The $l^{1/2}$ law and multifractal topography: theory and analysis

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Abstract. Over wide ranges of scale, orographic processes have no obvious scale; this has provided the justification for both deterministic and monofractal scaling models of the earth's topography. These models predict that differences in altitude (Δh) vary with horizontal seperation (l) as $\Delta h \approx l^H$. The scaling exponent has been estimated theoretically and empirically to have the value H=1/2. Scale invariant nonlinear processes are now known to generally give rise to multifractals and we have recently empirically shown that topography is indeed a special kind of theoretically predicted "universal" multifractal. In this paper we provide a multifractal generalization of the $l^{1/2}$ law, and propose two distinct multifractal models, each leading via dimensional arguments to the exponent 1/2. The first, for ocean bathymetry assumes that the orographic dynamics are dominated by heat fluxes from the earth's mantle, whereas the second - for continental topography - is based on tectonic movement and gravity. We test these ideas empirically on digital elevation models of Deadman's Butte, Wyoming.

1. Introduction

The geometry and statistics of topography, coastlines and rivers have always provided stimulus to mathematics and geophysics. Modern treatments include Perrin's (1913) discussion of the "tangentless" coastline of Brittany, and Steinhaus's (1954) discussion of the "non-rectifiable" nature of rivers. Explicit applications of scaling ideas to topography go back at least to Vening-Meinesz (1951) who found that the spectrum of fluctuations (E(k)) at wavenumber k was of the scaling (power law) form $E(k) \approx k^{-\beta_h}$ with spectral slope $\beta_h \approx 2$ (in accord with Bell 1975, Bills and Kobrick 1985). Similarly, using yardsticks of varying lengths, Richardson (1961) found various coastlines to be scaling over wide ranges and determined the relevant exponents. Mandelbrot (1967) related Richardson's exponents to fractal dimensions.

Scaling models of topography have also been developed. To our knowledge, the first was the deterministic scaling model of ocean floor bathymetry (Turcotte and Orburgh, 1967, see Parsons and Sclater 1977 for a review), who developed a theory to explain the (near) $l^{1/2}$ law for ocean floor bathymetry at a distance l from a ridge. Soon after this, Mandelbrot (1975, 1977) independently proposed a fractional Brownian motion model of terrain which was nothing more than a fractional integration (power law filter) of order H of gaussian white noise (Voss (1983), see Fournier et al (1982) for a variant). This power law filter yielded a stochastic topography model whose typical altitude fluctuation (Δh) varies as $\Delta h \approx l^H$. It had the basic property that all iso-lines - independently of the altitude - had fractal dimensions D=2-H (it was hence "monofractal"). It also respected another monofractal relation[1] $\beta_h=1+2H$. Visually and statistically[2], these models seem most realistic for $D \approx 3/2$, $H \approx 1/2$, $\beta_h \approx 2$.

Surprisingly, direct estimates of fractal dimensions of various topographic sets (e.g. lines of constant altitude) have not been numerous[3]. The main examples of which we are aware are Goodchild (1980), Aviles et al (1987), Okubo and Aki (1987), Turcotte (1989), De Cola (1989, 1990), Klinkenberg and Clarke (1992), Klinkenberg and Goodchild (1992) and Gaonac'h et al (1992). The first multifractal analyses of topography have been even more recent (Lovejoy and Schertzer (1990), Lavallée et al (1993)), as have been the corresponding multifractal

[1] These relations have often been used in the form $D=(\beta-5)/2$, to infer D from β. Unfortunately, they break down completely for multifractals since D is no longer unique (see discussion in Lavallée et al 1993).

[2] Actually, whereas "spectral methods" (using the monofractal formula $H=(\beta-1)/2$) give $H \approx 0.5$, the direct measurements of fractal dimension have tended to give values of $D \approx 1.3$ hence, assuming monofractality; $H=2-D \approx 0.7$. The discrepancy in estimates is resolved with multifractals, see footnote 3, and the discussion below and in Lavallée et al 1993.

