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Prebifurcation noise amplification and noise-dependent hysteresis as indicators of bifurcations in nonlinear geophysical systems

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Abstract. The phenomena of prebifurcation noise amplification and noise-dependent hysteresis are studied as prospective indicators of bifurcations ("noisy precursor") in nonlinear Geophysical systems. The phenomenon of prebifurcation noise amplification arises due to decreasing of damping coefficients just before bifurcation. A simple method for the estimation of the forced fluctuation variance is suggested which is based on results of linear theory up to the boundary of its validity. The upper level for the fluctuation variance before the onset of the bifurcation is estimated from the condition that the contribution of the non-linear term becomes comparable (in the sense of mean squares) with that of the linear term. The method has proved to be efficient for two simple bifurcation models (period doubling bifurcation and pitchfork bifurcation) and might be helpful in application to geophysics problems. The transition of a nonlinear system through the bifurcation point offers a new opportunity for estimating the internal noise using the magnitude of the noise-dependent hysteretic loop, which occurs when the control parameter is changed in the forward and backward direction.

1 Introduction

This paper studies two fundamental phenomena, which could be observed in geophysical systems near the threshold of bifurcation: prebifurcation noise amplification and noise-dependent hysteresis.

Analysis of prebifurcation noise amplification, performed in paper (Wiesenfeld, 1987), is based on linear theory, which demonstrates unlimited growth of fluctuations in the immediate vicinity of the bifurcation point. According to (Wiesenfeld, 1987), prebifurcation noise amplification might be an effective diagnostic instrument for nonlinear systems (so named "noisy precursor" of bifurcation), see also (Wiesenfeld, 1985). Nonlinear saturation of fluctuations in the vicinity of bifurcation point was studied in the papers (Kravtsov and Surovyatkina, 2001; Kravtsov et al., 2003) for the case of sufficiently slow (quasi-stationary) bifurcation. In this paper we shall show, that rapid bifurcation transitions cause decreasing of the noise amplification.

At fast bifurcation transitions ("dynamical bifurcations") another phenomenon occures – the noise-dependent hysteresis. This phenomenon was first mentioned in paper (Kapral and Mandel, 1985) where it was shown that under forward bifurcation transitions, the time-delay $\Delta\tau = \tau_+ - \tau_c$ between the break-away point τ_+ and critical bifurcation value τ_c is proportional to $\sqrt{s}$, where s is the rate of the control parameter change. Hysteretic loops in systems with bifurcations were observed experimentally (Morris and Moss, 1986). The most detailed investigation of the delay phenomenon was performed in paper (Baesens, 1991), where asymptotic analytical expressions for bifurcation diagram branches were found. In addition, in (Baesens, 1991) the asymptotic scaling was obtained for time-delays of period doubling bifurcations in conditions of very weak noise, caused by the rounding errors. In this work the phenomenon of hysteresis is studied for the nonlinear oscillator, experiencing spontaneous symmetry breaking. We shall show that characteristics of the histeretic loop essentially depend on the noise level and might be of practical interest for analysis of the nonlinear geophysical systems.

The main goal of this publication is to attract the attention of geophysicists to this phenomenon and thereby to stimulate searches for specific mechanisms that might be responsible for noise amplification and noise-dependent hysteresis in geophysics.

The phenomenon of prebifurcation noise amplification is shortly outlined by the examples of period doubling bifurcation in nonlinear map (Sect. 2) and a pitchfork bifurcation in nonlinear oscillator (Sect. 3) with a special emphasis on the transition from linear regime to the regime of nonlinear saturation of amplification. Opportunity for the weak noise measuring on basis of the nonlinear saturation of the prebifurcation noise amplification are discussed in Sect. 4.

