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Oscillating forcings and new regimes in the Lorenz system: a four-lobe attractor

V. Pelino[1], **F. Maimone**[1], **and A. Pasini**[2]

[1]Italian Air Force, CNMCA, Pratica di Mare (Rome), Italy
[2]CNR, Institute of Atmospheric Pollution Research, Rome, Italy

Correspondence to: A. Pasini (pasini@iia.cnr.it)

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Abstract. It has been shown that forced Lorenz models generally maintain their two-lobe structure, just giving rise to changes in the occurrence of their regimes. Here, using the richness of a unified formalism for Kolmogorov-Lorenz systems, we show that introducing oscillating forcings can lead to the birth of new regimes and to a four-lobe attractor. Analogies within a climate dynamics framework are mentioned.

1 Introduction

Some years ago, in a pioneering study about the influence of external forcings on patterns of climate variability, Corti et al. (1999) suggested that these forcings led to a change in the frequency of occurrence of dominant regimes of the Northern Hemisphere atmospheric circulation in the second half of the twentieth century. The authors also showed that this situation is consistent with the simple dynamical-system picture obtained by the insertion of a constant forcing term in the Lorenz system. In fact, even in the latter case, one observes no creation of new regimes/lobes on the Lorenz attractor, but a change in the frequency of residence of the state in the two lobes is clearly detectable via calculation of the two associated values of the probability density function. Even increasing the forcing value does not lead to new regimes but just to the disappearing of chaos: after a certain threshold the attractor becomes a fixed point (see, for instance, Pasini, 2008 and references therein).

Obviously, the constant forcing term introduced in the Lorenz system is a rough analogue of the amount of surplus greenhouse effect created by human activities, which is almost monotonically increasing. But, what happens if the Earth experiences an amplified oscillating forcing endowed with a period much greater than climate variability? This certainly happened, for instance, when the tilt of Earth's axis was greater than today, leading to an enhanced annual oscillation in the solar radiation at every point on the Earth surface and to an increase of the thermal contrast between winter and summer. Unfortunately, one does not possess data about these ancient periods, which would permit analysing this impact on circulation regimes.

In this framework, here we restrict ourselves to adopting a simple dynamical system approach and low-dimensional toy models for investigating the relationships between forcings and the resulting regimes; in particular, we are led to study the impact of oscillating forcings on regimes in the Lorenz model, as detected on its attractor. In doing this simple exercise, we do not aim to explain what happened to the real Earth system, but, as we will see, we supply the evidence that new regimes are possible under oscillating forcings.

As a matter of fact, there is extensive literature on the topic of small periodic perturbations, which often follows the Ruelle approach (Ruelle, 1998, 2009; Reick, 2002; Lucarini, 2009). In the present case, instead, we are dealing with large oscillating forcings and the Ruelle approach cannot be applied.

Furthermore, we would like to point out that a geometrical approach to Lorenz dynamics can be fruitful. In particular, in the past the Lorenz system had been reformulated in some elegant mathematical framework, as that of Nambu dynamics (Nambu, 1973; Nevir and Blender, 1994), which introduces a generalized Hamiltonian and adopts a n-fold product. In our case, we use the Lie-Poisson structure of Lorenz system, which corresponds to the simplest Nambu dynamics.