[3] We specifically exclude variogram methods such as those reported in Mark and Aronson (1984), Rees and Muller (1990) and Clarke and Schweizer (1991), or spectral methods such as those reported in Gilbert (1989) and Mareschal (1989). These measure the scaling exponent of the second order moments (K(2) defined below) - which is not a fractal dimension. In order to infer fractal dimensions they must assume monofractality.

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simulations (Sarma et al (1990), Wilson et al (1991), Pecknold et al 1993). Below, we put the $l^{1/2}$ law in a multifractal framework and show how the exponent can be derived from physical models of orographic processes. We also provide a new estimate for H as well as other universal multifractal parameters.

2. Multifractal processes and topography

Scale invariant geophysical systems are typically nonlinear, we anticipate that they will be associated with multiplicative processes and multifractals. The relevant multiplicative processes were first developed as models of turbulent cascades[4], initially proposed as a description of atmospheric dynamics by Richardson (1922). If we add the idea that the fundamental physical quantity in turbulence is the conserved energy flux from large scales to small[5], we render Richardson's cascades quantitative, since dimensional analysis yields the famous Kolmogorov (1941) $l^{1/3}$ law for velocity fluctuations:

$$\Delta v_l \approx \varepsilon_l^{1/3} l^{1/3} \qquad (1)$$

where Δv_l is a velocity fluctuation and ε_l is the energy flux to smaller scale through structures ("eddies") of size l. Note that the value of the exponent H=1/3 results purely from dimensional considerations which follwos from the assumptions a) l is within the scaling regime, b) in this range, ε is the fundamental physical quantity. In its original formulation, ε was considered spatially homogeneous, hence on a given realization of a turbulent flow, ε_l could be treated as constant and the standard deviation of the random velocity fluctuations (Δv_l) depended essentially on l. Although Kolmogorov did not suggest this[6], a fractional Brownian motion (e.g. a monofractal) with H=1/3 would be a model for such velocity fluctuations. Later, with the recognition of the importance of the intermittency (extreme variability) of turbulence, ε itself was considered highly variable (e.g. Kolmogorov (1962) and Obhukhov (1962) suggested that it was a lognormal variable) and cascade models were later constructed in order to model the variability.

In these models, the large scale multiplicatively modulates the smaller scales; it is now known that these cascades generically lead to multifractals (Schertzer and Lovejoy 1984, 1987). For the moment, the significant point is that throughout this historical development eq. 1 continued to hold, although its interpretation changed significantly. In the original model, a single ("universal") exponent H was sufficient, whereas in the general multifractal/multiplicative cascade model, an infinite number of exponents is needed to characterize ε (we shall see below, that in actual fact, only two more fundamental "universal exponents" are likely to be required). Indeed, it is now clear that whereas fractal geometry is adequate for dealing with scaling geometric sets of points, scaling fields (such as topography) will generally require multifractal fields. Even without direct empirical confirmation of this using using functional box-counting (Lovejoy and Schertzer 1990), it would indeed have been remarkable if all the different altitude isolines had the same unique fractal dimensions[7].

With this in mind, we can now consider the topography. In analogy with turbulence may expect the observable altitude fluctuations (Δh_l) to be related to the fundamental multifractal quantity φ_l and to l:

$$\Delta h_l \approx \varphi_l^{\,a}\, l^{H} \qquad (2)$$

A topography model with this behaviour (with H=1/2) was first proposed by Turcotte and Oxburgh (1967) who modelled the ocean floor bathymetry near mid-ocean ridges. This model is deterministic; i.e. in the model the difference in altitude Δh between the mid-ocean ridge and a point distance l from the ridge increases as $\Delta h \approx l^{1/2}$. This square root profile is the result of conductive cooling (and hence contraction) of the freshly formed crust as it moves away from ridges. In multifractal parlance, the model treats the ridge as a "regularity" (= a negative order "singularity") of order 1/2. The model uses a one dimensional heat equation, isostatic equilibrium, a constant velocity of the tectonic plates, and a constant temperature difference between the top and bottom of the plates. A monofractal model obeying eq. 2 but with Δh_l interepreted only as a standard deviation rather than a deterministic quantity can be obtained by considering the basic quanity φ as a gaussian white noise, and the l^H as a power law filter of k^{-H} (k is a wavenumber). This would yield a fractional Brownian motion topography[8].

