Nonlinear Processes in Geophysics (1995) 2: 131 - 135

Nonlinear Processes in Geophysics

Size-frequency relation of earthquakes in load-transfer models of fracture

J. B. Gómez[1], D. Iñiguez[2] and A. F. Pacheco[2]

[1] Department of Geological Sciences, University College London, Gower Street, London WC1 6BT, U.K.
[2] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

Received 10 June 1994 - Accepted 22 August 1994 - Communicated by D. Sornette

Abstract. Using Monte Carlo simulations of the process of breaking in arrays of elements with load-transfer rules, we have obtained the size-frequency relation of the avalanches occurring in 1- and 2-dimensional stochastic fracture models. The resulting power-law behaviour resembles the Gutenberg-Richter law for the relation between the size (liberated energy) of earthquakes and their number frequency. The value of the power law exponent is calculated as a function of the degree of stress dissipation present in the model. The degree of dissipation is implemented in a straightforward and simple way by assuming that only a fraction of the stress is transferred in each breaking event. The models are robust with respect to the degree of dissipation and we observe a consistent power-law behaviour for a broad range of dissipation values, both in 1D and 2D. The value of the power-law exponent is similar to the phenomenological b-value ($0.8 \leq b \leq 1.1$) for intermediate magnitude earthquakes.

1 Introduction

The load-transfer stochastic models of fracture were initially conceived to describe the strength of fibre bundles and extended later to many other fracture systems (Smalley et al., 1985; Turcotte et al., 1985; Sornette, 1989; Lomnitz-Adler et al., 1992; Gómez et al., 1993a, b). In these models, we deal with loaded sets of strong elements within which a process of correlated breaking takes place. It is assumed that the probability of failure of any element of the set, supporting a weight σ, is given by a Weibull distribution function

$$p_\sigma = 1 - e^{-(\sigma/\sigma_0)^\rho}, \qquad (1)$$

where σ_0 is a reference strength and ρ is an integer called the Weibull index (Weibull, 1939). The elements are positioned, say, on the sites of a lattice and the correlation among the elements is implemented by admitting that if an element fails, the load it supports is transferred, according to fixed rules, to other elements of the set, provoking new failures; the succession of induced failures ends either with a total or a partial collapse of the whole system.

A paradigmatic modality of the load-transfer rules is the equal load-sharing (ELS) model (Daniels, 1945; McCartney and Smith, 1983; Smith, 1980, 1981; Smith and Phoenix, 1981; Sornette, 1989), where the non-failed elements share the load equally and all the failed elements carry no load. Another case is that of the local load-sharing (LLS) model (Scop and Argon, 1969; Gotlib et al., 1973; Harlow and Phoenix, 1978a, 1978b, 1981a, 1981b; Phoenix and Smith, 1983; Gómez et al., 1993c), where the load of failed elements is given only to the nearest unbroken neighbours. Obviously, any intermediate (or hierarchical) option can be adopted according to the nature of the system to be modelled (Smalley et al., 1985; Turcotte et al., 1985; Gómez et al., 1993a).

Smalley et al. (1985), with the aim of accounting for the stick-slip behaviour of faults, used this type of static probabilistic models, assuming that the rule of stress transfer had a hierarchical (fractal) architecture. Later, we also explored these ideas emphasising the good properties of the simplest ELS models (Gómez et al., 1993a). As a qualitative progress within the group of LLS models, we should mention our recent proposal of a 'chiral' simplification of the 1-dimensional model (Gómez et al., 1993c), whose strength can then be obtained through a simple iterative method.

In the applications mentioned so far, the total load applied to the set is supposed to be constant. This seems reasonable when dealing, for example, with a loaded cable in a lab experiment, but not when one is trying to model the behaviour of an asperity set in a fault segment. In this case, it is reasonable to accept the existence of stress losses when breaking occur, so that only a part of the stress is redistributed among the surviv-

Correspondence to: J. B. Gómez

ing asperities. In this sense we have recently modelled (Gómez et al., 1994), in the most simple way, this concept of stress dissipation through a constant factor, α. We assume that in each event of stress transfer, only a fixed fraction α ($0 \le \alpha \le 1$) of the load supported by a failing element is actually transferred. The rest, i.e. the fraction $(1-\alpha)$, is lost. Thus, α acts as a sort of correlation parameter because in the limit $\alpha = 0$ the elements break independently and their assigned load is dissipated.