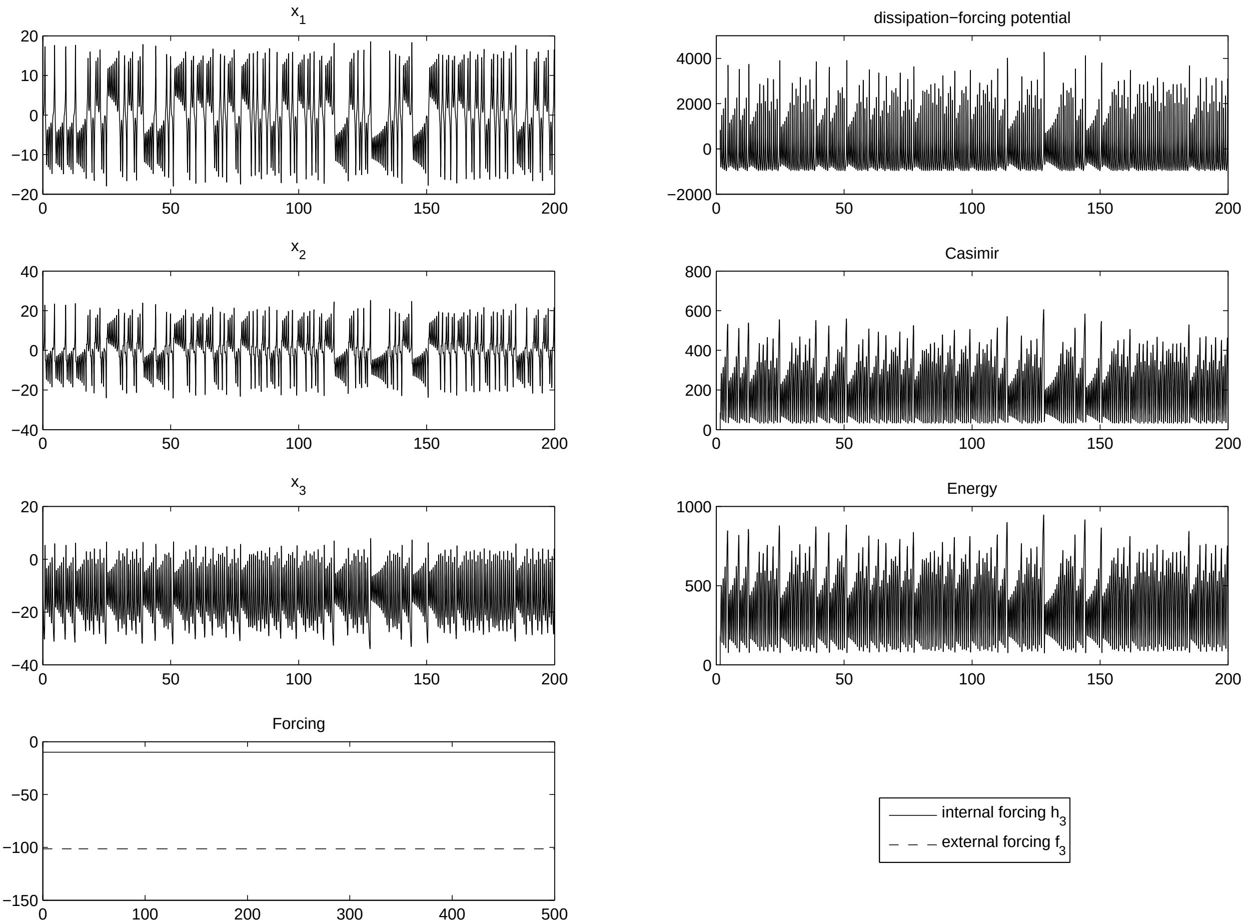

Fig. 1. Standard Lorenz-63 model. Left: trajectories. Right: evolution of dissipation forcing potential Φ, Casimir (C) and energy (H). Internal and external forcings are kept fixed, $h_3 = -\sigma$, $f_3 = -\beta(\rho + \sigma)$, where $\beta = 8/3, \rho = 28, \sigma = 10$.

In what follows, we adopt a unified formalism previously developed (Pasini et al., 1998; Pasini and Pelino, 2000; Pelino and Pasini, 2001; Pasini et al., 2010; Gianfelice et al., 2012) in order to clearly discuss the effect of different forcings and test some general cases, finally reaching the evidence of creation of new regimes and a four lobe Lorenz attractor.

2 A unified formalism for Kolmogorov-Lorenz systems

A typical equation describing dissipative forced dynamical systems can be written in Einstein notation as

$$\dot{x}_i = \{x_i, H\} - D_{ij} x_j + f_i \quad i = 1, 2..., n. \tag{1}$$

Equation (1) has been written by Kolmogorov, as reported in Arnold (1991), in a fluid dynamical context, but they are very common in simulating natural processes. Here, the brackets represent the algebraic structure of the Hamiltonian part of the systems, described by function H and the cosymplectic matrix $\mathbf{J}$ (Marsden and Ratiu, 1994):

$$\{F, G\} = J_{ik} \partial_i F \partial_k G. \tag{2}$$

The positive-definite diagonal matrix $\mathbf{D}$ represents the dissipation and the last term $\mathbf{f}$ is the external forcing. Such a formalism, as mentioned before, is particularly useful in fluid dynamics (Morrison, 1998), where Navier-Stokes equations show interesting properties in their Hamiltonian part (Euler equations). Furthermore, finite-dimensional systems like Eq. (1) represent the proper reduction of fluid dynamical equations (Pasini et al., 1998) in terms of conservation of the symplectic structures in the infinite domain (Zeitlin, 2004). This method, contrary to the classical truncation one, leads to the study of dynamics on Lie algebra, also known as Lie-Poisson equations, which is extremely interesting from the physical viewpoint and has a mathematical aesthetical appeal (Pelino and Pasini, 2001; Pelino and Maimone, 2007). Given a group $\mathbf{G}$ and a real-valued function (possibly time dependent) $H : T_e^* \mathbf{G} \rightarrow \mathbf{R}$, which plays the role of Hamiltonian, in the local co-ordinates x_i, the Lie-Poisson equations read as