We now outline a way of obtaining a multifractal physical model with this type of behaviour (i.e. still

[4]There is now a whole series of such phenomenological models: Novikov and Stewart (1964), Yaglom (1966), Mandelbrot (1974), Frisch et al. (1978), Schertzer and Lovejoy (1983, 1987), Benzi et al (1984), Meneveau and Sreenivasan (1987).

[5]The latter is exactly conserved by the nonlinear terms in the Navier-Stokes equations. Here and in the following, the term "conserved" refers to scale by scale conservation of the dynamical process.

[6]Indeed, he may have had such a model in mind since he mathematically described fractional Brownian motion the previous year (Kolmogorov 1940).

[7]A related consequence of multifractality is that theoretical frameworks for scaling surfaces involving self-affine fractal sets are no longer relevant.

[8] Since the gaussian white noise φ is space filling, in the notation used below, C_1=0. From a modelling point of view, the difference between a fractional Brownian motion model of topography and a multifractal model is that in the former case, the altitude is obtained by power law filtering of a gaussian white noise, while in the latter case, by power law filtering of a conserved multifractal (by k^{-H} in both cases).

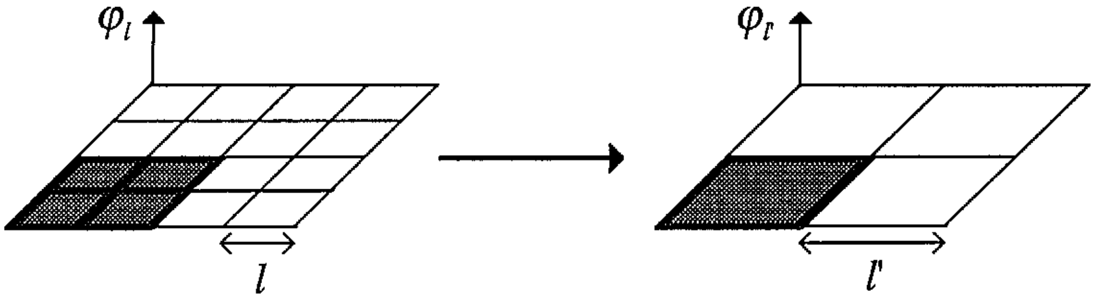

Fig. 1: Schematic illustration of the re-scaling operation performed to obtain the field at different resolutions. Example of a remapping operator $\varphi_{l'} \to \varphi_l$ is defined in eq. (4).

obeying eq. 2, but with φ and h multifractal[9]). Since, the analysis of the topographic data sets below indicate that $H\approx 1/2$, it turns out that we can use essentially the same dimensional quantities as Turcotte and Oxburgh (1967).

Take as the basic conservative quantity a multifractal heat flux Q_l from the mantle (which is postulated to be the basic flux across the mantle/crust boundary responsible for the forcing, it has units of energy per unit area per unit time, kg/s^3) and then (in the spirit of the Turcotte and Orburgh model) a number of dimensional parameters which essentially serve to convert heat fluctuations into altitude fluctuations. In this picture, the underlying horizontal variations in the heat flux from the mantle (arising from nonlinear but scaling mantle dynamics), lead to temperature changes and then finally to altitude variations via thermal expansion. In the simplest model incorporating this physics, it suffices to consider a mean conduction coefficient κ (m^2/s), and mean density ρ (kg/m^3), there is then a unique conserved quantity φ_l compatible with H=1/2. By adding other dimension quanitities such as, the thermal expansion coefficient (α, units K^{-1}) or the specific heat (C_p, units K/kg), the uniqueness is lost (for example, we obtain the dimensionless group ($\kappa^3 C_p \alpha \rho^2 Q_l^{-1}$)), but for our purposes here, any dimensionally correct combination will suffice. The result is a simple topography "flux" with correct dimensions given by:

$$\varphi_l \approx \left(\left(\frac{\rho}{Q_l} \right)^{\frac{1}{3}} \kappa \right)^{\frac{1}{2}} \tag{3}$$

$$\Delta h_l = \varphi_l \, l^{1/2}$$

(i.e. H=1/2, *a*=1).