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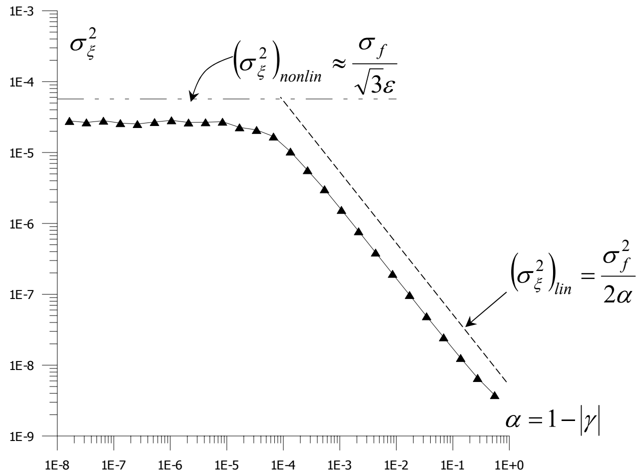

Fig. 1. Prebifurcation noise amplification for period doubling bifurcation in quadratic map $F(x)=\mu-x^2$ (σ_f^2=3.3·10^{-9}, ε=1).

The hysteretic phenomena in dynamic bifurcations in the nonlinear oscillator are described in Sect. 5. In Sect. 6, we shall point out an opportunity to use the noise-dependent hysteresis in bifurcation systems for noise level measurement. Conclusions (Sect. 7) summarize the main results of the work.

2 Prebifurcation noise amplification at the threshold of period doubling bifurcation

Nonlinear noisy map

$$x_{n+1} = F(x_n, \mu) + f_n, \tag{1}$$

where μ is a control parameter, and f_n is an external δ-correlated random process, $\langle f_n f_m \rangle = \sigma_f^2 \delta_{nm}$, has a noiseless fixed point $\bar{x}$ which obeys the equation $\bar{x}=F(\bar{x},\mu)$. Fluctuations $\xi_n = x_n - \bar{x}$ near stable state are governed by the equation

$$\xi_{n+1} = \gamma \xi_n + \varepsilon \xi_n^2 + ... + f_n, \tag{2}$$

where $\gamma = \frac{dF(\bar{x})}{dx}$ and $\varepsilon = \frac{1}{2}\frac{d^2F(\bar{x})}{dx^2}$. Bifurcation of period doubling arises when the modulus of multiplier γ exceeds a unit: $|\gamma|>1$.

In frame of the linear approach, when Eq. (2) takes the form $\xi_{n+1} = -\gamma\xi_n + f_n$, variance of fluctuations $\langle \xi_n^2 \rangle \equiv \sigma_f^2$ is given by

$$\left(\sigma_\xi^2\right)_{\rm lin} = \frac{\sigma_f^2}{1-\gamma^2} \cong \frac{\sigma_f^2}{2\alpha}, \tag{3}$$

where parameter $\alpha = 1-|\gamma|$ characterizes closeness of the nonlinear system to bifurcation threshold $|\gamma|$=1 (note, that parameter α can also be written as $\alpha=\mu_c-\mu$). According to Eq. (3), in frame of the linear approach fluctuation variance σ_ξ^2 tends to the infinity at $\alpha \to 0$, what corresponds to the case studied by Wiesenfeld (Wiesenfeld, 1987).

In the "linear" regime, the fluctuations ξ_n consist of a sum of a large number of independent terms. Hence, by virtue of the central limit theorem, the sequence ξ_n is nearly Gaussian.

Linear theory is valid until the linear term in Eq. (2) becomes comparable (in the statistical sense) with the quadratic term, that is when

$$(1-\gamma^2)\sigma_\xi^2 \approx \varepsilon^2 \left\langle \xi^4 \right\rangle. \tag{4}$$

Using relation $\langle \xi_n^4 \rangle \approx 3\sigma_\xi^2$, as for Gaussian values, one can estimate (from Eqs. 3 and 4) the minimal value $\alpha_{\rm min}$ limiting the area of linear theory validity: $\alpha_{\rm min} = \sqrt{3}\varepsilon\sigma_f/2$.

Substitution of $\alpha_{\rm min}$ into Eq. (3) allows estimating the level of saturated fluctuation (Eq. 4):

$$(\sigma_\xi^2)_{\rm nonlin} \approx \frac{\sigma_f}{\sqrt{3\varepsilon}}. \tag{5}$$

The dependence of σ_ξ^2 on α is presented on Fig. 1: at $\alpha > \alpha_{\rm min}$, one can use the results of linear theory, Eq. (3) (unlimited dashed line), whereas at $\alpha < \alpha_{\rm min}$, nonlinear estimate (Eq. 5) enters into the play (horizontal pointed line).