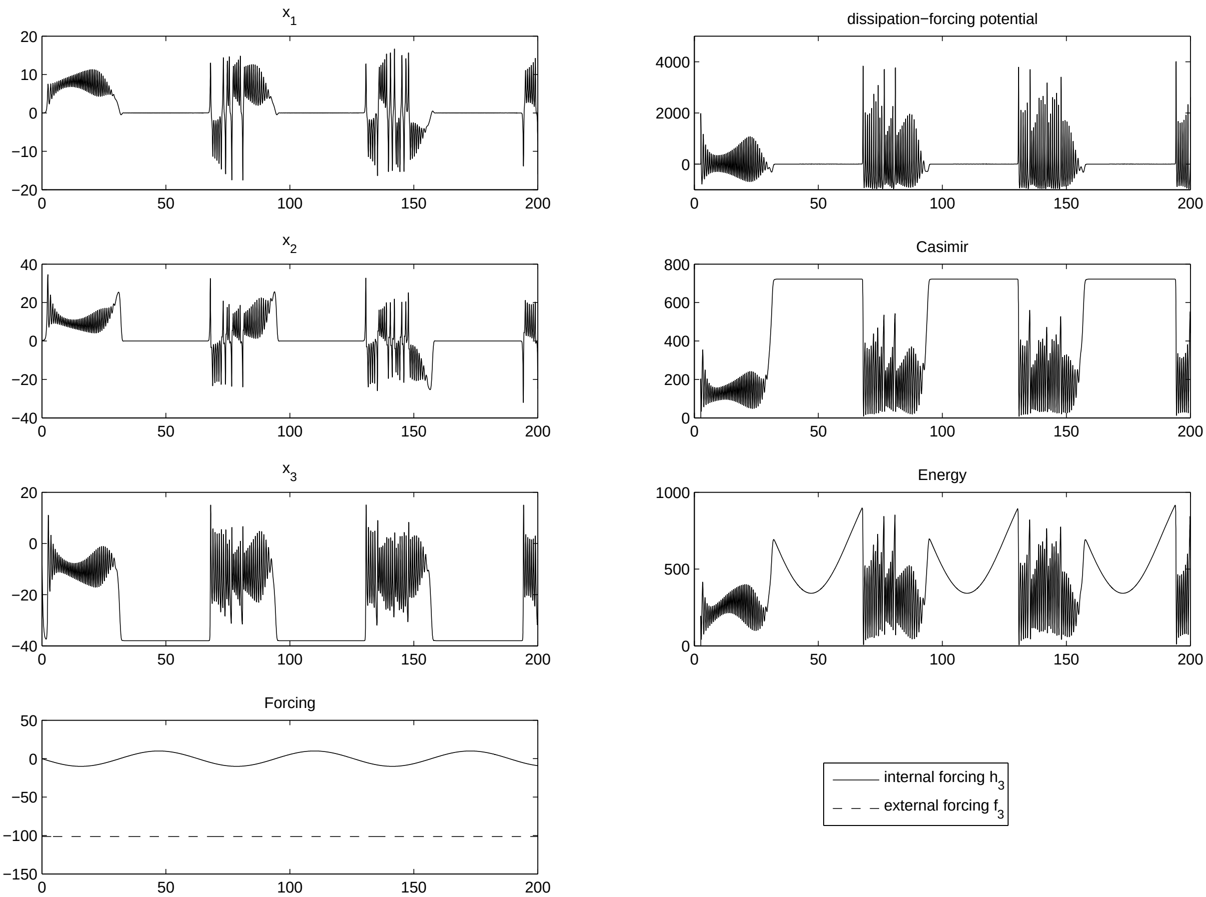

Fig. 2. Lorenz system forced by an internal periodic forcing. Left: trajectories. Right: evolution of dissipation forcing potential Φ, Casimir (C) and energy (H). Bottom: forcings $h_3 = -\sigma \cdot \sin(\omega_k t)$, $f_3 = -\beta(\rho+\sigma)$, where $\beta = 8/3$, $\rho = 28$, $\sigma = 10$, and $\omega_k = 0.1$.

$$\dot{x}_i = C_{ik}^j x_j \partial_k H, \tag{3}$$

where the tensor C_{ik}^j represents the constants of the structure of the Lie algebra $\mathbf{g}$ and the cosymplectic matrix assumes the form $J_{ik} = C_{ik}^j x_j$. Here, $T_e^* \mathbf{G} = \mathbf{g}^*$ is the dual of the Lie algebra $\mathbf{g}$ in a fibre bundle formalism. It is straightforward to show that, in this formalism, $\mathbf{g}$ is endowed with a Poisson bracket characterized by expression (2) for functions $F, G \in C^\infty(\mathbf{g}^*)$. Casimir functions C are given by the kernel of bracket (2), i.e. $\{C, G\} = 0, \forall G \in C^\infty(\mathbf{g}^*)$. Therefore, they represent constants of motion of the Hamiltonian system, $\dot{C} = \{C, H\} = 0$; moreover, they define a foliation of the phase space (Arnold and Khesin, 1988).

Here, we are interested in $G = SO(3)$, $J_{ik} = \varepsilon_{ik}^j x_j$; in the case of a quadratic Hamiltonian function,

$$H_0 = \frac{1}{2}\Omega_{ik} x_i x_k. \tag{4}$$

Equation (3) represents the Euler equations for the rigid body, with Casimir $C = x^i x_i$.

In a previous paper by Pasini and Pelino (2000), it has been shown that also the famous Lorenz-63 system (Lorenz, 1963), after a coordinate translation ($x_1 \to x_1, x_2 \to x_2, x_3 \to x_3 + \rho + \sigma$), can be written as

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = -x_1 x_3 - \sigma x_1 - x_2 \\ \dot{x}_3 = x_1 x_2 - \beta x_3 - \beta(\rho+\sigma) \end{cases} \tag{5}$$

if, in the Kolmogorov formalism – Eq. (1) – we assume the following gyrostat-like Hamiltonian:

$$H = H_0 + h_k x_k, \tag{6}$$

with $\Omega = \mathrm{diag}(2,1,1)$, dissipation $\mathbf{D} = \mathrm{diag}(\sigma, 1, \beta)$, an internal forcing given by an axisymmetric rotor $\mathbf{h} = (0, 0, -\sigma)$ and an external forcing $\mathbf{f} = (0, 0, -\beta(\rho+\sigma))$ (see Fig. 1). In this formalism, it is worthwhile to note that Corti et al. (1999) studied Eq. (1) for the case $f_1, f_2 \neq 0$. In this paper, we just modulate the intensity of the forcings already present inside this formalism.

