Nonlin. Processes Geophys., 18, 287–294, 2011
www.nonlin-processes-geophys.net/18/287/2011/
doi:10.5194/npg-18-287-2011

Nonlinear Processes in Geophysics

Multifractal two-scale Cantor set model for slow solar wind turbulence in the outer heliosphere during solar maximum

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Received: 16 February 2011 – Revised: 20 April 2011 – Accepted: 22 April 2011 – Published: 12 May 2011

Abstract. To quantify solar wind turbulence, we consider a generalized two-scale weighted Cantor set with two different scales describing nonuniform distribution of the kinetic energy flux between cascading eddies of various sizes. We examine generalized dimensions and the corresponding multifractal singularity spectrum depending on one probability measure parameter and two rescaling parameters. In particular, we analyse time series of velocities of the slow speed streams of the solar wind measured in situ by Voyager 2 spacecraft in the outer heliosphere during solar maximum at various distances from the Sun: 10, 30, and 65 AU. This allows us to look at the evolution of multifractal intermittent scaling of the solar wind in the distant heliosphere. Namely, it appears that while the degree of multifractality for the solar wind during solar maximum is only weakly correlated with the heliospheric distance, but the multifractal spectrum could substantially be asymmetric in a very distant heliosphere beyond the planetary orbits. Therefore, one could expect that this scaling near the frontiers of the heliosphere should rather be asymmetric. It is worth noting that for the model with two different scaling parameters a better agreement with the solar wind data is obtained, especially for the negative index of the generalized dimensions. Therefore we argue that there is a need to use a two-scale cascade model. Hence we propose this model as a useful tool for analysis of intermittent turbulence in various environments and we hope that our general asymmetric multifractal model could shed more light on the nature of turbulence.

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1 Introduction

Multifractality is commonly related to a probability measure that may have different fractal dimensions on different parts of the support of this measure Mandelbrot (1989). In this case the measure is multifractal. Here we propose a notion of multifractality based on an extended self-similarity that depends on scale. We consider the concept of the multiscale multifractality in the context of scaling properties of intermittent turbulence in astrophysical and space plasmas (Meneveau and Sreenivasan, 1987, 1991). To quantify scaling of this turbulence, we use a generalized weighted Cantor set with two different scales describing with various probabilities nonuniform intermittent multiplicative process of distribution of the kinetic energy between cascading eddies of various sizes (Macek, 2007; Macek and Szczepaniak, 2008).

The question of multifractality is of great importance for space plasmas because it allows us to look at intermittent turbulence in the solar wind (Burlaga, 1991, 2001; Carbone, 1993, 1994; Carbone and Bruno, 1996; Marsch et al., 1996; Marsch and Tu, 1997; Bruno et al., 2003). Starting from Richardson's (1922) scenario of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies at various scales, see for a review (e.g., Bruno and Carbone, 2005). In particular, the multifractal spectrum has been investigated using Voyager (magnetic field fluctuations) data in the outer heliosphere (Burlaga, 1991, 2001) and using Helios (plasma) data in the inner heliosphere (Marsch et al., 1996). The multifractal scaling has also been investigated using Ulysses observations, e.g., (Horbury and Balogh, 2001; Wawrzaszek and Macek, 2010) and with Advanced Composition Explorer (ACE) and WIND data, e.g., (Hnat et al., 2003; Szczepaniak and Macek, 2008).

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In general, the spectrum of generalized dimensions D_q as a function of a continuous index, $-\infty < q < \infty$, with a degree of multifractality $\Delta = D_{-\infty} - D_{\infty}$, quantify multifractality of a given system (e.g., Ott, 1993). The degree of multifractality is simply related to the deviation from a simple self-similarity. That is why Δ is also a measure of intermittency, which is in contrast to self-similarity (Frisch, 1995, ch. 8). The related multifractal singularity spectrum $f(\alpha)$ as a function of a singularity strength α is also often used.

In addition, a chaotic strange attractor has been identified in the solar wind data by Macek (1998) and examined by Macek and Redaelli (2000). We have considered the D_q spectrum for the solar wind attractor using a multifractal model with a measure of the self-similar weighted Cantor set with two parameters describing uniform compression in phase space and the probability measure of the attractor of the system. The spectrum of D_q is found to be consistent with the data, at least for positive index q (Macek, 2002, 2003, 2006; Macek et al., 2005, 2006). However, the full spectrum is necessary to estimate the degree of multifractality. Notwithstanding of the well-known statistical problems with negative q (Macek, 2006), we have succeeded in estimating the entire spectrum for solar wind attractor using a generalized weighted Cantor set with two different scales describing nonuniform compression (Macek, 2007).