The inclusion of the stress dissipation factor does not seem to be required only by the very nature of the geophysical systems to which we want to apply these models; indeed, when one analyses the effect of including an $\alpha \neq 1$, one observes that sets with local load-transfer rules acquire an approximately N-independent strength in the manner of the ELS sets, N being the number of elements composing the set.

One of the properties most intensively studied in these models is the set strength, i.e. the average value of stress level at which a set, with N fixed, fails (e.g., McCartney and Smith, 1983; Phoenix and Smith, 1983; Newman and Gabrielov, 1991). Another interesting property is the size-frequency relation (SFR) of the breaking bursts occurring during the process of loading (Hemmer and Hansen, 1992; Christensen and Olami, 1992; Ding and Yu, 1993). This paper refers to the Monte Carlo computation of this relation, for several load-transfer rules and devoting special emphasis to 2-dimensional local load-transfer systems, which are the most interesting to model real earthquakes. Our interest in this computation was basically triggered by the work of Hemmer and Hansen (1992), who obtained a universal power-law dependence for the SFR in ELS models. The structure of this paper is the following. In Sect. 2, we explain the Monte Carlo simulations for the various modalities of load-transfer, and in Sect. 3, the results for the SFR and our conclusions are presented.

2 Size-frequency relations

As explained in Sect. 1, we start with an array formed by N intact elements with a stochastic strength distribution, obeying Eq. 1. (Let us suppose, for definiteness that the set obeys LLS rules). Now we proceed to slowly increase the stress acting on the elements until reaching the level of failure of the weakest element in the set. We will denote this first stress increase as $\Delta\sigma_1$. After this increase, that element fails and its load is transferred to its nearest neighbours. As a consequence, new breakings can occur and let us denote by n_1 the number of elements broken in this first burst, earthquake or avalanche (we will use thereafter the three words as synonymous). After this, the stress distribution in the set will be non-homogeneous, and every element will have a strength higher than the local stress acting on it. Now we proceed to increase homogeneously again the stress on the whole system until reaching the strength of the new weakest element. This increase will be denoted by $\Delta\sigma_2$, and a second avalanche will come about, composed by n_2 elements. This process is iterated and a sequence of stress increases $\Delta\sigma_1$, $\Delta\sigma_2$, $\Delta\sigma_3$, $\dots$, and avalanche sizes n_1, n_2, n_3, $\dots$ will generated up to the concluding event in which a final avalanche destroys the remainder of the set. Then a new cycle can be initiated along similar lines.

We suppose that the process of stress increase proceeds at a constant rate and therefore the magnitude of any of the steps $\Delta\sigma_i$, is proportional to the time interval involved in that step. Accordingly $\sum \sigma_i$ is a measure of the total time taken by that cycle. On the other hand, the number n_i of elements broken in burst $\#i$ measures the area broken in that avalanche, and will be called the size of the burst. (In seismology, the area broken in a fault during an earthquake, is proportional to the energy release in that event, e.g. Scholz, 1990). Thus, in each cycle we will have a distribution of burst sizes, or in other words, a size-frequency relation, which will depend on the assumed details of the model (as load-transfer rules, degree of stress dissipation, and, of course, the spatial dimensionality of the model).

The calculation of the SFR for the usual —dissipationless— ELS model was first done by Hemmer and Hansen (1992) who found a quite interesting result, namely: in that case the SFR has a power-law relation with an exponent, $\xi = 5/2$, independent of the distribution of breaking strength of the individual elements (in the case of assuming a Weibull distribution function, Eq. 1, for the strength of the elements this means that the exponent ξ is ρ-independent). Expressing the SFR as the number of bursts, f, with a size bigger than a given one, n, they found that

$$f \propto n^{-b} \qquad b = 1.5. \tag{2}$$

(Note that if one considers instead the number of bursts of size n *vs.* n, the resulting relation is also of power law type but with an exponent equal to $\xi = b + 1$. In this case $\xi = 2.5$. Figures 1 and 2 are expressed in this way).