The presence of the second term in Eq. (6) is the source of the much richer mathematical and physical properties of

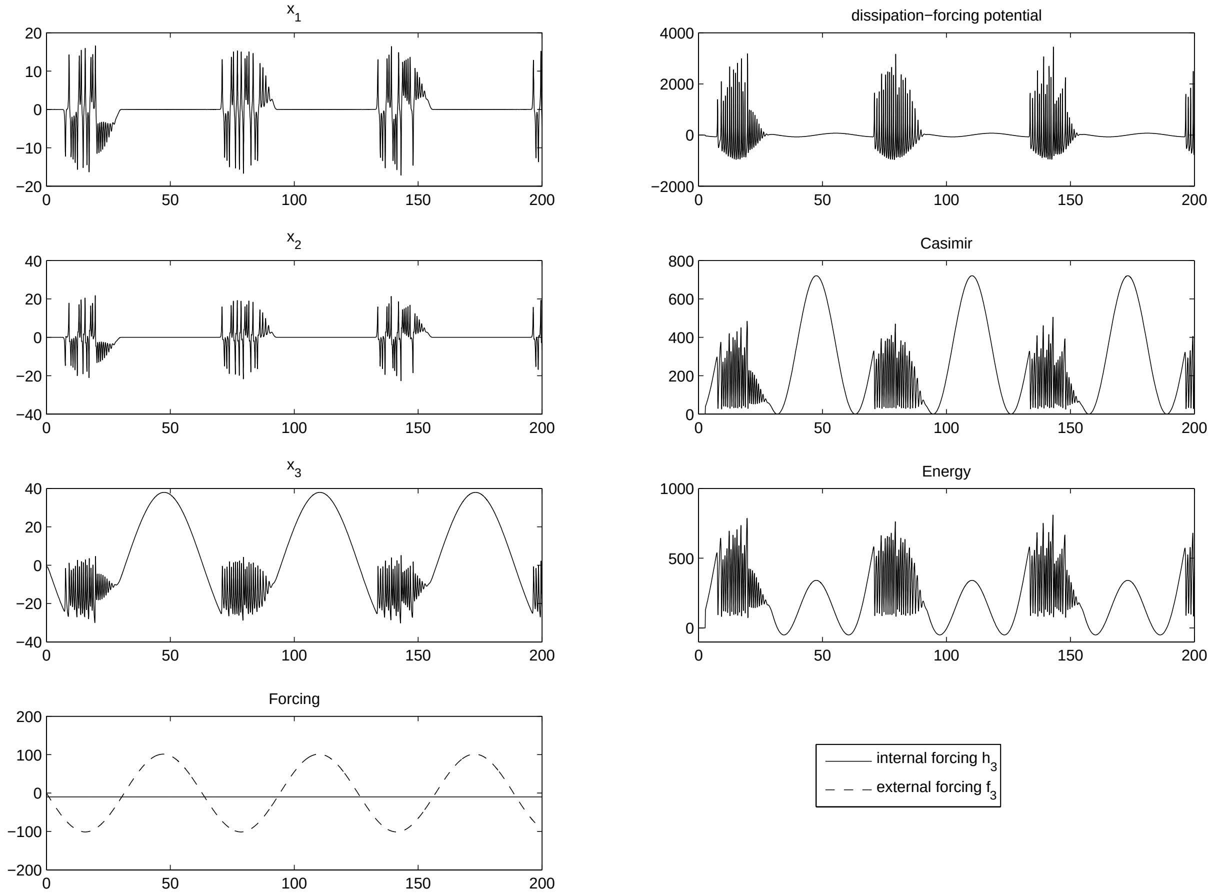

Fig. 3. Lorenz system forced by an external periodic forcing. As in Fig. 2, bottom: forcings $h_3 = -\sigma$, $f_3 = -\beta\,(\rho+\sigma)\cdot\sin\left(\omega_f t\right)$, where $\beta = 8/3$, $\rho = 28$, $\sigma = 10$, and $\omega_f = 0.1$.

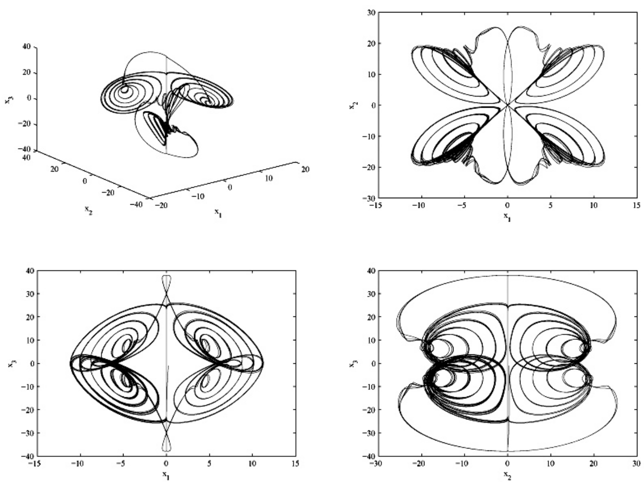

Fig. 4. 3-D view of the four-lobe Lorenz structure and its projection on different planes.