Therefore, to quantify scaling of solar wind turbulence, we have developed a generalized weighted two-scale Cantor set model using the partition technique (Macek and Szczepaniak, 2008). This model and the rank-ordered multifractal analysis lead to complementary information about the multifractal nature of the fluctuations (cf. Lamy et al., 2010). We have already studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behavior of solar wind turbulence in the inner (Helios and ACE) and outer heliosphere (Voyager) using fluctuations of the velocity of the flow of the solar wind at various scales. We have investigated the resulting spectrum of generalized dimensions and the corresponding multifractal singularity spectrum depending on the model parameters (Macek and Szczepaniak, 2008; Macek and Wawrzaszek, 2009). By using the cascade model with two different scaling parameters we have shown that the degree of multifractality of the velocity fluctuations of the solar wind in the inner and outer heliosphere is different for slow and fast streams. Moreover, during solar minimum both the degree of multifractality and the degree of asymmetry of the singularity spectrum are correlated with the heliospheric distance, and we have observed the evolution of multifractal scaling in the heliosphere (Macek and Wawrzaszek, 2009).

In this paper, we would like to investigate the degree of multifractality and asymmetry of the multifractal scaling provided by a deep space mission also during solar maximum. Namely, we further consider in fuller detail the question of scaling properties of intermittent turbulence using velocities of the slow speed streams of the solar wind measured in situ by Voyager 2 at various distances from the Sun. Namely, by using our cascade model with two different scaling parameters we investigate the degree of multifractality of the slow solar wind in the outer heliosphere during solar maximum looking at the evolution of multifractal scaling in the outer heliosphere (cf. Burlaga, 1991; Burlaga et al., 2003; Burlaga, 2004). We show that in contrast to the solar minimum both the degree of multifractality and asymmetry are only weakly correlated with the heliospheric distance.

This paper is organized as follows. In Sect. 2 a generalized two-scale Cantor set model is summarised, and the data are presented in Sect. 3. The methods related the concept of the generalized dimensions and the singularity spectrum in the context of turbulence scaling are reviewed in Sect. 4. The results of our analysis are presented and discussed in Sect. 5. The importance of our new more general asymmetric multifractal cascade model is underlined in Sect. 6.

2 Two-scale Cantor set cascade model

The Cantor set with weight p and two scales is an example of multifractals, as discussed in several textbooks (e.g., Falconer, 1990; Ott, 1993). Namely, at each stage of construction of this generalized Cantor set we have two rescaling parameters l_1 and l_2, where $l_1 + l_2 \le L = 1$ (normalized) and two different probability measure $p_1 = p$ and $p_2 = 1 - p$. To obtain the generalized dimensions $D_q \equiv \tau(q)/(q-1)$ for this multifractal set we use the following partition function (a generator) at the n-th level of construction (Hentschel and Procaccia, 1983; Halsey et al., 1986)

$$\Gamma_n^q(l_1, l_2, p) = \left(\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} \right)^n = 1. \quad (1)$$

Namely, we see that $\tau(q)$ does not depend on n, and after n iterations we have $\binom{n}{k}$ intervals of width $l = l_1^k l_2^{n-k}$, where $k = 1, \ldots, n$, visited with various probabilities. The resulting set of 2^n closed intervals (more and more narrow segments of various widths and probabilities) for $n \to \infty$ becomes the weighted two-scale Cantor set.

In our model of turbulence we consider a standard scenario of cascading eddies, each breaking down into two new ones, but not necessarily equal and twice smaller, as proposed by Macek and Szczepaniak (2008). In particular, space filling turbulence could be recovered for $l_1 + l_2 = 1$. Naturally, in the inertial region of the system $\eta \ll l \ll 1$ the energy is not allowed to be dissipated directly, assuming $p_1 + p_2 = 1$, until the Kolmogorov scale η is reached. However, in this range at each n-th step of the binomial multiplicative process, the flux of kinetic energy density ε transferred to smaller eddies (energy transfer rate) could be divided into nonequal fractions p and $1 - p$ (cf. Meneveau and Sreenivasan, 1987).

The multifractal normalized measure (Mandelbrot, 1989) $\mu = \varepsilon / \langle \varepsilon_L \rangle$ on the unit interval generated at the twelfth step