More directly relevant to the continental topography analysed here (Deadman's Butte Wyoming, 512X512 pixels at 50m resolution, see Lavallée et al 1993 for more information), are the standard orographic models which assume plate collisions and isostatic equilibrium (nonlinearly) coupled with sediment transport and erosion processes. As in the bathymetry model, the physical processes which determine the topography act over a wide range of scales: a priori there is no reason to assume that they break the scale invariant symmetry (i.e. that they introduce a characteristic length). A simple model compatible with this is to consider that the only dimensional quantities responsible for the altitude fluctuations are the force of gravity (g), and a velocity shear Δv_l arising from the horizontal velocity field associated with the movement of the tectonic plates, subplates etc. In this case, the only dimensionally consistent combination yielding H=1/2 is $\varphi_l = \frac{\Delta v_l}{\sqrt{g}}$ i.e. *a*=1 as above.

In either case, we will consider φ_l to be the result of scale invariant multifractal cascade process which generically leads to the following multiscaling property[10]:

$$\langle \varphi_l^q \rangle \propto l^{-K(q)} \tag{4}$$

Note that in additive monofractals such as fractional Brownian motion, K(q)=0, whereas for intermittent multiplicative "β model" (Novikov and Stewart 1964, Frisch et al 1978) monofractals, K(q) is linear.

3. Analysis of scaling properties of the statistical moments

We now show by analysing DEM's of Deadman's Butte that K(q) is both nonzero and nonlinear -hence φ_l is indeed multifractal - and that $H\approx 1/2$. This empirical result was recently reported in Lavallée et al 1993 using a somewhat different empirical analysis technique. The DEM is systematically rescaled to lower (or coarsen) the resolution (see fig. 1) by averaging:

[9]In multifractal models, structures such as ocean ridges are singularities of various orders. Near them, it will be possible to discern various power law behaviours. Note that multifractality means that each level set will be characterized by a different fractal dimension and the statistical moments will be multiscaling (i.e. follow eq. 4). This should not be confused with broken scaling (e.g. scale dependent fractal dimensions).

[10]The (intrinsic) codimension notation K(q) is related to the more familiar dimension notation (Halsey et al. (1986)) by the following: $\tau_D(q) = (q-1)D-K(q)$. See Schertzer and Lovejoy (1991) for a discussion.

$$\varphi_l = \frac{\int_{B_l} \varphi_{l'} d^D x}{\int_{B_l} d^D x} \tag{5}$$

where the l' is the intrinsic resolution of the data, l the coarse grained resolution. The integration (or sum) is performed over all the events in the box B_l of volume l^D and D is the dimension of the support of the fields (=2 here). The q^{th} statistical moment[11] $\langle \varphi_l^q \rangle$ is then estimated by summing φ_l^q over all $((L/l')^D$ boxes) needed to disjointedly cover the entire data set (size L) followed by a sum over all N independent samples (=1 here).