dynamical systems (1), which we call Kolmogorov-Lorenz (K-L) systems. An important result of this formalism is that there is no chaotic behaviour in the system for $\mathbf{h} = \mathbf{0}$ (Pasini and Pelino, 2000); moreover, there is much more information on the dynamics of the system otherwise hidden in Eq. (5), as for example shown in K-L application in the field of synchronization of chaotic systems (d'Anjou et al., 2005). This formalism has been also analyzed by means of the fruitful concept of tangent bundles (Yajima and Nagahama, 2010).

Finally, introducing a dissipation forcing potential,

$$\Phi = \frac{1}{2} D_{ij} x_i x_j - f_i x_i, \tag{7}$$

Eq. (1) can be also written as (McLachlan et al., 1998)

$$\dot{x}_i = \{x_i, H\} - \partial_i \Phi. \tag{8}$$

3 Periodically driven K-L systems

Lorenz equations with an external periodic driving term have been extensively considered in literature (Ahlers et al., 1984;

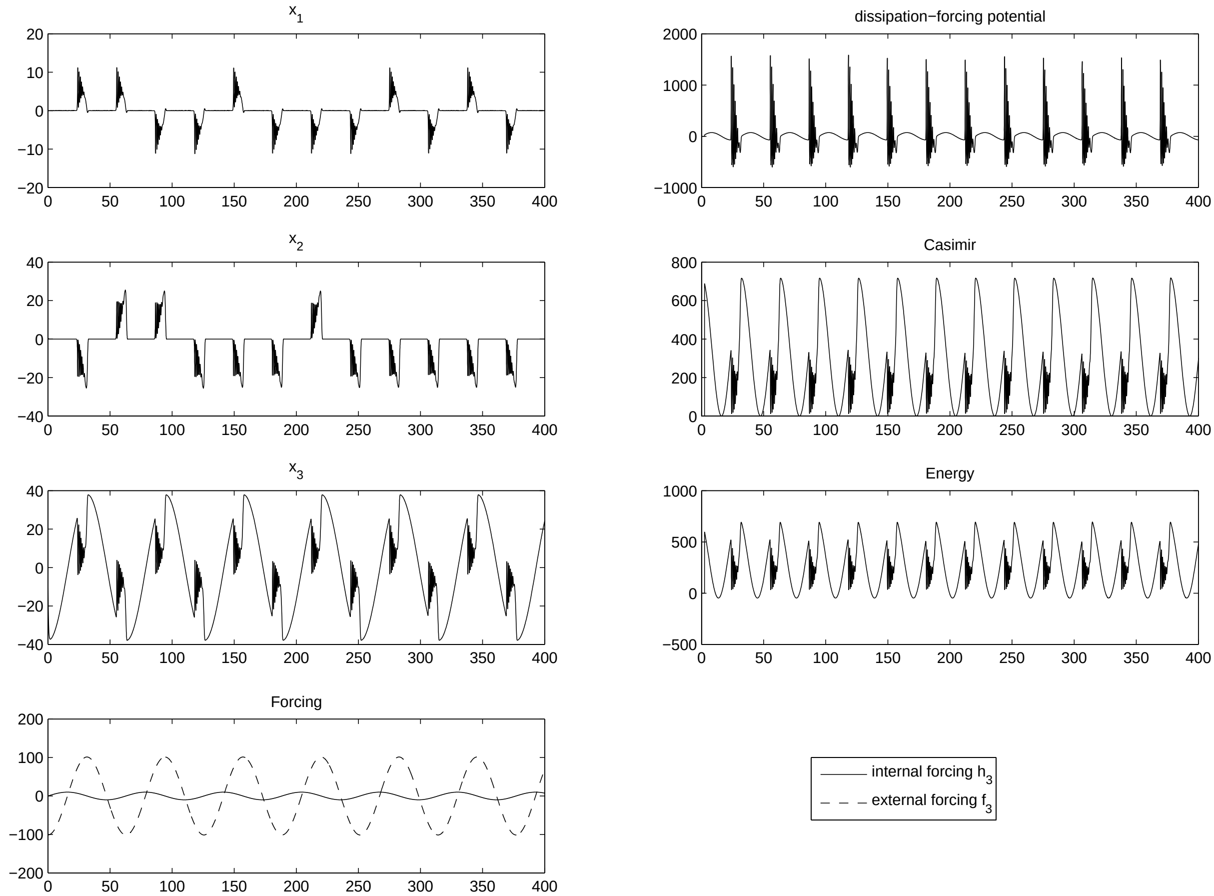

Fig. 5. Lorenz system forced by both periodic forcings. Left: trajectories. Right: evolution of dissipation forcing potential Φ, Casimir (C) and energy (H). Bottom: forcings $h_3=-\sigma\cdot\sin(\omega_k t)$, $f_3=-\beta(\rho+\sigma)\cdot\cos(\omega_f t)$, where $\beta=8/3, \rho=28, \sigma=10$, and $\omega_k=\omega_f=0.1$.