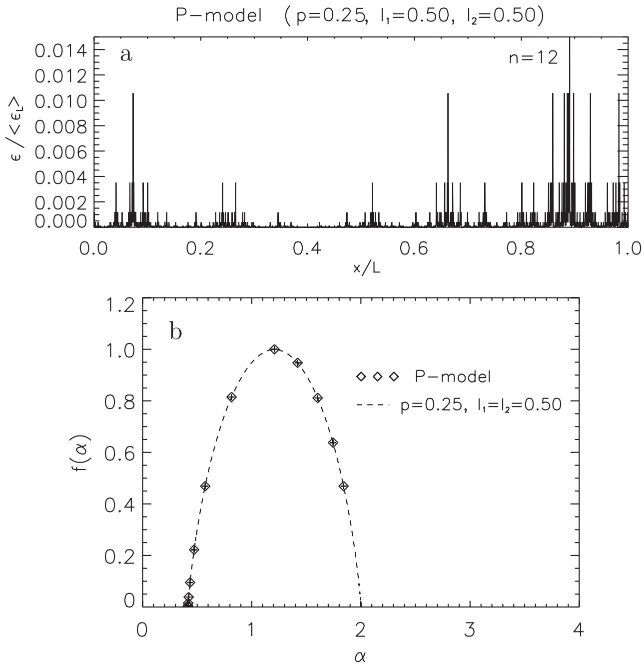

Fig. 1. **(a)** The theoretical multifractal measure $\mu = \varepsilon/\langle\varepsilon_L\rangle$ on the unit interval for twelfth step of the construction of the usual one-scale Cantor set. **(b)** The multifractal spectrum $f(\alpha)$ obtained directly for this measure (diamonds) with a fit (dashed line) to the p-model.

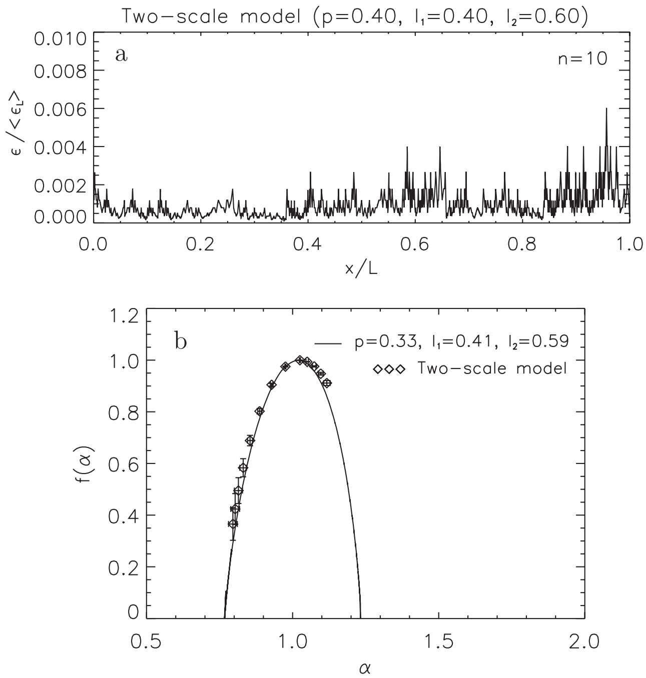

Fig. 2. **(a)** The theoretical multifractal measure $\mu = \varepsilon/\langle\varepsilon_L\rangle$ on the unit interval for the tenth step of the construction of the weighted two-scale Cantor set. **(b)** The multifractal spectrum $f(\alpha)$ obtained directly for this measure (diamonds) with a fit (continuous line) to the generalized p-model.

of the construction ($n = 12$) for the usual one-scale p-model (Meneveau and Sreenivasan, 1987) is shown in Fig. 1a. The corresponding measure for the generalized two-scale cascade model with the following assumed parameters: $p = 0.4$, $l_1 = 0.4$ and $l_2 = 0.6$ is illustrated in Fig. 2a. The multifractal spectrum $f(\alpha)$ calculated directly (Chhabra and Jensen, 1989; Chhabra et al., 1989) for this measure (diamonds) with a fit (dashed line) to the p-model and the generalized p-model (continuous line) obtained with the similar parameters are also presented in Figs. 1b and 2b, correspondingly.

3 Solar wind data

We have already analysed Helios, ACE, and Voyager data using plasma parameters measured in the inner and outer heliosphere. Namely, to study turbulence cascade Macek and Szczepaniak (2008) have selected four-day time intervals of v_x samples in 1976 (solar minimum) for both slow and fast solar wind streams measured by Helios 2 at various distances from the Sun. The results for ACE data at 1 AU and dependence on solar cycle are discussed by Szczepaniak and Macek (2008). We have analysed time series of velocities of the solar wind measured by Voyager 2 at various distances from the Sun, 2.5, 25, and 50 AU, selecting longer (13-day) time intervals, each of 2^{11} data points, interpolated with sampling time of 192 s for both slow and fast solar wind streams during the following solar minima: 1978, 1987–1988, and 1996–1997 (Macek and Wawrzaszek, 2009). In this paper the same analysis is repeated for the Voyager 2 data for the slow solar wind during the solar maxima: 1981, 1989, 2001, at 10, 30, and 65 AU, correspondingly. This will allow us to investigate the dependence of the multifractal spectra on the phase of the solar cycle.