The estimates presented above, are in good agreement with the results of numerical simulation, performed in (Kravtsov and Surovyatkina, 2003) and shown by a continuous line at Fig. 1. Estimate Eq. (5) happens to be only 30–40% less as compared to numerical data.

3 Prebifurcation noise amplification in nonlinear oscillator experiencing a pitchfork bifurcation

Nonlinear oscillator, described by the equation

$$\ddot{\xi} + 2\beta\dot{\xi} + B\xi + A\xi^3 = f(t), \tag{6}$$

where β is damping coefficient, admits pitchfork (symmetry breaking) bifurcation at critical value B_c=0: unique stable state $\bar{\xi}$=0 at $B>0$ converts into pair of stable states $\bar{\xi} = \pm\sqrt{-\frac{B}{2A}}$ at $B<0$.

In the linear approximation (nonlinear term ξ^3 is considered to be negligibly small) variance $\sigma_\xi^2 = \langle \xi^2 \rangle$ is given by

$$(\sigma_\xi^2)_{\rm lin} = \frac{\sigma_f^2 \tau_f}{2\beta B}, \tag{7}$$

where τ_f is correlation time of the process $f(t)$. According to Eq. (7), in frame of linear theory fluctuation varianceσ_ξ^2 tends to infinity, when B is approaching to the bifurcation point B_c=0. The results of linear theory are shown at Fig. 2 by a dashed curve.

Variance of the elastic force $B\xi + A\xi^3$ is equal to

$${\rm Var}(B\xi + A\xi^3) = B^2\left\langle \xi^2 \right\rangle + 2AB\left\langle \xi^4 \right\rangle + A^2\left\langle \xi^6 \right\rangle. \tag{8}$$

The contribution of nonlinear term $2AB\langle\xi^4\rangle \approx 6AB\sigma_\xi^4$ in Eq. (8) ($\langle\xi^4\rangle$is considered to be $3\sigma_\xi^4$, as for Gaussian process) becomes comparable with the contribution of linear term $B^2\sigma_\xi^2$ when

$$\sigma_\xi^2 \approx \frac{B}{6A}. \tag{9}$$

Taking for σ_ξ^2 linear expression (Eq. 7), one can estimate the minimal value of $B_{\min}$, restricting the area of applicability of linear theory:

$$B_{\min} \approx \sigma_f \sqrt{\frac{3A\tau_f}{\beta}}. \quad (10)$$

This value, substituted in Eq. (9) provides nonlinear estimate for fluctuation saturation at bifurcation threshold B=0:

$$\left(\sigma_\xi^2\right)_{\mathrm{nonlin}} \approx \sigma_f \sqrt{\frac{\tau_f}{12A\beta}}. \quad (11)$$

This estimate is presented by a horizontal dotted line in Fig. 2. Nonlinear estimate (Eq. 11) is valid only at the bifurcation point in interval $B[0, B_{\min}]$.

It is convenient to define pre-bifurcation noise amplification coefficient as the ratio of fluctuation intensity $(\sigma_\xi^2)_{\mathrm{nonlin}}$ in the area of saturation to the intensity of fluctuations $(\sigma_\xi^2)_{\mathrm{lin}}$ at ω^2=B

$$K = \frac{(\sigma_\xi^2)_{\mathrm{nonlin}}}{(\sigma_\xi^2)_{\mathrm{lin}}} = \frac{1}{\sigma_f}\left(\frac{\beta}{3A\tau_f}\right)^{1/2} \omega^2. \quad (12)$$

This quantity shows, by how many times fluctuation intensity in the saturation zone exceeds stationary fluctuations in the oscillator.

We discussed above fluctuations under slow (quasi-stationary) change of parameter B. In the case of a rapid bifurcation transition, one can expect some decrease of $(\sigma_\xi^2)_{\mathrm{nonlin}}$ as compared to slow (quasi-stationary) bifurcation transitions.