We now now seek to determine the scaling of the conservative φ in terms of the observed h. Since the spectrum is the fourier transform of the autocorrelation function (a second order moment) its energy spectrum is a power law with spectral exponent $\beta_\varphi = 1 - K(2)$. Furthermore, since l^H in eq. (1) corresponds to a further filter by k^{-H} in Fourier space (hence of the modulus by (k^{-2H}), the spectrum of h will have a scaling exponent given by the following expression:

$$\beta_h = \beta_\varphi + 2H = 1 - K(2) + 2H \tag{6}$$

Whereas the spectrum of the conservative φ is always less steep than a "1/f noise" (since $K(2) \geq 0$), and empirically, for this data set, $\beta_h \approx 1.93 \pm 0.03 > 1$ (Lavallée et al 1993), h cannot be a conservative process[12]. To obtain an estimate of φ_l, it is sufficient to fractionally differentiate either in Fourier space, or (for simplicity) here, by studying the modulus of the gradient of the topography[13]:

$$\varphi_{l'} = |\nabla h|_{l'} \tag{7}$$

The resulting scaling behavior of the $\log\langle \varphi_l^q \rangle$ as function of $\log(L/l)$ is illustrated in fig. 2 for several values of q. The straight line behaviour indicates that the scaling is well respected over a range[14] of L/l going from 512/1 to 512/64. The slopes of these curves give the (nonlinear) behavior of K(q) which shows that the topography is multifractal (see fig. 3). Using eq. (5), the value of H can now be inferred from the observed β_h and K(2). Fig. 3 shows that $K(2) \approx 0.10 \pm 0.02$, combined with[15] $\beta_h \approx 1.93 \pm 0.03$ we obtain $H \approx 0.51 \pm 0.03 \approx 1/2$. This last result confirms fairly well the $l^{1/2}$ law mentioned earlier and is in accord with the estimates of Lavallée et al 1993.

4. Universal Multifractals

We have already mentioned that stable, attractive behavior of multifractal processes leads K(q) to follow certain "universal" forms[16] (Schertzer and Lovejoy 1987, 1988, 1989, 1991, Fan 1989, Schertzer et al 1991). This means that under fairly general circumstances, independently of many of the dynamical details, we may expect to obtain special "universal" multifractals with the following K(q):

$$K(q) = \begin{cases} \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right) & ,\alpha \neq 1 \\ C_1 \log q & ,\alpha = 1 \end{cases} \tag{8}$$

where α $(0 \leq \alpha \leq 2)$ and C_1 are the fundamental parameters needed to characterize the processes. The Lévy index[17] α indicating the class to which the probability distribution belongs[18]; it tells us how far we are from monofractality: α=0 corresponds to the monofractal "β model", α=2 is the maximum, the "lognormal" universal multifractal[19]. The parameter C_1 is the fractal codimension of the singularities contributing to the average values of the field - it tells us about the sparsity of the average level of intensity. Furthermore, if C_1 >D (the dimension of space in which the process is observed), then the multifractal is "degenerate" on the space i.e. it almost surely vanishes everywhere.

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11 Generalization to the trace-moment is discussed in Schertzer and Lovejoy (1987) and Schertzer et al. (1991). Further generalization to the double trace moments (DTM) were introduced in Lavallée (1991) and Lavallée et al. (1992).

12 The spectral slope is connected with the apparent stationarity or nonstationarity of the process. For $H \leq 0$ all normalized powers of the corresponding field will be apparently stationary, whereas for $H>0$ certain powers will be apparently nonstationary (i.e. for scales within the scaling regime). Note that in multifractals, unlike (the perhaps more familiar) quasi-gaussian systems, second order moments play no special role.

13 This could be considered to be a standard "pre whitening" procedure, although the resulting field need not generally be white, it need only be at least as spectrally flat as the conserved process φ_l. The assumption is that the estimate of $K(q)$ is not affected by residual nonconservation as long as the spectrum is fairly flat (when $\beta \leq 1 - K(2)$). This approximation has been discussed with the help of numerical simulations in Lavallée 1991 and Lavallée et al 1993.

14 Scaling is not well observed for the larger values of l, l=128, 256 and 512. This may be due to the small number of events included in the sum to approximated statistical moments $\langle \overline{\varphi}_l^{\,q} \rangle$ (see eq. 6) at large scale length l respectively 16, 4 and 1.

