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Relativistic whistler oscillitons – do they exist?

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Abstract. We examine the effects of relaxing the assumption of quasi-charge neutrality on the propagation of stationary whistler waves. This necessitates that in the wave frame the equations of motion must be made fully relativistic and when this is done, the correct form of the relativistic dispersive effects is recovered. Our numerical solutions indicate that fully relativistic effects can prevent the formation of oscillitons by virtue of the weakening of dispersive effects required to counterbalance those of nonlinear steepening.

1 Introduction

Recently, Dubinin et al. (2003) studied nonlinear stationary whistler waves propagating in a cold plasma parallel to the magnetic field. Using quasineutrality, which requires $V_{Ae}^2 \ll c^2$ (or $\omega_{pe}^2 \gg \Omega_e^2$), ($V_{Ae}$ and c are, respectively, the Alfvén speed based on the electron density and the speed of light; ω_{pe} and Ω_e are, respectively, the electron plasma frequency and electron gyrofrequency) the complete system of nonlinear equations can be reduced to one highly nonlinear first order differential equation for the total amplitude of the transverse motion of the particles (u_e or u_p). The treatment involved was exact, in which the inclusion of the proton dynamics was crucial to the formation of nonlinear whistler waves. This point is manifested in the reference frame moving with the nonlinear structure, since in this frame the amplitude of the transverse momentum carried by the electrons and protons are equal, $m_p u_p = m_e u_e$, and therefore the proton dynamics must be included in a self-consistent manner. The phases of the transverse motion of the electrons and protons are different and the momentum exchange between the protons and electrons is mediated by the Maxwell stresses. Webb et al. (2005)[1]) showed that, in fact, the system is Hamiltonian and u_e^2 and the phase difference $\phi = \phi_p - \phi_e$ are the canonical coordinates. Soliton-type along with other nonlinear solutions with a core filled by smaller-scale oscillations (and therefore called oscilliton) were found. Although such a solution appears like an envelope soliton, the phase of the oscillations is stationary, in contrast to the envelope soliton.

Since the minimum speed of the oscilliton is $V_{Ae}/2$, it is clear that in a plasma with $\omega_{pe} \le \Omega_e$ (such a situation is rather typical for the auroral magnetosphere) the soliton speed can approach the light speed so that in transforming to the wave frame relativistic effects must be included and correspondingly the quasineutrality condition $n_p \approx n_e$ is no longer valid so that the Poisson equation must also be included in a self consistent manner. Solitary Alfvén and whistler waves in relativistic electron-positron plasmas were discussed in (Verheest, 1996; Verheest and Cattaert, 2004). In this paper we generalize the results of Dubinin et al. (2003) to the study the nonlinear relativistic whistlers.

2 Dispersion for stationary whistler waves

The linear dispersion equation provides us with the necessary condition for the existence of nonlinear stationary waves. The classical form of the dispersion relation for whistlers propagating parallel to the magnetic field in cold plasma is

$$N'^2 \equiv \frac{k'^2 c^2}{\omega'^2} = 1 + \frac{\omega_{pe}'^2}{\omega'(\Omega_e' - \omega')}, \tag{1}$$

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[1]Webb, G. M., McKenzie, J. F., Dubinin, E. and Sauer, K.: Hamiltonian formulation of nonlinear travelling whistler waves, Nonlin. Proc. Geophys., submitted, 2005.

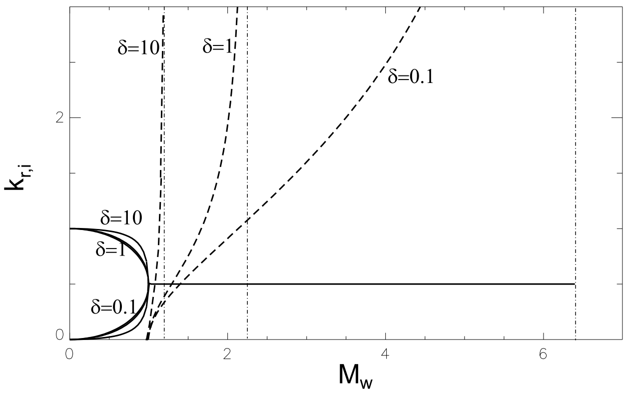

Fig. 1. The dispersion of stationary relativistic whistlers. The solid (dashed) curves show the real (imaginary) parts of the wave number $k_{r,i}(M_w)$ as a function of the wave speed at different values δ.