Let parameter B depends on time: B=$B(t)$ (dynamic bifurcations) and assume, that parameter B takes positive values B>0 at $t<t^*$, where t^* is the bifurcation point, and negative values at $t>t^*$. For example, such a behavior is demonstrated by function

$$B(t) = -\omega_0^2 \operatorname{arctg} \eta(t - t^*). \quad (13)$$

Coefficient η characterizes here the speed of the control parameter change.

The slow-down tendency of σ_ξ^2 growth rate with the speed of transition through the point of bifurcation η can be illustrated by way of considering the limit $\eta\to\infty$. At $t<t^*$, $B(t)$ takes a constant value ω_0^2, while at $t>t^* B$=$-\omega_0^2$. In both cases, the intensity of fluctuations is identical and, in line with Eq. (7), equals to

$$(\sigma_\xi^2)_0 = \frac{\sigma_f^2\tau_f}{2\beta B},$$

so that the fluctuation amplification factor turns into a unit: K=1.

Obviously, at $t>t^*$, an exponential fluctuation growth will be observed due to the loss of stability of the equilibrium state ξ=0, but this has no effect on fluctuation behavior at $t<t^*$.

In Sect. 5, we will discuss in more detail the influence of noise on dynamic bifurcations in nonlinear oscillator.

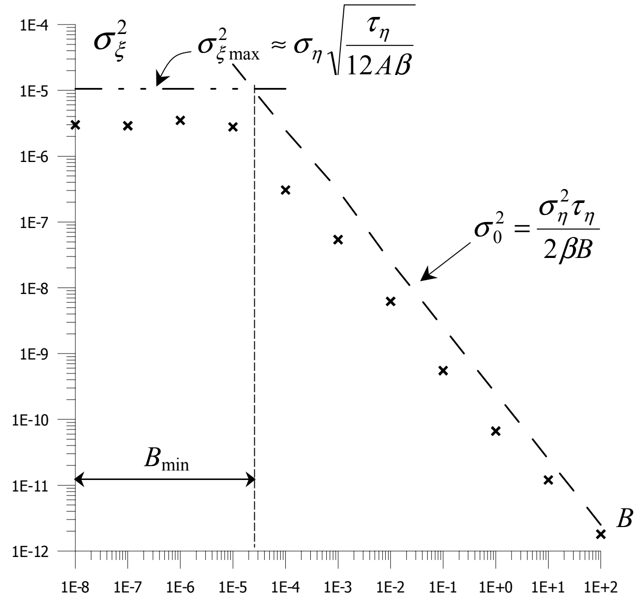

Fig. 2. Prebifurcation noise amplification for bifurcation of spontaneous symmetry breaking (σ_η^2=10^{-8}, τ_η=3.5·10^{-3}, β=0.1, A=0.5).

4 On weak noise measurements on basis of nonlinear saturation of prebifurcation noise amplification

A noticeable increase in the variation of fluctuations σ_ξ^2 near the bifurcation threshold might be an effective diagnostic instrument for nonlinear geophysical systems. Signal amplification in data time series can be connected with the approach of the system to bifurcation threshold. Obviously, the hypothesis of bifurcations present in the system should be thoroughly examined.

One of possible methods of examination can be the method for weak noise measuring on the basis of phenomenon of the prebifurcation noise amplification under study. Our approach is based on a comparison of the maximal variation $(\sigma_\xi^2)_{\max}$~σ_f at the bifurcation point with the variation σ_ξ^2≈σ_f^2 far away from that point.

One can suggest another approach: having measured variance σ_ξ^2 of variable ξ in the very vicinity of bifurcation point, variance σ_f^2 of the noise process $f(t)$ can be determined according to Eqs. (5) and (11) by ratio

$$\sigma_f^2 \sim \sigma_\xi^4.$$

This is true not only for period-doubling bifurcations (Kravtsov and Surovyatkina, 2003), but also for pitchfork bifurcations.