4 Methods of data analysis

The generalized dimensions D_q as a function of index q (Grassberger, 1983; Grassberger and Procaccia, 1983; Hentschel and Procaccia, 1983; Halsey et al., 1986) are important characteristics of *complex* dynamical systems; they quantify multifractality of a given system (Ott, 1993). In the case of turbulence cascade these generalized measures are related to inhomogeneity with which the energy is distributed between different eddies (Meneveau and Sreenivasan, 1991). In this way they provide information about dynamics of multiplicative process of cascading eddies: high positive values of q emphasize regions of intense energy transfer rate, while negative values of q accentuate low-transfer rate regions.

Let us consider the generalized weighted Cantor set, where the probability of providing energy for one eddy of size l_1 is p (say, $p \leq 1/2$), and for the other eddy of size l_2 is $1-p$ as depicted in Fig. 1 of the paper by Macek and Wawrzaszek (2009). For any q one obtains $D_q = \tau(q)/(q-1)$ by solving numerically the following transcendental equation (e.g., Ott, 1993)

$$\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} = 1. \tag{2}$$

In the inertial range the transfer rate of the energy flux $\varepsilon(x,l)$ is widely estimated by the third moment of structure function of velocity fluctuations (e.g., Marsch et al., 1996)

$$\varepsilon(x,l) \sim \frac{|u(x+l)-u(x)|^3}{l}, \tag{3}$$

where $u(x)$ and $u(x+l)$ are velocity components parallel to the longitudinal direction separated from a position x by a distance l. Therefore to each ith eddy of size l in the turbulence cascade ($i=1,\ldots,N=2^n$) we associate a probability measure defined by

$$p_i(l) = \frac{\varepsilon(x_i,l)}{\sum_{i=1}^{N}\varepsilon(x_i,l)}. \tag{4}$$

This quantity can roughly be interpreted as a probability that the energy is transferred to an eddy of size $l = v_{\mathrm{sw}}t$.

Now, one can further associate a generalized average probability measure of cascading eddies

$$\bar{\mu}(q,l) \equiv \sqrt[q-1]{\langle (p_i)^{q-1}\rangle_{\mathrm{av}}}, \tag{5}$$

and identify D_q as scaling of the measure with size l,

$$\bar{\mu}(q,l) \propto l^{D_q}. \tag{6}$$

Hence, the slopes of the logarithm of $\bar{\mu}(q,l)$ of Eq. (6) versus $\log l$ (normalized) provides

$$D_q = \lim_{l\to 0} \frac{\log\bar{\mu}(q,l)}{\log l}. \tag{7}$$

The singularity spectrum $f(\alpha) = q\alpha - \tau(q)$ as a function of $\alpha = \tau'(q)$ could also be obtained by using Legendre transformation, or directly from the slopes of the generalized measures (Halsey et al., 1986; Jensen et al., 1987). We can take

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} \tag{8}$$

as the degree of multifractality, see (e.g., Macek, 2006, 2007). Farther, using the value of the strength of singularity α_0 at which the singularity spectrum has its maximum $f(\alpha_0) = 1$, one can define a measure of asymmetry

$$A \equiv \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}. \tag{9}$$

5 Results and discussion

In order to estimate the multifractal spectrum for solar wind turbulence, we should first calculate the multifractal measure given in Eq. (4). The values obtained using data of the velocity components $u = v_x$ measured by Voyager 2 spacecraft during solar minimum (1978, 1987–1988, and 1996–1997) at 2.5, 25, and 50 AU are presented in Fig. 2 of the paper by Macek and Wawrzaszek (2009) for the slow (a), (c), and (e) and fast (b), (d), and (f) solar wind, correspondingly.

In this paper our calculations are repeated for velocity fluctuations measured in the slow wind by Voyager 2 at various distances of 10, 30, and 65 AU during solar maximum. Namely, using the slopes of $\bar{\mu}(q,l)$, given in Eq. (5), the corresponding results for the generalized dimensions D_q as a function of q defined by Eq. (7), with the statistical errors of the average slopes over the scaling range, are shown in Fig. 3a, c, and e. In addition, we can calculate the generalized average logarithmic probability and pseudoprobability $\mu(q,l)$ of cascading eddies taken with respect to the generalized measure $\langle \log_{10} p_i(l)\rangle$ and $\langle \log_{10}\mu_i(q,l)\rangle$ versus $\log_{10} l$, as given in Eqs. (5) and (10) of the paper by Macek and Wawrzaszek (2009). In this way, the singularity spectra $f(\alpha)$ are calculated directly from the data as functions of singularity strength α. The obtained results are also presented in Fig. 3b, d, and f. In fact, both values of D_q and $f(\alpha)$ for one-dimensional turbulence are calculated using the radial velocity components $u = v_x$ (in time domain) in Eq. (3) (cf. Macek and Szczepaniak, 2008, Fig. 3). Admittedly, as is well known, for $q < 0$ we have some basic statistical problems Macek (2006, 2007). Nevertheless, in spite of some statistical errors in Fig. 3, especially for $q < 0$, the multifractal character of the measure can still clearly be discerned. Therefore one can confirm that the spectrum of dimensions still exhibits the multifractal structure of the solar wind in the outer heliosphere.