15 For comparison, standard Brownian motion has H=1/2, K(q)=0, hence β=2. In spite of the fact that the spectral slope (=1.93) is close to the Brownian motion value (=2), the difference in statisics is important since in the multifractal case, the probability distributions will be long tailed log-Levy distributions whereas in the former case they will be gaussian and large fluctuations will be rare.

16 For the recent debate about strong v.s. weak multifractal universality, see Schertzer and Lovejoy, 1994a.

17 From eq. 8, it can be seen that α is also the degree of nonanalyticity of K(q) as $q \rightarrow 0$; Mandelbrot et al 1990 has stressed this aspect, calling the corresponding φ a "left handed measure".

18 In technical parlance, the hypsographic curve (the histogramme of elevation values) will be log-Levy with the parameters indicated below.

19 The widespread geophysical phenomenology of lognormal distributions can thus be explained by universality.

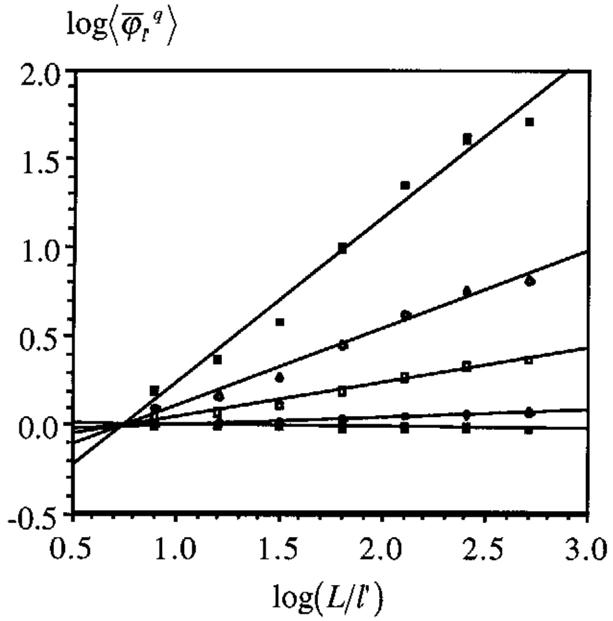

Fig. 2: The scaling behavior of the statistical moments of the Deadman's Butte data is illustrated here by the straightness of the curves of the $\log\langle\overline{\varphi}_l{}^q\rangle$ as functions of the $\log(L/l)$, with L the largest scale in the DEM: from bottom to top q=0.5, 1.5, 2.5, 3.5 and 5. The slopes correspond to the scaling exponent K(q).

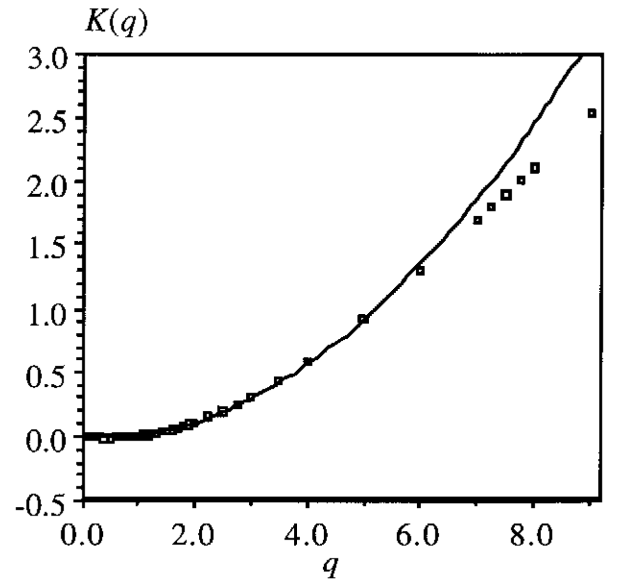

Fig. 3: The scaling exponent K(q) against q for the Deadman's Butte data (see fig. 2). The continuous curve is the theoretical universal multifractal fit with α=1.9, C_1=0.05 (see eq. (6)). For q>6, the asymptotic behavior of K(q) becomes linear. This is a second order multifractal phase transition predicted from the finite sample size.