This approach looks to be prospective for analysis of different nonlinear geophysical systems in which period-doubling bifurcations and pitchfork bifurcations take place. Such a system is, for example, suggested in (Seongjoon and Ghil, 2002) where a model of atmospheric zonal-flow vacillation with successive period-doubling bifurcations is suggested. The model describes the origin of zonal-flow vacillation in the Southern Hemisphere (Kidson, 1988, 1991;

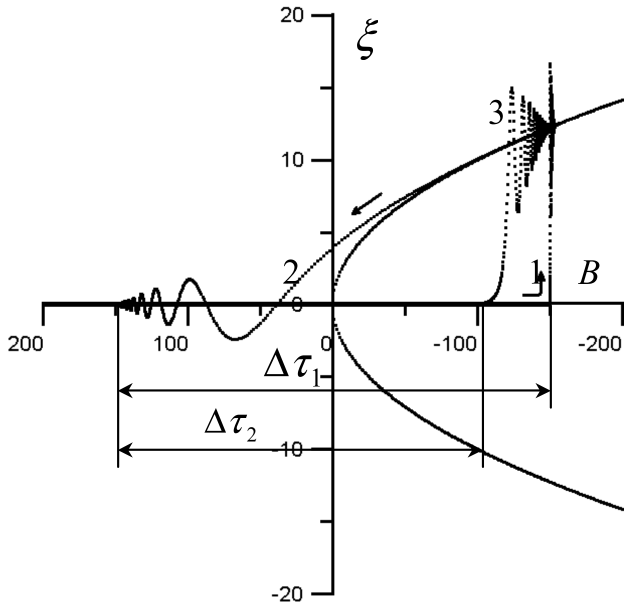

Fig. 3. Bifurcation diagram and typical orbits of nonlinear oscillator (Eq. 6) in the case of constant sweep rate: orbit 1 – positive sweep rate (η=3); orbit 2 – negative sweep rate (η=−3); orbit 3 – influence of noise σ_η^2=1, 87·10^{-7} at η=3. $\Delta\tau_1$ – hysteretic loop size without noise, $\Delta\tau_2$ – hysteretic loop size decrease due to noise.

Hartmann and Lo, 1998) and Northern Hemisphere (Charney and DeVore, 1979, 1981) observations. Another example is the ocean's overturning circulation between cold regions, where water is heavier and sinks, and warm regions where it is lighter and rises (Ghil, 2001) . The effect of temperature on the water's mass density and hence, motion, is in competition with the effect of salinity. These competing effects can also give rise to two distinct equilibria. In a simplified mathematical setting, these two equilibria arise by a pitchfork bifurcation that breaks the problem's mirror symmetry (Quon and Ghil, 1992; Thual and McWilliams, 1992).

5 Noise-dependent hysteresis

The hysteretic phenomenon in dynamic bifurcations is manifested in that, after passing over the bifurcation point, the system for a considerable time remains on the unstable branch and only later makes a rather quick transition to another stable state (Kapral and Mandel, 1985; Morris and Moss, 1986; Baesens, 1991). Similar process takes place under the backward transition, when control parameter decreases. Periodical change in the control parameter results in that the system delays in the vicinity of the former stable points under both forward and backward motions. The delay phenomenon causes the emergence of the hysteretic loop. The higher is the control parameter change speed, the more distinct is the hysteretic phenomenon.

We observed the phenomenon when considering the model of non-linear oscillator subject to a pitchfork bifurcation (see Sect. 3). In a quasi-stationary mode, when parameter B changes slowly, bifurcation in the system happens at critical value B=B^*=0. Under a swift change of B, a pitchfork bifurcation happens only some time after the critical value B^*=0 is passed, the delay time being dependent on the speed of change of parameter η.

Figure 3 presents results of numerical modeling of bifurcation transition in nonlinear oscillator when parameter B conforms to Eq. (13). For illustrative purposes, the figure combines the bifurcation diagram of the model, that is stable values at constant B, and the typical orbit of the nonlinear oscillator, that is the dependence $x(B)$ with changing B. With the change of B, the system having passed the value B=B^*, still resides for some time in the vicinity of the unstable branch (this time depends considerably on speed η), and only after that switches to one of the two possible stable states of equilibrium. Similar process happens under backward transition through the bifurcation point, with the delay resulting in the emergence of hysteresis a hysteretic loop. Figure 3 illustrates the phenomenon of hysteresis in nonlinear oscillator at η=3 (forward sweep, orbit 1) and η=−3 (backward sweep, orbit 2). The size of the emerged hysteretic loop is denoted as $\Delta\tau_1$.