For $q \geq 0$ these results agree with the usual one-scale p-model fitted to the generalized dimensions as obtained analytically using $l_1 = l_2 = 0.5$ in Eq. (2) and the values of the parameter $p \simeq 0.19$, 0.19, and 0.17, for various distances correspondingly, as shown by dashed lines. The values of parameter p are related to the usual models, which are based on the p-model of turbulence (e.g., Meneveau and Sreenivasan, 1987). On the contrary, in general for $q < 0$ (right part of the singularity spectrum in Fig. 3) the p-model cannot describe the observational results. Admittedly, a deviation from one-scale multifractal scaling can sometimes be attributed to special characteristics of turbulence. But here we show that the experimental values are consistent with the generalized dimensions obtained numerically from Eq. (2) for the weighted two-scale Cantor set using an asymmetric scaling, i.e., using unequal scales $l_1 \neq l_2$, as is depicted in Fig. 3 by continuous lines. Evolution of the parameters of the two-scale (l_1, l_2) weighted (p) Cantor set model as a function of the heliospheric distance is shown in Fig. 4 for solar maximum

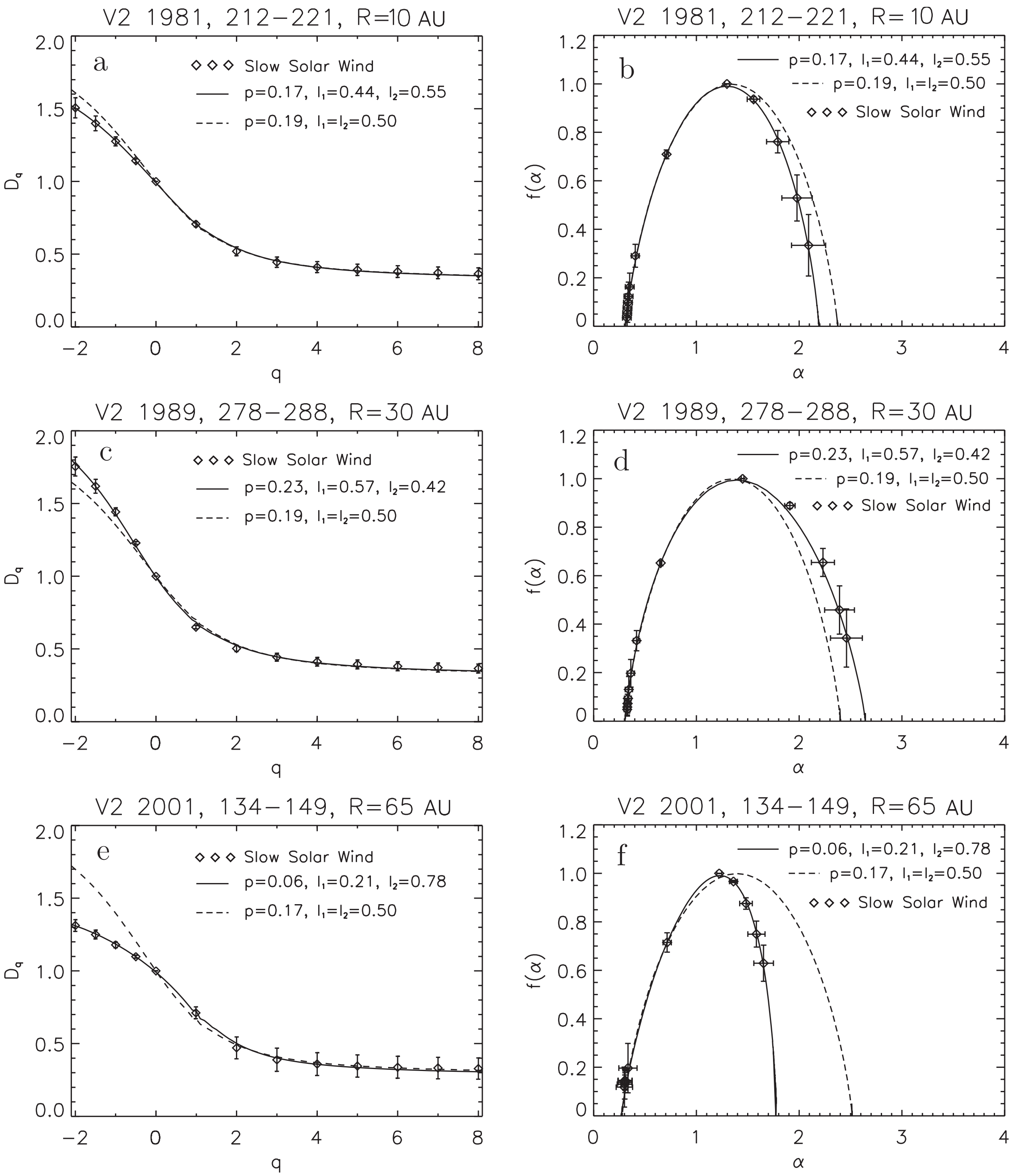

Fig. 3. The generalized dimensions D_q **(a, c, e)** and singularity spectra $f(\alpha)$ **(b, d, f)** calculated for the one-scale *p*-model (dashed lines) and the generalized two-scale (continuous lines) models with parameters fitted to the multifractal measure $\mu(q,l)$ using data measured by Voyager 2 during solar maximum (1981, 1989, 2001) at 10, 30, and 65 AU (diamonds) for the slow solar wind.

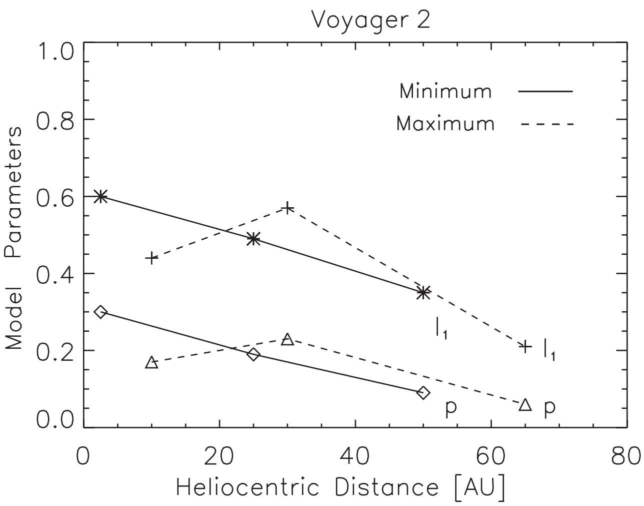

Fig. 4. Evolution of the parameters of the two-scale (l_1, l_2) weighted (parameter p) Cantor set model as a function of the heliospheric distance during solar maximum (dashed lines) and solar minimum (continuous lines).

(dashed lines) and solar minimum (continuous lines). We see that beyond the planetary orbits somewhat different scale parameters are involved depending on the phase of the solar cycle.

Table 1. Degree of multifractality Δ and asymmetry A for solar wind data in the outer heliosphere during solar maximum.

Heliocentric distance (Year)	Slow solar wind Δ	A
10 AU (1981)	1.88 ± 0.16	1.1 ± 0.18
30 AU (1989)	2.34 ± 0.14	0.96 ± 0.11
65 AU (2001)	1.51 ± 0.10	1.71 ± 0.29