where the superscript $'$ refers to quantities in the rest frame S', and N' is the refractive index in S'. In the nonrelativistic case, the relation between the frequency and the wavenumber (ω', k') in the rest frame S' and the frequency and wavenumber (ω, k) in the wave frame S (moving to the left with the velocity U) is determined by the Doppler shift, $\omega'=\omega-kU$, $k'=k$. In the relativistic case, however, we have,

$$\omega' = (\omega - kU)\gamma, \; k' = (k - \frac{\omega U}{c^2})\gamma, \tag{2}$$

where γ is the Lorentz factor $\gamma=1/\sqrt(1-U^2/c^2)$.

The first term i.e. unity on the righthand side of Eq. (1), corresponds to the displacement current contribution which automatically implies the use of the Poisson equation and quasi-charge neutrality can no longer be assumed. Note that div of Ampere's law yields

$$0 = \mu_o div\boldsymbol{j} + \frac{1}{c^2}\frac{\partial div\boldsymbol{E}}{\partial t} \tag{3}$$

so that on using the continuity equation for the charge q and current $\boldsymbol{j}$, namely

$$\frac{\partial q}{\partial t} + div\boldsymbol{j} = 0 \tag{4}$$

yields the Poisson equation.

The dispersion equation for stationary waves in S, $k(U)$, is readily obtained by the simple substitution $\omega'/k' \to U$ in Eq. (1) which gives

$$\bar{k}(1-\bar{k}) = \frac{U^2}{V_{Ae}'^2}\gamma^2, \tag{5}$$

where $\bar{k}=\dfrac{Uk'}{\Omega_e'}$ is the wave number normalized to Ω_e'/U and V_{Ae}' is the Alfven speed in S' based on the electron mass.

Equation (5) can written in the form

$$\bar{k}=\frac{1}{2}\left(1\pm i\sqrt{\frac{4U^2\gamma^2}{V_{Ae}'^2}-1}\right)=\frac{1}{2}\left(1\pm i\sqrt{\frac{1}{V_m^2}\frac{(U^2-V_m^2)}{1-\dfrac{U^2}{c^2}}}\right), \tag{6}$$

where

$$\frac{1}{V_m^2}=\frac{1}{c^2}+\frac{4}{V_{Ae}'^2}, \quad \left(V_m^2=\frac{V_{Ae}'^2/4}{1+V_{Ae}'^2/4c^2}\right).$$

V_m is the maximum phase speed with $V_m \to V_{Ae}'/2$ (in the limit $V'^2_{Ae}/c^2 \ll 1$) and $V_m \to c$ (in the relativistic limit $V_{Ae}'/c \to \infty$). Introducing the proper Mach number $M_w=U/V_m$ Eq. (6) may be written

$$\bar{k} = \frac{1}{2}\left(1 \pm i\sqrt{\frac{M_w^2-1}{1-\dfrac{V_{Ae}'^2}{c^2}M_w^2}}\right). \tag{7}$$

Thus the appearance of the relativistic factor shows that relativistic effects must be taken into consideration if deviations from charge-neutrality become essential.

Note that for "supersonic" case ($c/V_m \geq M_w > 1$) $\bar{k}$ is complex and stationary structures (if they exist) are characterized by spatial oscillations superimposed on the spatial growth or decay given by $Im(\bar{k})$. Figure 1 shows the real and imaginary parts of $\bar{k}$ as a function of of M_w for stationary whistlers. The curve for $M_w<1$ describes the usual linear whistler waves. However for $M_w>1$ the wave number is complex and evanescent type solutions which contain an oscillating core are possible. The characteristic wave number k_r of the oscillations embedded into these waves is equal to $1/2L^{-1}$, where the length scale L here is

$$L = U/\Omega_e' = M_w V_m/\Omega_e' = \frac{M_w(V_{Ae}'/2)}{\Omega_e'}\frac{1}{\sqrt{1+V_{Ae}'^2/4c^2}}$$

$$= \frac{c}{\omega_{pe}}\frac{M_w}{2\sqrt{1+V_{Ae}^2/4c^2}}. \tag{8}$$

In the limit $V_{Ae}'^2 \ll c^2$ at the expression for stationary nonrelativistic whistlers (Eq. 23 in Dubinin et al., 2003) is recovered.