As seen from the figure, hysteretic loop diminishes with noise (orbit 3). Notice, that under forward transition through the bifurcation point, the system is more sensitive to noise than under backward transition. This effect can be used to measure weak noise in nonlinear systems as it was suggested earlier for systems with period doubling bifurcations (Butkovskii et al., 1997).

The phenomenon of the hysteretic was observed experimentally in the case of transition from laminar to vortex shedding flow in soap films (Horvath et al., 2000).

6 Use of noise-dependent hysteresis in bifurcation systems for noise measurement

The phenomenon of noise-dependent hysteresis can appear useful for measuring weak noise in nonlinear systems with bifurcation parameter changing with time.

Since pre-bifurcation fluctuation growth slows down under swift transitions, noise measurement techniques described in Sect. 4 become ineffective. In these conditions, the dynamical method to infer noise levels from the characteristics of hysteretic loops ensures a much higher sensitivity because the size of hysteretic loop is very sensitive to noise level.

In order to estimate noise level σ_f^2, it is suggested to measure the size of a hysteretic loop and, based on a corresponding calibration graph, estimate the variance of internal noise in the system.

An example in Fig. 4 shows the dependence of the hysteretic loop size $\Delta\tau$ on noise level σ_f^2 in a system for nonlinear oscillator. One can see that the size of the loop decreases with noise.

7 Conclusions

In this paper, nonlinear effects of noise at the threshold of quasi-stationary and dynamic bifurcations are investigated. In the case of slowly (quasi-stationary) changing control parameter, the variance of forced fluctuations increases with the approach to the threshold of bifurcation. In the immediate vicinity of the threshold, the variance saturates at a level proportional to mean square of noise. It is shown, that prebifurcation noise amplification is most distinct under slow transition of a system through the bifurcation point, whereas under swift transitions, the amplification effect diminishes.

In the case of dynamic bifurcations, periodical change in the control parameter results in that the system delays in the vicinity of the former bifurcation points under both forward and backward sweeps. The delay phenomenon causes the emergence of a hysteretic loop, whose characteristics are very sensitive to the noise level.

The phenomenon of noise-dependent hysteresis takes place not only in period doubling bifurcations but in a pitchfork bifurcation as well.

Noise-dependent nonlinear effects at the bifurcation threshold are of practical interest for measuring noise in nonlinear systems:

– If the bifurcation parameter does not change, it is possible to use the method based on the phenomenon of prebifurcation noise amplification.

– In the case of changing bifurcation parameter, it is possible to use the method on the basis of noise-dependent hysteresis.

Such methods are promising for measuring very weak noise in various nonlinear systems.

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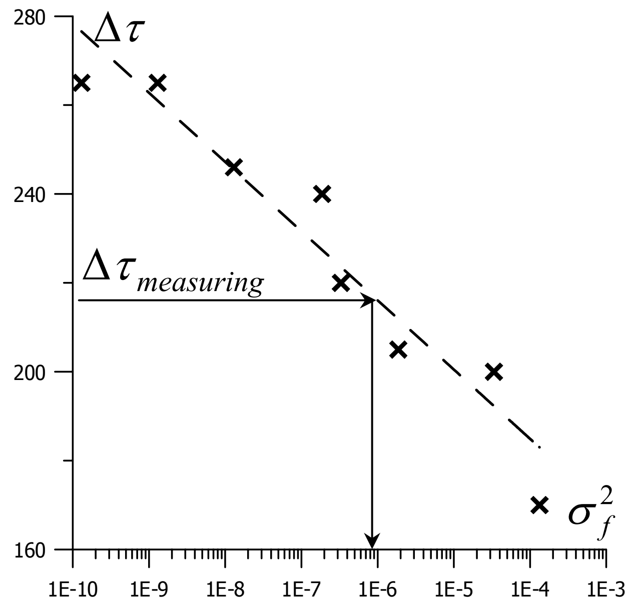

Fig. 4. The calibration line for the nonlinear oscillator: shows the dependence of the hysteresis loop size $\Delta\tau$ on noise level σ_f^2.

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