In Fig. 3 we also show the calculated universal multifractal singularity spectra. We can take the degree of multifractality Δ and measure of asymmetry A defined in Eqs. (8) and (9), respectively. Both values of Δ and A are presented in Table 1. As expected the multifractal scaling could sometimes be rather asymmetric, and we can analyse the evolution of the multifractality in the outer heliosphere (cf. Burlaga, 1991; Burlaga et al., 2003; Burlaga, 2004), which has also been noticed in the inner heliosphere, e.g., by Bruno et al. (2003). However, it appears that the degree of multifractality and asymmetry are rather weakly correlated with the heliospheric distance. In fact, as we see from Table 1 the obtained values of Δ are rather similar for the solar wind in the distant heliosphere, cf. Fig. 3.

In particular, one should note that the degree of multifractality for the slow solar wind during solar maximum is roughly constant similarly as for the case of solar minimum (cf. Macek and Wawrzaszek, 2009). Moreover, it seems that the degree of asymmetry for the slow wind is only weakly correlated with the moderate heliospheric distances. One can say that in the slow streams during solar maximum the scaling is rather symmetric within our planetary system. Only in the very distant solar wind at $\sim$ 65 AU, i.e., beyond the planetary orbits, the observed spectrum becomes substantially asymmetric with $A = 1.7$ (cf. Macek and Wawrzaszek, 2010). This is also consistent with the evolution of the parameters of our model displayed in Fig. 4. Apparently, this shows that also the slow wind can exhibit asymmetric scaling.

6 Conclusions

We have studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behaviour of solar wind turbulence in the outer heliosphere. In particular, we have demonstrated that for the model with two different scaling parameters a better agreement with the real data is obtained, especially for $q < 0$. One can expect that the degree of multifractality and asymmetry should be correlated with the heliospheric distance and we can observe the evolution of multifractal scaling in the outer heliosphere (cf. Burlaga, 1991; Burlaga et al., 2003; Burlaga, 2004). However, by investigating the Voyager 2 data we have demonstrated that the degree of multifractality of the solar wind in the outer heliosphere does not vary too much with the heliospheric distance. It appears that the degree of multifractality for the slow solar wind during solar maximum is somewhat similar to that for solar minimum. It is worth noting that the multifractal scaling could be asymmetric. In particular, the slow wind during solar maximum exhibits some asymmetric scaling in the very distant heliosphere, beyond the planets.