3 Equations for relativistic whistler-stationary waves

In one dimensional progressive waves of the form $f(x+Ut)$, it is convenient to carry out the analysis in the wave frame where the various species appear to flow steadily ($\dfrac{\partial}{\partial t}=0$) from left to right with speed U. In the nonrelativistic case (Dubinin et al., 2003), the governing nonlinear equations were derived from the differential equations of motion for the transverse velocities of the electrons and protons, $u_{e,y,z}$, $u_{p,y,z}$. The remaining variables (the longitudinal components of the velocities and the transverse components of the magnetic field) were expressed as functions of the transverse velocity components by using the constants of motion along the lines described by McKenzie et al. (2004). In the relativistic case, the equations of motion become

$$u_{ix}\frac{d\boldsymbol{p}_i}{dx} = \pm e(\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{B}), \tag{9}$$

where $\boldsymbol{p}_i$ is the momentum of the $i{=}p$, e species

$$\boldsymbol{p}_i = m_i \boldsymbol{u}_i / \sqrt{\left(1 - \frac{u_i^2}{c^2}\right)} \tag{10}$$

Introducing the new variables

$$\boldsymbol{v}_i = \boldsymbol{u}_i / \sqrt{\left(1 - \frac{u_i^2}{c^2}\right)}, \tag{11}$$

where $u_i^2{=}u_{ix}^2{+}u_{iy}^2{+}u_{iz}^2$, yields, in the normalized form

$$\mu_i \frac{dv_{ix}}{dx} = \pm\left(\frac{E_x}{u_{ix}} + \frac{v_{iy}}{v_{ix}} b_z - \frac{v_{iz}}{v_{ix}} b_y\right) \tag{12a}$$

$$\mu_i \frac{dv_{iy}}{dx} = \pm\left(\frac{v_{iz}}{v_{ix}} - b_z\right) \tag{12b}$$

$$\mu_i \frac{dv_{iz}}{dx} = \pm\left(b_y - \frac{v_{iy}}{v_{ix}}\right). \tag{12c}$$

Here $\mu_p{\equiv}\mu{=}m_p/m_e$, $\mu_e{=}1$, $B_x{=}B_o{=}1$, $b_{z,y}{=}B_{z,y}/B_o$, and the velocities and the electric field are normalized to U and UB_o, respectively, with the spatial variable x normalized to U/Ω_e. Note that in the relativistic case, Ω_e is no longer the relativistic gyrofrequency since the latter becomes a function of particle energy.

Ampere's law gives the relationship between the velocity and magnetic field variations,

$$\frac{db_y}{dx} = M_{Ae}^2\left(\frac{v_{pz}}{v_{px}} - \frac{v_{ez}}{v_{ex}}\right) \tag{13a}$$

$$\frac{db_z}{dx} = -M_{Ae}^2\left(\frac{v_{py}}{v_{px}} - \frac{v_{ey}}{v_{ex}}\right) \tag{13b}$$

Here $M_{Ae}{=}U/V_{Ae}$ (V_{Ae} is the Alfvén speed in S based on the electron mass density).

The Poisson equation becomes

$$\delta\frac{dE_x}{dx} = \frac{1}{u_{px}} - \frac{1}{u_{ex}}, \tag{14}$$

where $\delta{=}V_{Ae}^2/c^2$ and we assume that at $x{=}-\infty$, $n_p{=}n_e{=}1$ and $u_{px}{=}u_{ex}{=}U{=}1$, and we have used the conservation of the particle flux $n_i u_{ix}{=}1$. The relationship between $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ in the dimensionless variables becomes

$$\boldsymbol{u}_i = \boldsymbol{v}_i\sqrt{1 - M_{Ae}^2 \delta u_i^2} \tag{15}$$

Note that in the wave frame, Faraday's law implies that E_y=const, E_z=const, so that for motion parallel to the magnetic field $E_y{=}0$, $E_z{=}0$.