The power law form for the SFR has a close analogy with the empirical law found by Gutenberg and Richter (1954) in seismology:

$$\log f_M = \text{const} - bM, \tag{3}$$

where f_M is the number of earthquakes occurring in a given region, for a definite time interval, with a magnitude bigger than M. As it is known that the earthquake magnitude is related to the area broken in the event by $M = \log n + \text{const}$, we have that (3) implies

$$\log f_n = \text{const} - b \log n, \tag{4}$$

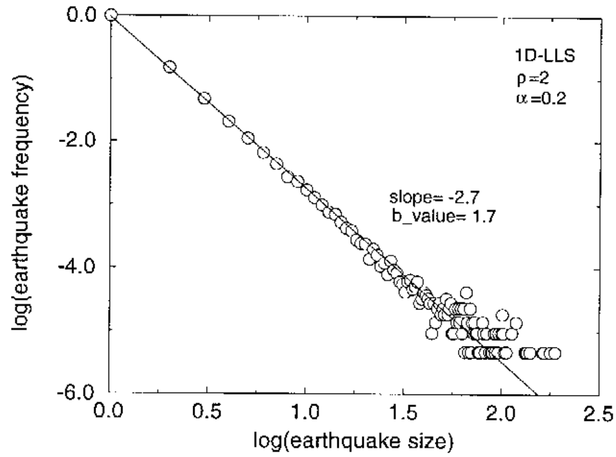

Fig. 1. Frequency vs. size for avalanches, in a characteristic case of the 1-D local model.

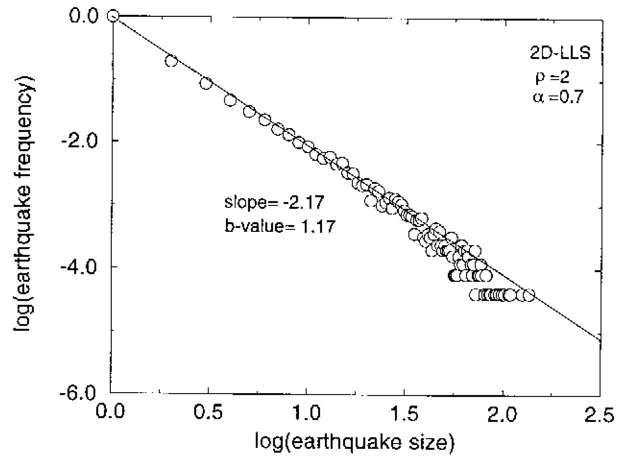

Fig. 2. Frequency vs. size for avalanches, in a characteristic case of the 2-D local model. The axis scaling is the same as in Fig. 1 in order to easily compare the slopes.

which coincides with the form of Eq. 2. Note that the value of the slope b, defined in Eq. 2 corresponds to the Gutenberg-Richter slope, which phenomenologically is $0.8 \leq b \leq 1.1$. Under this optics, the theoretical result obtained from the conservative ELS model is a bit too big.

Our task in the next Section will be to find out if this power-law behaviour holds or not, in the various modalities of load transfer models.

3 Results and Conclusions

The type of SFR obtained through Monte Carlo simulations for load transfer models is illustrated in Figs. 1 and 2. Figure 1 corresponds to the results emerging from a LLS model in 1D, in which the transfer occurs in both directions in the chain. For this figure a Weibull index $\rho = 2$ and a dissipation factor $\alpha = 0.2$ were assumed. As indicated above, f *vs.* n are both plotted in logarithmic scales. The slope of the least square fit is -2.7, and the corresponding b-value 1.7. Thus the power law dependence is well manifested, except in the region of very big avalanches where it shows a characteristic scatter. Note that for big sizes although it seems, there is no multivalueness in the frequency. In this graph, the size of the set used was 4100 elements and a statistics of 100 cycles of total breaking were performed.

Figure 2 shows the results obtained for 2D rectangular arrays of 100×50 elements, with LLS rules of transfer. The parameters chosen were $\rho = 2$, $\alpha = 0.7$. The number of cycles analysed was 20. Again, a power-law dependence is well apparent, with a slope of -2.17 and a b-value of 1.17.

These two figures illustrate the behaviour observed in many cases, which proves the important result that this family of models has SFR of the type of Eq. 2 for a broad range of α-values.

In Fig. 3, the b-value (i.e. the modulus of the power-law exponent and the slope of the best curve fit) obtained for the different models is shown as a function of the degree of stress dissipation, α. Note first that for $\alpha = 1$, i.e. in the non dissipation limit, in the ELS model, $b = 1.5$ which perfectly agrees with the Hemmer and Hansen (1992) prediction. And what is remarkable, the b-value is practically independent of the value of α. This type of universality observed in the ELS case, i.e. b does not depend either on ρ or on α, will not hold in the LLS models. For this group of models, in 1 dimension, we show in Fig. 3 the value of b for $\rho = 2$ and $\rho = 5$. We observe a rather strong dependence on α, reaching a minimum value of $b \simeq 1.4$ at $\alpha \simeq 0.2$ for $\rho = 5$, and $b \simeq 1.5$ at $\alpha \simeq 0.4$ for $\rho = 2$. In 2D, again b varies with α, but we observe that there exists a comfortable range, say between 0.5 and 0.9 (see Fig. 3) where b is within the phenomenological range.