Naturally, the generalized dimensions for solar wind are consistent with the generalized p-model for both positive and negative q, but rather with different scaling parameters for sizes of eddies, while the usual p-model can only reproduce the spectrum for $q \geq 0$. Hence we hope that our more general asymmetric multifractal model could shed light on the nature of turbulence and we therefore propose this model as a useful tool for analysis of intermittent turbulence in space environments.

Acknowledgements. We would like to thank the plasma instruments team of Voyager mission, NASA National Space Science Data and the Space Science Data Facility for providing velocity data. This work has been supported by the Ministry of Science and Higher Education (MNiSW) through Grant NN 307 0564 40.

Edited by: H. Lamy
Reviewed by: two anonymous referees

References

Bruno, R. and Carbone, V.: The solar wind as a turbulence laboratory, Living Rev. Sol. Phys., 2, lrsp2005–4, 2005.

Bruno, R., Carbone, V., Sorriso-Valvo, L., and Bavassano, B.: Radial evolution of solar wind intermittency in the inner heliosphere, J. Geophys. Res., 108, 1130, doi:10.1029/2002JA009615, 2003.

Burlaga, L. F.: Multifractal structure of the interplanetary magnetic field - Voyager 2 observations near 25 AU, 1987–1988, Geophys. Res. Lett., 18, 69–72, 1991.

Burlaga, L. F.: Lognormal and multifractal distributions of the heliospheric magnetic field, J. Geophys. Res., 106, 15917–15927, doi:10.1029/2000JA000107, 2001.

Burlaga, L. F.: Multifractal structure of the large-scale heliospheric magnetic field strength fluctuations near 85AU, Nonlin. Processes Geophys., 11, 441–445, doi:10.5194/npg-11-441-2004, 2004.

Burlaga, L. F., Wang, C., and Ness, N. F.: A model and observations of the multifractal spectrum of the heliospheric magnetic field strength fluctuations near 40 AU, Geophys. Res. Lett., 30, 50–1, doi:10.1029/2003GL016903, 2003.

Carbone, V.: Cascade model for intermittency in fully developed magnetohydrodynamic turbulence, Phys. Rev. Lett., 71, 1546–1548, doi:10.1103/PhysRevLett.71.1546, 1993.

Carbone, V.: Scaling exponents of the velocity structure functions in the interplanetary medium, Ann. Geophys., 12, 585–590, doi:10.1007/s00585-994-0585-3, 1994.

Carbone, V. and Bruno, R.: Cancellation exponents and multifractal scaling laws in the solar wind magnetohydrodynamic turbulence, Ann. Geophys., 14, 777–785, doi:10.1007/s00585-996-0777-0, 1996.

Chhabra, A. and Jensen, R. V.: Direct determination of the f(α) singularity spectrum, Phys. Rev. Lett., 62, 1327–1330, doi:10.1103/PhysRevLett.62.1327, 1989.

Chhabra, A. B., Meneveau, C., Jensen, R. V., and Sreenivasan, K. R.: Direct determination of the f(α) singularity spectrum and its application to fully developed turbulence, Phys. Rev. A, 40, 5284–5294, doi:10.1103/PhysRevA.40.5284, 1989.

Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications, J. Wiley, New York, 1990.

Frisch, U.: Turbulence. The legacy of A.N. Kolmogorov, Cambridge: Cambridge Univ. Press, ch. 8, 1995.

Grassberger, P.: Generalized dimensions of strange attractors, Phys. Lett. A, 97, 227–230, doi:10.1016/0375-9601(83)90753-3, 1983.

Grassberger, P. and Procaccia, I.: Measuring the strangeness of strange attractors, Physica D, 9, 189–208, doi:10.1016/0167-2789(83)90298-1, 1983.

Halsey, T. C., Jensen, M. H., Kadanoff, L. P., Procaccia, I., and Shraiman, B. I.: Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A, 33, 1141–1151, doi:10.1103/PhysRevA.33.1141, 1986.

Hentschel, H. G. E. and Procaccia, I.: The infinite number of generalized dimensions of fractals and strange attractors, Physica D, 8, 435–444, doi:10.1016/0167-2789(83)90235-X, 1983.

Hnat, B., Chapman, S. C., and Rowlands, G.: Intermittency, scaling, and the Fokker-Planck approach to fluctuations of the solar wind bulk plasma parameters as seen by the WIND spacecraft, Phys. Rev. E, 67, 056404, doi:10.1103/PhysRevE.67.056404, 2003.