The system of Eqs. (12)–(14) admits the following constants of motion (momentum flux conservation)

$$\mu(v_{px} - 1) + (v_{ex} - 1) + \frac{\left(b_y^2 + b_z^2\right)}{2M_{Ae}^2} - \delta\frac{E_x^2}{2} = 0 \tag{16a}$$

$$\mu v_{py} + v_{ey} = \frac{b_y}{M_{Ae}^2} \tag{16b}$$

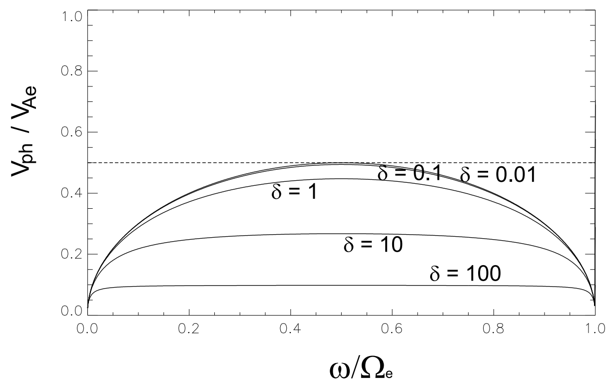

Fig. 2. The dispersion of whistler waves at the different values $\delta{=}V_{Ae}^2/c^2$. The curves reveal maxima of the phase velocity, which determine the threshold values for the existence of oscilliton structures (the necessary condition).

$$\mu v_{pz} + v_{ez} = \frac{b_z}{M_{Ae}^2}. \tag{16c}$$

Therefore the magnetic field components can be expressed as functions of the transverse velocities. Correspondingly, the electric field E_x can be expressed in the terms of the velocity variables.

Multiplying Eqs. (12b) and (12c) by $\mu_i v_{iy}$ and $\mu_i v_{iz}$, respectively and adding yields after elimination b_y and b_z with the help of Eqs. (16b) and (16c):

$$\mu(v_{py}^2 + v_{pz}^2)^{1/2} = \left(v_{ey}^2 + v_{ez}^2\right)^{1/2}, \tag{17}$$

i.e. a similar relation between the amplitudes of the transverse velocity variables as in nonrelativistic case. The difference is that the variables $v_{iy,z}$ depend now explicitly on the total speed of the species (Eq. 11).

Before analysing the properties of the remaining nonlinear differential equations for the species velocities we check that these equations do indeed reduce to the dispersion equation for stationary waves in the wave frame as described in Sect. 2. Linearizing the Eqs. (12b)–(12c) of transverse motion and introducing circular polarized variables $u_{i\pm}{=}u_{iy}{\pm}iu_{iz}$, yields

$$\gamma\frac{du_e}{dx} = \mp i\left[M_{Ae}^2\gamma(u_{e\pm} + \mu u_{p\pm}) - u_{e\pm}\right] \tag{18a}$$

$$\gamma\mu\frac{du_p}{dx} = \pm i\left[M_{Ae}^2\gamma(u_{e\pm} + \mu u_{p\pm}) - u_{p\pm}\right]. \tag{18b}$$

Here $\gamma{=}1/\sqrt{1{-}U^2/c^2}$.

Seeking solutions of the form $\sim u_i \exp(ikx)$, gives

$$u_e\left(\gamma k + \left(M_{Ae}^2\gamma - 1\right)\right) = -M_{Ae}^2\gamma\mu u_p \tag{19a}$$

$$u_p\left(\gamma\mu k - \left(M_{Ae}^2\gamma\mu - 1\right)\right) = M_{Ae}^2\gamma u_e \tag{19b}$$