The values of the parameters α and C_1 can be determined from the estimated curves of K(q). The behavior of K(q), given in eq. (7), allows us to deduce two relations, each dependent on only one of the two universal parameters, α or C_1. The minimum values q_{min} of K(q) is a transcendental function α (see Lavallée, 1991):

$$\left.\frac{dK(q)}{dq}\right|_{q=q_{\min}} = 0,$$

$$q_{\min} = \left(\frac{1}{\alpha}\right)^{1/\alpha - 1}, \alpha \neq \{0,1\} \tag{9}$$

The slope of K(q) for q=1 is simply equal to C_1:

$$\left.\frac{dK(q)}{dq}\right|_{q=1} = C_1 \tag{10}$$

From the empirically estimated K(q) function, we estimate q_{min}=0.49. From eq. (9) we obtain $\alpha\approx1.9$. The value of C_1 is obtained with this approximation of eq. (10), C_1=(K(1.1)-K(0.9))/0.2=0.055. These values are very close to, and confirm, previous estimates obtained[20] in Lavallée et al., 1993: $\alpha\approx1.9\pm0.1$, $C_1\approx0.045\pm0.005$.

The universal behaviour of K(q) is illustrated by the continuous curve in fig. 3, using eq. (6) with α=1.9 and C_1=0.055, showing the good fit to the empirical K(q) up until the breakdown at $q_s\approx6$. Departure from the universal behaviour is expected for q> min(q_D, q_s) where q_D, q_s characterize respectively the divergence of moments[21] and the undersampling respectively (Lavallée et al (1991)). In these cases K(q) becomes linear for q greater than these critical values, leading to respectively first and second order multifractal phase transitions (Schertzer et al 1993, Schertzer and Lovejoy 1994b). When the number of samples $N=1$, the value of q_s is given by the following relation (see Lavallée, 1991):

$$q_s \approx \left(\frac{D}{C_1}\right)^{1/\alpha} \tag{11}$$

which leads to a value $q_s \approx 7$, close to the value[22] in fig. 3.

5. Conclusions

Scaling laws for the variation in the elevation of the earth's topography have been proposed in both deterministic and then in stochastic, but monofractal frameworks in which lines of coonstant altitude all had the same fractal

[20]This can be compared with the corresponding estimates for French topography over the range 1-1000km; α=1.7±0.1, C_1=0.075±.005; Lavallée et al 1993.

[21]The divergence of moments is now considered the hallmark of self-organized critical (SOC) phenomena. Multifractals generically involve the divergence of high order moments, hence they provide a nonclassical route to SOC, see Schertzer and Lovejoy 1994b.

[22]Examination of the probability distribution shows that q_D>8.

dimensions. In both cases, scaling exponents (H) of value ≈1/2 have been invoked for either theoretical or empirical reasons. Drawing on recent theoretical and empirical work showing that topography is multifractal rather than monofractal, we outline multifractal models based on mantle heat fluxes and on shears of tectonic plates to argue that the law reappears in this new framework. We then empirically test the multifractal law on DEM's of Deadman's Butte, confirming a recent estimate that $H \approx 0.51 \pm 0.03$. Finally, we show that the multiple scaling exponent K(q) is well described by universal multifractals, and we give a new estimate of the basic parameters.

The values of K(q), and H that we have obtained indicate that the height field has seemingly paradoxical properties that go a long way towards explaining both the successes and the limitations of monofractal analyses and models (which generally assume K(q)=0). First, when q is not far from 1 (the mean gradient), K(q) is small (since $C_1 \approx 0.05$) compared to H (≈0.5) implying that corresponding topography properties are nearly constant and have fractal dimensions ≈2-H=1.5 for lines of constant altitude for topography of average smoothness. However, once we start examining isolines at for very rough, wild topography (or high order moments of gradients), the rapid increase of K(q) with q indicates[23] that the multifractal nature of the process will become dominant.

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