Christensen and Olami (1992) proposed a 2D cellular automata model equivalent to a block-spring model and studied its scaling behaviour as a function of a conservation parameter in the frame of self-organized criticality (SOC). The level of conservation in their model depends on the elastic constant of the 'springs' connecting the blocks, and in the isotropic case we have $\alpha_{Ch} = \frac{1}{4}\alpha_G$, where α_{Ch} is the parameter used by Christensen and Olami (1992) and α_G is the parameter used in this paper. With this scale change we can compare our Fig. 3 with their Fig. 7, noting that the overall behaviour of the b-value is similar in both models for levels of conservation between 0.3 and 0.8. In the range 0.8-1.0 the model of Christensen and Olami (1992) predits a monotonically decreasing value for the b-value to almost zero, whereas our model for the 2D case has a b-value minimum at $\alpha \simeq 0.7$ followed by a steady increase to $b \simeq 2.5$ at $\alpha = 1.0$ (cf. Fig. 3).

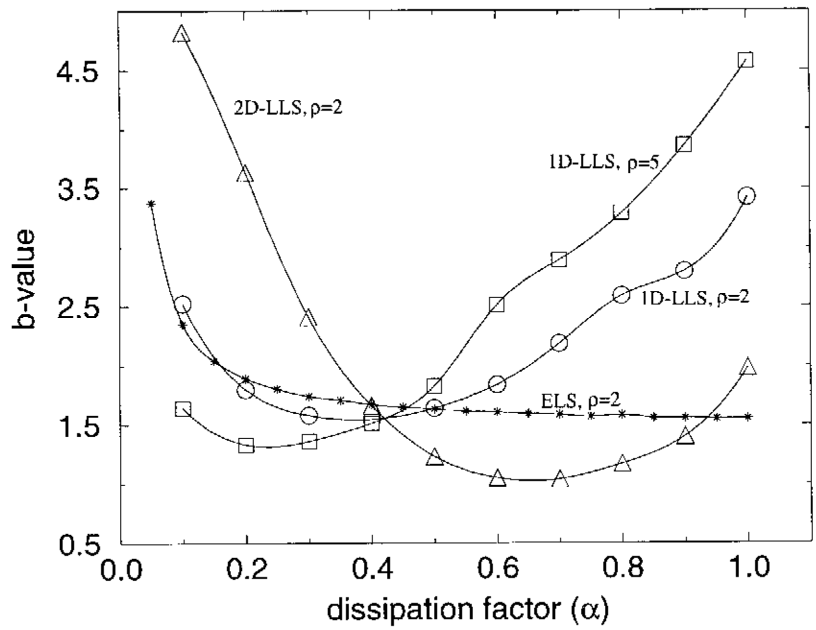

Fig. 3. Magnitude-frequency exponent, b, *vs.* α, for the various models described in the text.

The discrepancy arises because Christensen and Olami's model is a continuously driven system where a sliped element can slip again many times during a single simulation. In our model once a site is broken it remains so for the rest of the simulation. We average the results over many simulations instead of runing the same simulation for longer periods of time. The SFR emerging from our model reflects the cummulative effect of fluctuations, fluctuations which diverge in size when the final instability (total collapse of the system) is approached. In that sense, the 'sweeping of an instability' theory (Sornette, 1994) could be a better theoretical framework than SOC to interpret our results.

Thus the main conclusions that can be drawn from these results are: i) load-transfer models of breaking fulfil the Gutenberg-Richter relation for the spectrum of their avalanches for a broad range of α-values. And ii) the value of the power-law exponent, b, is a function of the degree of stress dissipation one is assuming, when local load-transfer rules are used. For this modality of transfer, in two spatial dimensions, the b-value is of the order of that observed in real earthquakes. This stimulates positively the use of thése models in seismology.

Acknowledgements. This work was supported by the Spanish DGICYT (Project PB90-0916). One of the authors (JBG) is supported by a Spanish post-doctoral fellowship (PF93-25428779). We thank the RTN collaboration (University of Zaragoza) for computer support.

References

Christensen, K., and Olami, Z., Scaling, phase transitions, and nonuniversality in a self-organized critical cellular-automaton model, *Phys. Rev. A*, *46*, 1829–1838, 1992

Daniels, H. E., The statistical theory of the strength of bundles of threads, *Proc. Roy. Soc. Lond. A*, *183*, 404–435, 1945.