Hence we obtain the dispersion relation in S,

$$\gamma k^2 - k + M_{Ae}^2 = 0 \tag{20a}$$

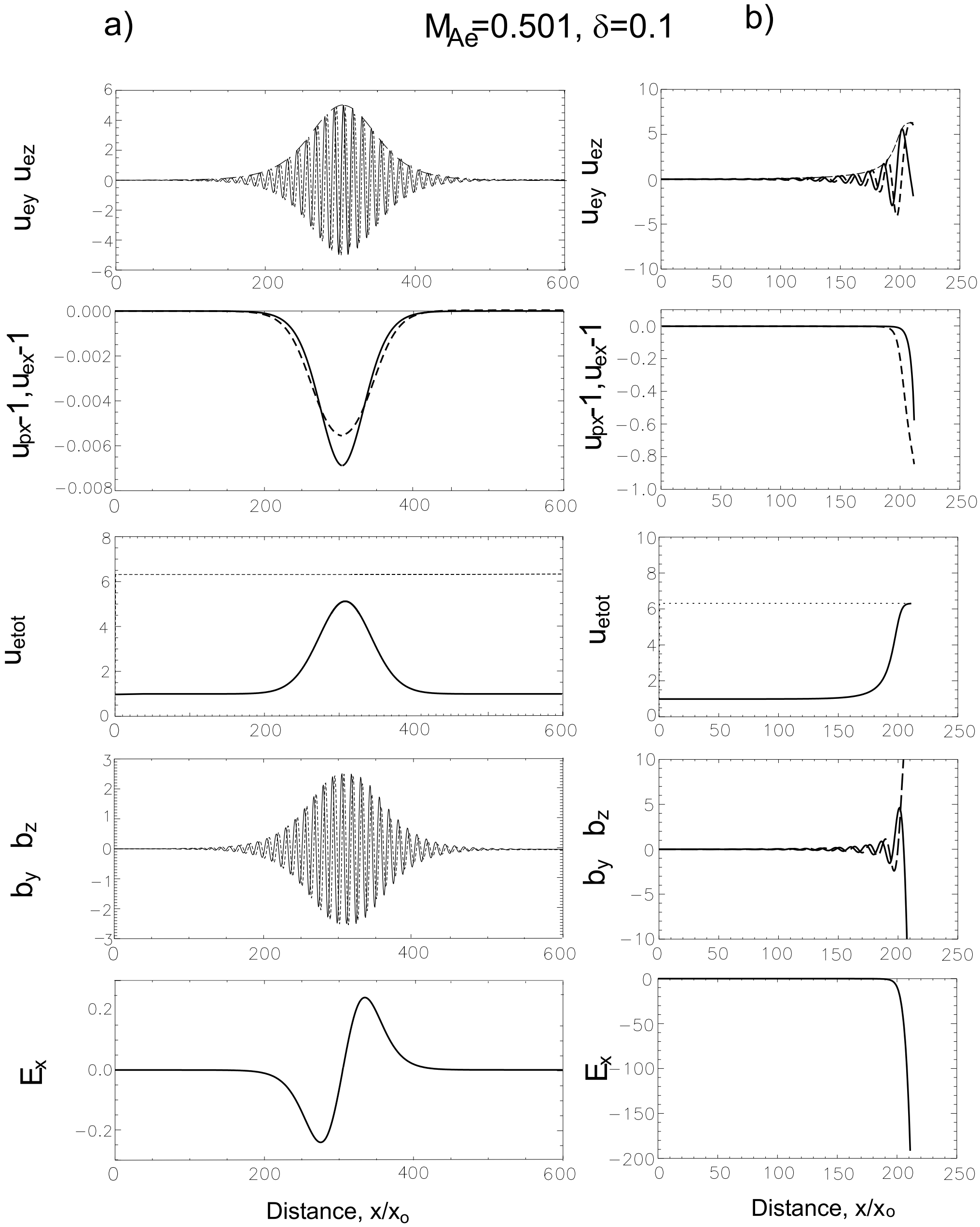

Fig. 3. (a) Oscilliton structure found in the approximation of nonrelativistic equations of the particles motion, while the charge-neutrality is not imposed from the onset. From the top to the bottom are (i) the transverse velocities and the total value of the transverse speed of the electrons, (ii) the longitudinal velocities of the protons (solid curve) and electrons (dashed curve), (iii) the total value of the electron velocity and the light speed, (iv) two transverse components of the magnetic field, (v) the longitudinal electric field. **(b)** The same parameters for the relativistic case.

or

$$k=\frac{1}{2\gamma}\left(1\pm\sqrt{1-4M_{Ae}^2\gamma}\right) \tag{20b}$$

Using the relativistic result $k'=k\gamma$ which follows from Eq. (2) in the wave frame (ω=0), and $V_{Ae}'^2=V_{Ae}^2\gamma$, Eq. (20b) does indeed coincide with the dispersion Eq. (6).