Horbury, T. S. and Balogh, A.: Evolution of magnetic field fluctuations in high-speed solar wind streams: Ulysses and Helios observations, J. Geophys. Res., 106, 15929–15940, doi:10.1029/2000JA000108, 2001.

Jensen, M. H., Kadanoff, L. P., and Procaccia, I.: Scaling structure and thermodynamics of strange sets, Phys. Rev. A, 36, 1409–1420, doi:10.1103/PhysRevA.36.1409, 1987.

Lamy, H., Wawrzaszek, A., Macek, W. M., and Chang, T.: New multifractal analyses of the solar wind turbulence: Rank-ordered multifractal analysis and generalized two-scale weighted Cantor set model, Twelfth International Solar Wind Conference, 1216, 124–127, doi:10.1063/1.3395816, 2010.

Macek, W. M.: Testing for an attractor in the solar wind flow, Physica D, 122, 254–264, doi:10.1016/S0167-2789(98)00098-0, 1998.

Macek, W. M.: Multifractality and chaos in the solar wind, in: Experimental Chaos, edited by: Boccaletti, S., Gluckman, B. J., Kurths, J., Pecora, L. M., and Spano, M. L., 622, American Institute of Physics Conference Series, 74–79 , doi:10.1063/1.1487522, 2002.

Macek, W. M.: The multifractal spectrum for the solar wind flow, in: Solar Wind Ten, edited by: Velli, M., Bruno, R., Malara, F., and Bucci, B., 679, American Institute of Physics Conference Series, 530–533, doi:10.1063/1.1618651, 2003.

Macek, W. M.: Modeling multifractality of the solar wind, Space Sci. Rev., 122, 329–337, doi:10.1007/s11214-006-8185-z, 2006.

Macek, W. M.: Multifractality and intermittency in the solar wind, Nonlin. Processes Geophys., 14, 695–700, doi:10.5194/npg-14-695-2007, 2007.

Macek, W. M. and Redaelli, S.: Estimation of the entropy of the solar wind flow, Phys. Rev. E, 62, 6496–6504, doi:10.1103/PhysRevE.62.6496, 2000.

Macek, W. M. and Szczepaniak, A.: Generalized two-scale weighted Cantor set model for solar wind turbulence, Geophys. Res. Lett., 35, L02108, doi:10.1029/2007GL032263, 2008.

Macek, W. M. and Wawrzaszek, A.: Evolution of asymmetric multifractal scaling of solar wind turbulence in the outer heliosphere, J. Geophys. Res., 114, A03108, doi:10.1029/2008JA013795, 2009.

Macek, W. M. and Wawrzaszek, A.: Multifractal turbulence at the termination shock, Twelfth International Solar Wind Conference, 1216, 572–575, doi:10.1063/1.3395930, 2010.

Macek, W. M., Bruno, R., and Consolini, G.: Generalized dimensions for fluctuations in the solar wind, Phys. Rev. E, 72, 017202, doi:10.1103/PhysRevE.72.017202, 2005.

Macek, W. M., Bruno, R., and Consolini, G.: Testing for multifractality of the slow solar wind, Adv. Space Res., 37, 461–466, doi:10.1016/j.asr.2005.06.057, 2006.

Mandelbrot, B. B.: Multifractal measures, especially for the geophysicist, Pure Appl. Geophys., 131, 5–42, doi:10.1007/BF00874478, 1989.

Marsch, E. and Tu, C.-Y.: Intermittency, non-Gaussian statistics and fractal scaling of MHD fluctuations in the solar wind, Nonlin. Processes Geophys., 4, 101–124, doi:10.5194/npg-4-101-1997, 1997.

Marsch, E., Tu, C.-Y., and Rosenbauer, H.: Multifractal scaling of the kinetic energy flux in solar wind turbulence, Ann. Geophys., 14, 259–269, doi:10.1007/s00585-996-0259-4, 1996.

Meneveau, C. and Sreenivasan, K. R.: Simple multifractal cascade model for fully developed turbulence, Phys. Rev. Lett., 59, 1424–

1427, doi:10.1103/PhysRevLett.59.1424, 1987.
Meneveau, C. and Sreenivasan, K. R.: The multifractal nature of turbulent energy dissipation, J. Fluid Mech., 224, 429–484, doi:10.1017/S0022112091001830, 1991.
Ott, E.: Chaos in dynamical systems, Cambridge: Cambridge Univ. Press, ch. 9, 1993.
Szczepaniak, A. and Macek, W. M.: Asymmetric multifractal model for solar wind intermittent turbulence, Nonlin. Processes Geophys., 15, 615–620, doi:10.5194/npg-15-615-2008, 2008.
Wawrzaszek, A. and Macek, W. M.: Observation of the multifractal spectrum in solar wind turbulence by Ulysses at high latitudes, J. Geophys. Res. (Space Physics), 115, A07104, doi:10.1029/2009JA015176, 2010.