Franz and Zhang, 1995; Broer et al., 2002). In the formalism described above, it results in two types of different forcings acting in the K-L system: an *internal* spinning rotor given by the h_3 component and a forcing *external* to the system given by f_3. It is then natural to study the behaviour of Eq. (1) under internal and/or external periodic driving forcing.

First, we consider the case of a periodic internal forcing, starting with the study of the Hamiltonian function

$$H(x_i,t)=H_0+h_3x_3\cdot\sin\omega_h t \tag{9}$$

associated with a constant external forcing $f_3=-\beta(\rho+\sigma)$.

For $(\sigma,\beta,\rho)=(10,8/3,28)$, the characteristic frequency of the unforced Lorenz-63 system has been found to be $\omega_0\approx 8.3$ (Park et al., 1999). Resonance to external forcing for $\omega_0=8.3$ has been found also in Reick (2002) and Lucarini (2009). It is then interesting to choose the internal forcing in cases of low and high frequencies with respect to ω_0. Within the range $\omega_h<\omega_0$, by solving numerically (1) we find chaos suppression when the forcing is in its positive phase, $h_3(t)>0$, with an associated constant Casimir and a continuous energy trend, as shown in Fig. 2.

In the high frequency range, $\omega_h\gg\omega_0$, numerical studies show that this intermittency is lost and chaos is completely suppressed, giving rise to limit-cycle orbits or fixed points. This fact can be shown also analytically using Kapitza's method, as done in Choe et al. (2005). In fact, when writing $x_i(t)=\bar{x}_i+\xi_i$ as the sum of a "slow" and a "fast" component, it is easy to show that the system of equations for $\bar{x}_i$ is

$$\begin{cases}\dot{\bar{x}}_1\cong-\sigma\bar{x}_1+\sigma\overline{\xi_2\sin\omega_h t}\\ \dot{\bar{x}}_2\cong-\bar{x}_1\bar{x}_3-\bar{x}_2-\sigma\overline{\xi_1\sin\omega_h t}\\ \dot{\bar{x}}_3\cong\bar{x}_1\bar{x}_2-\beta\bar{x}_3\end{cases} \tag{10}$$

which, being $\overline{\sin\omega_h t\cos\omega_h t}=0$ over a period $2\pi/\omega_h$, reduces to a system of two coupled linear equations plus an equation for the (uncoupled) x_1. Here $\xi_1(t)=-\frac{\sigma\bar{x}_2}{\omega_h}\cos\omega_h t$ and $\xi_2(t)=\frac{\sigma\bar{x}_1}{\omega_k}\cos\omega_h t$.

A similar behaviour has been found for $h_3=-\sigma$ and the case of a harmonic external driving force:

$$f_3(t)=-\beta(\rho+\sigma)\sin\omega_f t. \tag{11}$$

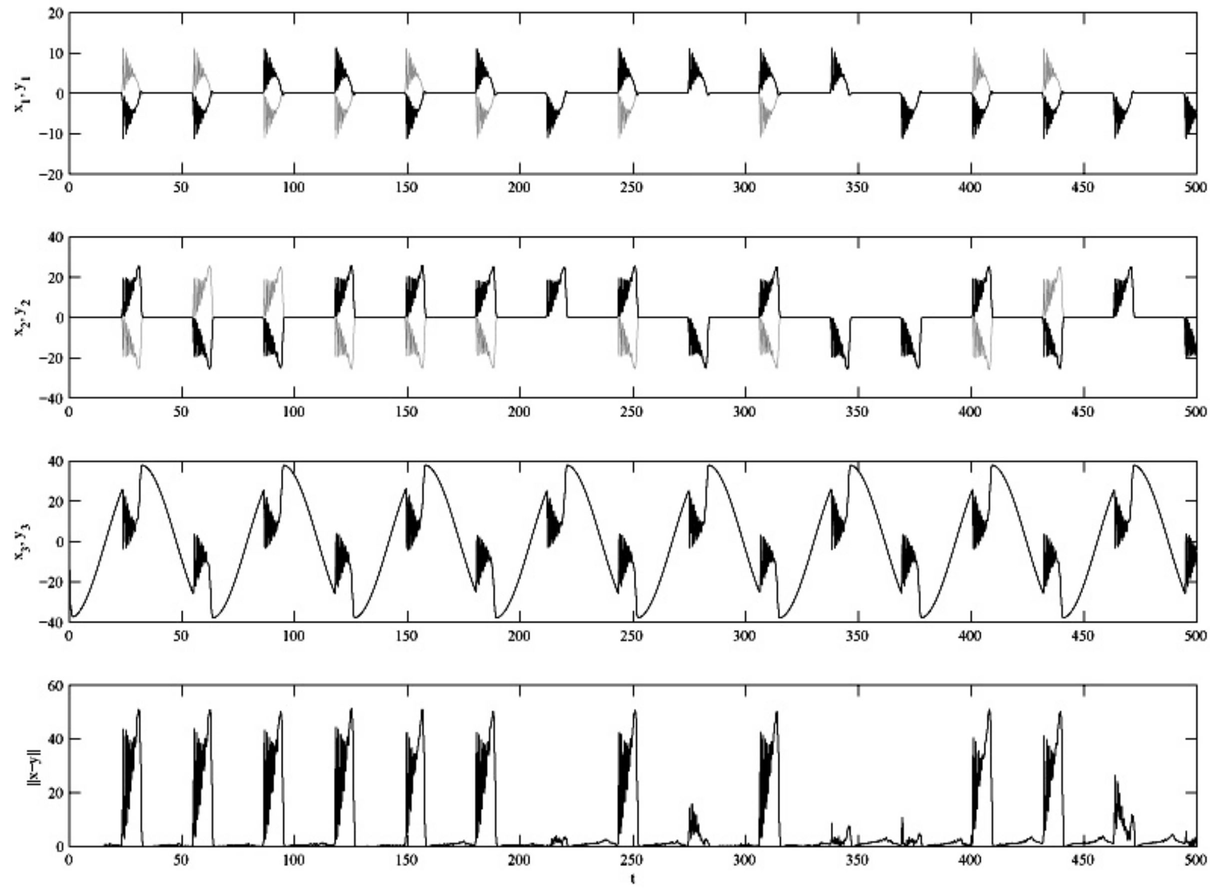

Fig. 6. Two different trajectories on the four-lobe Lorenz structure, differing by $\varepsilon = 0.001$ in initial conditions: $\mathbf{X}(0) = (15, 46, 60)$, black; $\mathbf{X}(0) = \mathbf{Y}(0) + \varepsilon$, gray. Bottom: Euclidean distance between the trajectories.

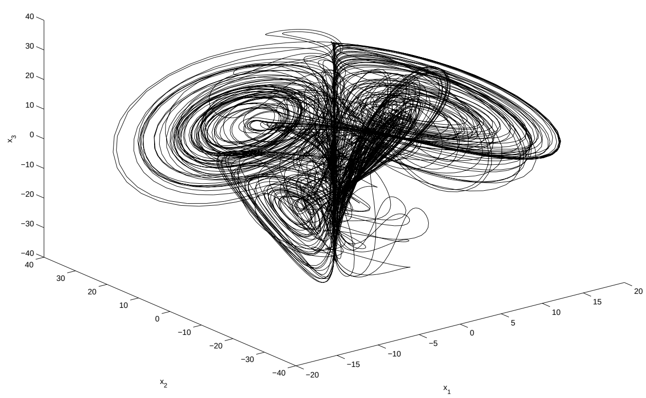

Fig. 7. A chaotic four-lobe attractor.

In the range of low frequency, $\omega_f < \omega_0$, chaos is suppressed for $f_3(t) > 0$, giving rise to periodic trend for Casimir and energy staggered by chaotic shots, as shown in Fig. 3.

A brief comment on Figs. 2 and 3 can be given relatively to the system's trajectories. For both types of forcings, x_1 and x_2 are fixed points of the system in the phase of chaos suppression. A different behaviour occurs for x_3, where a periodic motion takes place in the external forcing case. In the chaotic phases the typical two-lobe structure reappears in all the components of the system, giving rise to an intermittence of predictable and unpredictable regimes.

4 A four-lobe Lorenz attractor

The behaviour illustrated in the last section is interesting for predictability of chaotic systems. As a matter of fact, it has been shown elsewhere (Crisanti et al., 1997) that, under a periodic variation – with appropriate frequency – of a control parameter, a chaotic system undergoes into epochs of large unpredictability alternated with periodic behaviour. It is also known that the dynamics of many systems, such as climate, is characterised by both forcings and internal variability, possibly coupled with each other. In the K-L formalism, it has been shown that actually the behaviour of Lorenz attractor dynamics is related to two different forcings; therefore, in this section we study Eq. (1), assuming