Equations (12) and (16) which completely describe the dynamical system in the relativistic case are solved numerically. Recall that in the nonrelativistic case, in which quasi-neutrality holds, the type of nonlinear stationary solutions depends on the wave speed. For $U>V_{\max}=V_{Ae}/2$ ($V_{\max}$ is the maximum phase speed which follows from the linear dispersion analysis), there is a family of periodic solutions around an O-type point limited by a heteroclinic orbit connecting two saddle points with a family of free orbits outside. The orbit connecting two saddle points corresponds to a soliton type solution for the longitudinal u_{ix} and total transverse

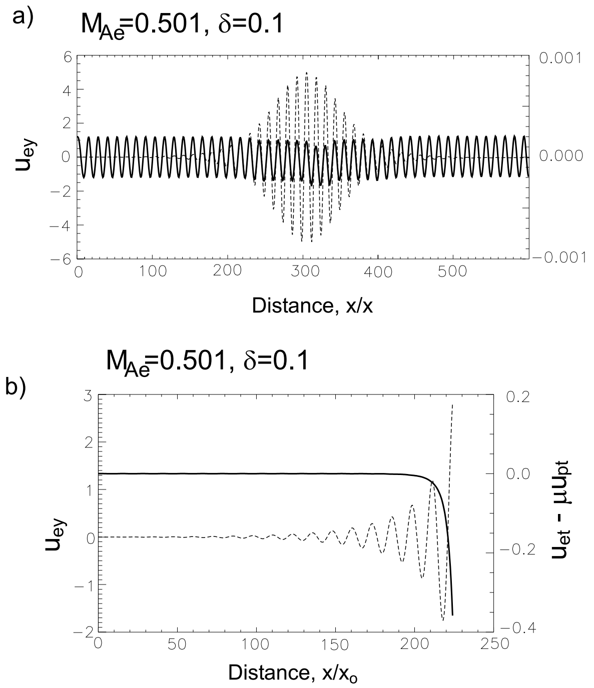

Fig. 4. The difference $u_{et}-\mu u_{pt}$ (the solid curves) within the 'oscilliton' sructures for nonrelativistic **(a)** and relativistic **(b)** cases. The dashed curves depict the velocity component u_{ey}.

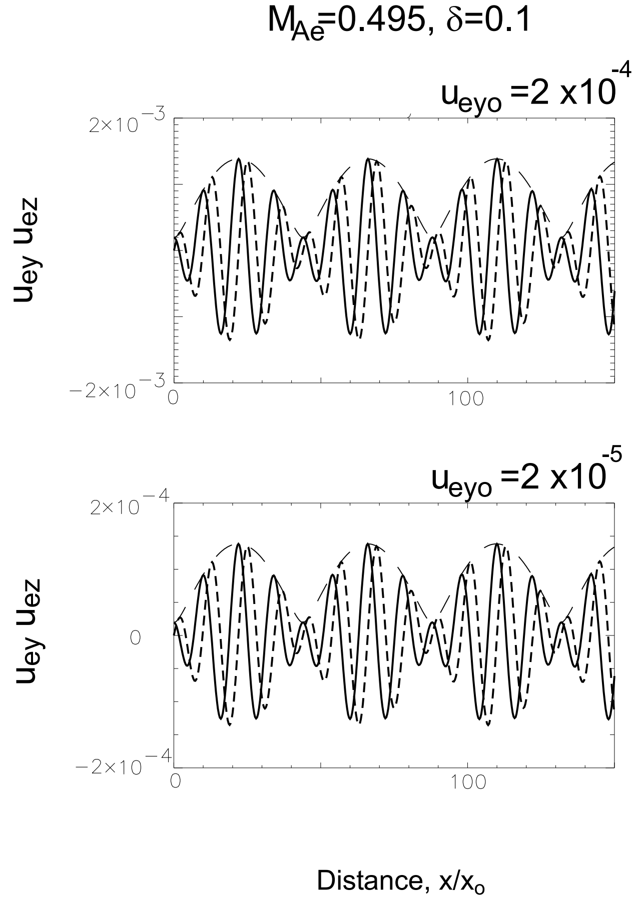

Fig. 5. Nonlinear periodic waves in the relativistic case.