Ding E. J., and Yu, Y. N., Analytical treatment for a spring-block model, *Phys. Rev. Lett.*, *70*, 3627–3630, 1993.

Gotlib, Y. Y., El'yashevich, A. M., and Svetlov, Y., Effects of microcracks on the local stress distribution in polymers and their deformation properties, *Soviet Phys. Solid State*, *14*, 2672–2677, 1973.

Gómez, J. B., Pacheco, A. F., and Seguí, A. J., Statistical model for the stick-slip behaviour of faults, *Geophys. J. Int.*, *113*, 115–124, 1993a.

Gómez, J. B., Pacheco, A. F., and Seguí, A. J., Instability of loaded fractal trees, *J. Appl. Mech.*, *60*, 959–961, 1993b.

Gómez, J. B., Iñiguez, D., and Pacheco, A. F., Solvable fracture model with local load-transfer, *Phys. Rev. Lett.*, *71*, 380–383, 1993c.

Gómez, J. B., Iñiguez, D., and Pacheco, A. F., Load-transfer models of fracture with stress dissipation, Submitted to *Int. J. Fracture*, 1994.

Gutenberg, B., and Richter, C. F., *Seismicity of the Earth and associated phenomenon*, Princeton University Press, Princeton, 2nd edition, 1954.

Harlow, D. G., and Phoenix, S. L., The chain- of-bundles probability model for the strength of fibrous materials, I: analysis and conjectures, *J. Composite Mater.*, *12*, 195–214, 1978a.

Harlow, D. G., and Phoenix, S. L., The chain-of-bundles probability model for the strength of fibrous materials, II: a numerical study of convergence, *J. Composite Mater.*, *12*, 314–334, 1978b.

Harlow, D. G., and Phoenix, S. L., Probability distributions for the strength of composite materials, I: two level bounds, *Int. J. Fracture*, *17*, 347–372, 1981a.

Harlow, D. G., and Phoenix, S. L., Probability distributions for the strength of composite materials, II: a convergent sequence of tight bounds, *Int. J. Fracture*, *17*, 601–630, 1981b.

Hemmer, P. C., and Hansen, A., The distribution of simultaneous fiber failures in fiber bundles, *J. Appl. Mech.*, *59*, 904–914, 1992.

Lomnitz-Adler, J., Knopoff, L., and Martínez-Meckler, G., Avalanches and epidemic models of fracturing in earthquakes, *Phys. Rev. A*, *45*, 2211–2221, 1992.

McCartney, L. N., and Smith, R. L., Statistical theory of the strength of fibre bundles, *J. Appl. Mech.*, *50*, 601–608, 1983.

Newman, W. I., and Gabrielov, A. M., Failure of hierarchical distribution of fibre bundles, I, *Int. J. Fracture*, *50*, 1–14, 1991.

Phoenix, S. L., and Smith, R. L., A comparison of probabilistic techniques for the strength of fibrous materials under local load-sharing among fibres, *Int. J. Solid Struct.*, *19*, 479–496, 1983.

Scholz, C. H., *The mechanics of earthquakes and faulting*, Cambridge Univ. Press, Cambridge, 437pp, 1990.

Scop, P. M., and Argon, A. S., Statistical theory of strength of laminated composites, II, *J. Composite Mater.*, *3*, 30–47, 1969.

Smalley, R. F., Turcotte, D. L., and Solla, S. A., A renormalization group approach to the stick-slip behavior of faults, *J. Geophys. Res.*, *90*, 1894–1900, 1985.

Smith, R. L., A probability model for fibrous composites with local load-sharing, *Proc. Roy. Soc. Lond. A*, *372*, 539–553, 1980.

Smith, R. L. and Phoenix, S. L., Asymptotic distributions for the failure of fibrous materials under series-parallel structure and equal load-sharing, *J. Appl. Mech.*, *48*, 75–82, 1981.

Sornette, D., Elasticity and failure of a set of elements loaded in parallel, *J. Phys. A: Math. Gen.*, *22*, L243–L250, 1989.

Sornette, D., Sweeping of an instability: an alternative to self-organized criticality to get powerlaws without parameter tuning, *J. Phys. I France*, *4*, 209-221, 1994.

Turcotte, D. L., Smalley, R. F., and Solla, S. A., Collapse of loaded fractal trees, *Nature*, *313*, 617–672, 1985.

Weibull, W., A statistical theory of the strength of materials, *Proc. Ing. Vetenskapakad Handl, Stockholm*, *151*, 5–45, 1939.