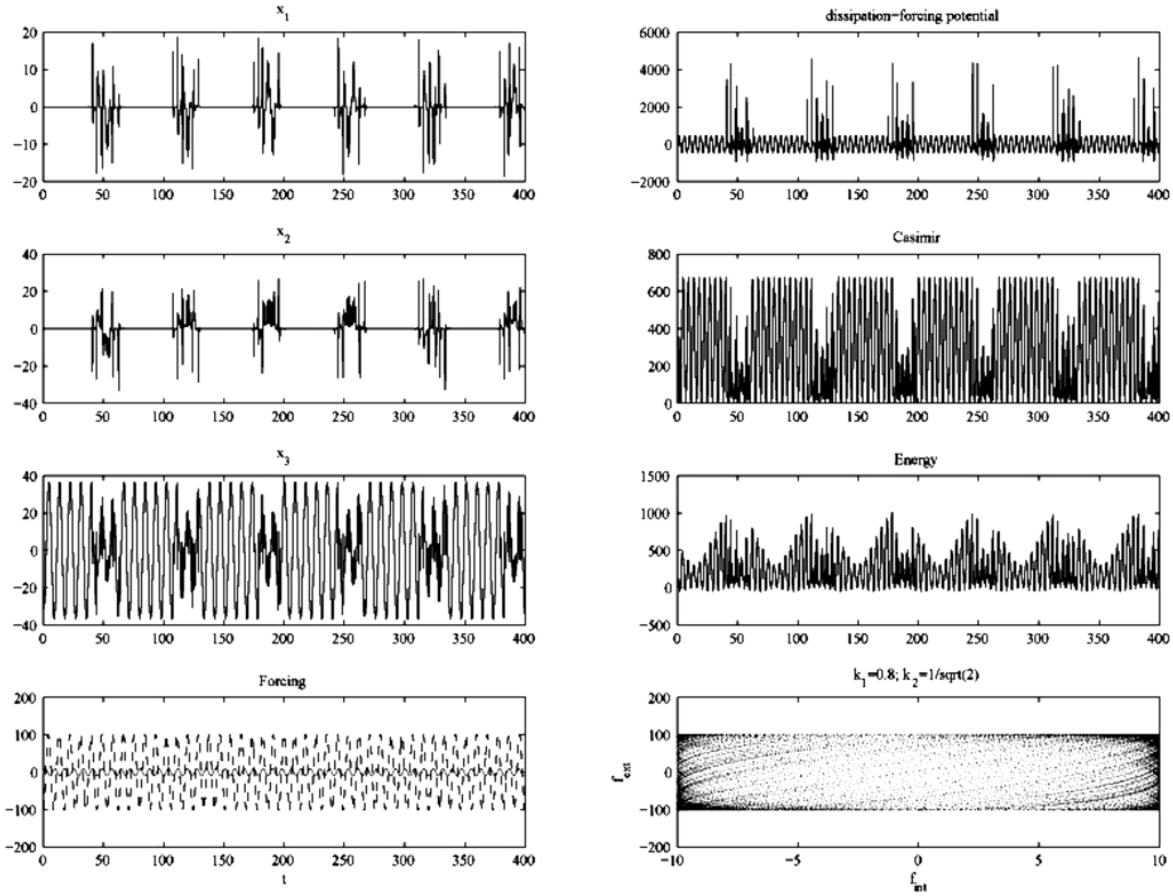

Fig. 8. Trajectories and dynamics of the attractor, as shown in Fig. 7. Here, the forcings frequencies are not rationally dependent: $\omega_k = 0.8$ and $\omega_f = 1/\sqrt{2}$.

both external and internal oscillating forcings. We fix our attention on the case of forcings characterized by the same frequency: $\omega_h = \omega_f < \omega_0$. Numerical simulations, assuming $\omega_h = \omega_f = 0.1$, show a kind of homoclinic trajectory, whose geometry is given by a double Lorenz structure (Fig. 4).

An analysis of the output illustrated in Fig. 5 shows a slow periodic behaviour of energy, Casimir and dissipation forcing potential. Inside this porting signal of slow modulation, a faster oscillation pulse term is inserted, giving rise to an unpredictable behaviour for x_1 and x_2 components. This is found before x_3 approaches its maximum or minimum value.

Here, because of a saddle point, trajectory chaotically visits the right or left lobe of the bottom or top Lorenz structure. In this way, chaos appears every time the trajectory leaves the x_3 axis, as illustrated in Fig. 6.

Other numerical simulations show a four-lobe Lorenz attractor in the range of many combinations of forcing parameters (see, for instance, Figs. 7 and 8).

As a further remark, it is interesting to see how the very natural insertion of internal and external forcings in the K-L formalism can be "translated" into the more usual classical Lorenz formalism. In our case the system (5) becomes

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \sin\omega_h t + x_1 x_3 - x_2 x_3 \\ \dot{x}_2 = -x_1 x_3 - \sigma x_1 \sin\omega_h t - x_2 \\ \dot{x}_3 = x_1 x_2 - \beta x_3 + \beta(\rho+\sigma)\sin\omega_f t. \end{cases} \tag{12}$$

In this context, these forcings seem ad hoc and we are no longer able to appreciate the naturalness of our choice.

Finally, the analysis performed in this section is essentially numerical. However, it would be very interesting to see how the four-lobe attractor can emerge analytically from the geometrical structure of the K-L system, for instance, by considering the trajectory as the intersection between Casimir and energy time-varying ellipsoids. This will be done in a future work.

5 Conclusions and prospects

The introduction of a forcing (either time dependent or not) into a dynamical system can have a multiplicity of effects on the underlying attractor. It can act to change the dynamics within the attractor while leaving invariantly its form (Corti et al., 1999), or it can change the form but not the topology, or, more radically, it can change topology. Furthermore, it can induce changes in the stability of fixed points and then, ultimately, in the chaotic character of the motion. Also, we want to point out that nonlinear quadratic dynamical systems, as Eq. (1), are dynamical cores of a large family of atmospheric/oceanic circulation models (Majda and Wang, 2006).

In particular, here we supply the evidence that, differently to constant forcings, oscillating ones can lead to the birth of new regimes/lobes in the Lorenz system. Doubtless, we are considering this behaviour just as a test case in Lorenz dynamics, without strict reference to precise implications in real physical systems. However, complex systems, such as climate, are subject to different kinds of forcings and internal feedbacks, so that the richness – shown here – displayed by Kolmogorov-Lorenz equations at different variations of internal and external forcings can be considered as a prototype framework, in which to simply analyze the dynamical features of changes in regimes.

Finally, an interesting prospect of future investigations regards the study of the transitions between two- and four-lobe attractors with the increase of amplitude in oscillating forcings.

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