speed u_{it} variables. However the transverse velocity and magnetic field components $u_{iy,iz}$, $b_{y,z}$ reveal smaller scale oscillations embedded within the "envelope" soliton structures. At $U \leq V_{Ae}/2$ two saddle points degenerate to one and only nonlinear periodic solutions remain. These solutions correspond to a periodic sequence of wave packets.

When quasi-neutrality is not imposed at the outset but relativistic effects are assumed negligible (γ_i=1), a similar picture appears although the system of nonlinear equations cannot now be reduced to the form which admits a simple phase-portrait analysis (Verheest et al., 2004). In the relativistic case, the maximum phase speed $V_{\max}=V_m$ (see Sect. 2) decreases with δ as shown in Fig. 2. Moreover the presence of relativistic γ-factors drastically changes the character of the nonlinear stationary solutions. In fact we did not find oscilliton-type solutions, although at the initial stage the system evolved as an oscilliton wave with growing periodic oscillations. Thus, in contrast to the nonrelativistic case it appears that dispersion cannot prevent nonlinear steepening and the amplitude of the wave continues to grow until the total electron speed reaches the light speed. Figure 3 (right panels) depicts the spatial growth of the wave. At $x \sim 210$, the amplitudes of the transverse components of the electron speed (the upper panel) and the longitudinal velocity (the second panel) achieve certain values and growth ceases. The third panel shows the total value of the electron velocity. The dashed curve gives the light speed. It is observed that a 'saturation' occurs when $u_e \rightarrow c$. The lower two panels depict the magnetic and electric field variations. For comparison, the left panels present the corresponding parameters when the relativistic factors $\gamma_i \rightarrow 1$ where solitary solutions are possible.

Figure 4 shows the deviation from zero of the value $u_e - \mu u_p$ (where u_i is the total value of the transverse velocity $u_i = (u_{iy}^2 + u_{iz}^2)^{1/2}$) characterizing a "balance" between the electron and proton transverse oscillating motions. In the nonrelativistic case, in the quasi-neutral approximation, this value is exactly zero showing that the total values of the transverse momentum flux carried by the protons and electrons are equal. If the relativistic factors are neglected, while not retaining charge-neutrality, this value oscillates near zero ($u_e - \mu u_p \sim 2 \cdot 10^{-4}$) (Fig. 4, the top panel), i.e. the amplitudes of the transverse momentum fluxes carried by the protons and electrons are almost the same. In the relativistic case, this balance is violated and the value $|u_e - \mu u_p|$ sharply increases since the electrons, whose velocity is much larger than that of the protons, experience a relativistic decrease in the velocity (Fig. 4, the bottom panel). Although the transverse momentum of the system is conserved (Eqs. 16b and 16c) the transverse motion of the particles is not in balance to conserve an oscilliton structure.

In contrast to oscillitons, nonlinear wave packets continue do exist in the relativistic case. Figure 5 gives an example of such a structure. The amplitude of the waves is proportional

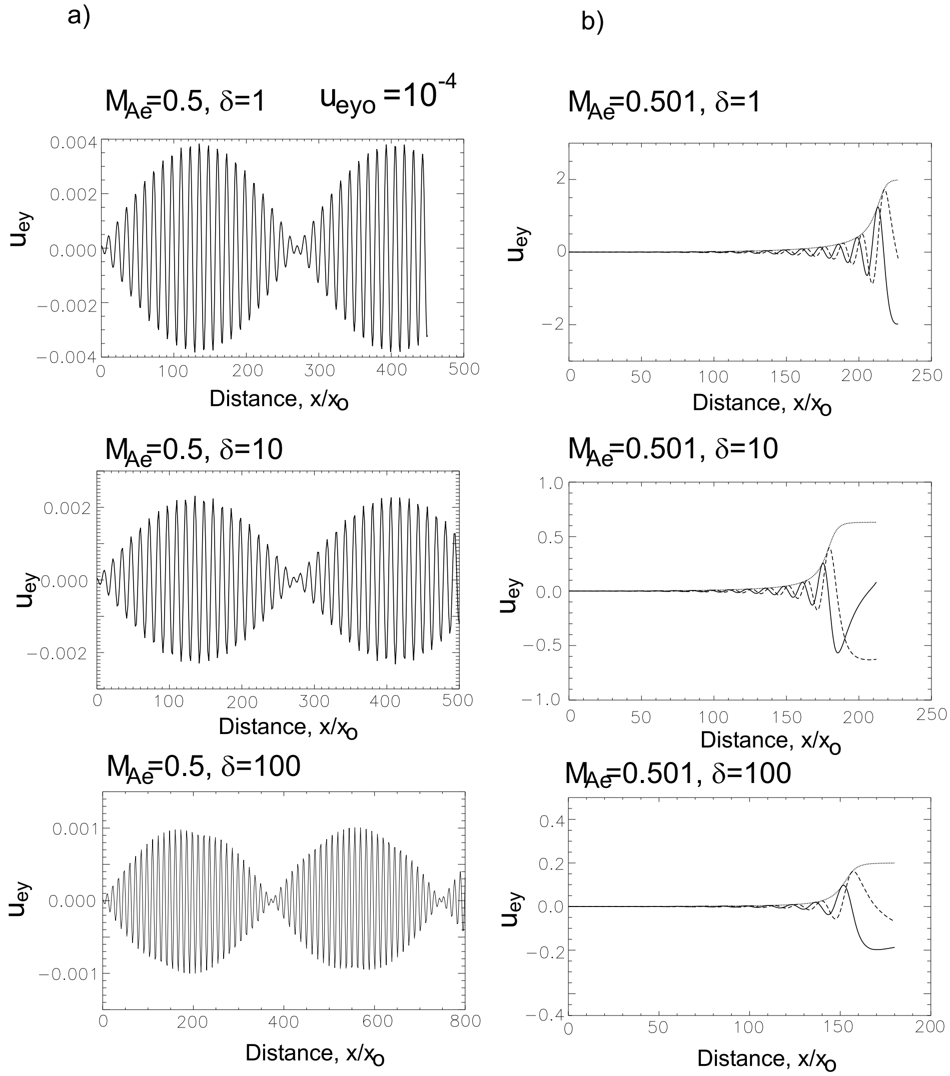

Fig. 6. **(a)** Periodic waves at different values of V_{Ae}^2/c^2 and the threshold value of the wave speed U=$0.5V_{Ae}$. **(b)** For U>$0.5V_{Ae}$, the electron speed in the oscillating structures reaches the light speed and stationary solutions are absent.

to the "initial" perturbation value at x=$-\infty$ ("the integration constant") and the amplification coefficient A=$u_{ey_{\max}}/u_{ey_\infty}$ is about of 10. It is interesting to note that periodic solutions only exist at $M_{Ae}\leq 0.5$, independently of the value of δ=V_{Ae}^2/c^2 (Fig. 6). The coefficient A reaches a maximum at M_{Ae}=0.5 which increases with decreasing δ ($\sim$10 ÷ 45 for δ=100 ÷ 0.03). At M_{Ae}>0.5, no stationary oscilliton-type solutions is found and the electron velocity increases up to the light speed.

4 Conclusions

This analysis shows that if quasi charge neutrality is not imposed at the outset the relativistic equations of motion must be used to describe the system consistently. This stems from the fact that non-charge neutrality implies the inclusion of relativistic effects through the displacement current and the associated Poisson equation. Only if these effects are included can the proper stationary wave dispersion equation be correctly derived. Moreover it would appear that fully relativistic effects may prevent the formation of oscilliton-like structures by virtue of dispersive effects being unable to compete with and balance nonlinear steepening. This requires a further careful study, but it is already apparent in Fig. 2, where $\gamma = 100$, that the phase speed is practically constant for all values of $\omega/\Omega_e \leq 1$.

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