A mixture theory for geophysical fluids

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Abstract. A continuum theory is developed for a geophysical fluid consisting of two species. Balance laws are given for the individual components of the mixture, modeled as micropolar viscous fluids. The continua allow independent rotational degrees of freedom, so that the fluids can exhibit couple stresses and a non-symmetric stress tensor. The second law of thermodynamics is used to develop constitutive equations. Linear constitutive equations are constituted for a heat conducting mixture, each species possessing separate viscosities. Field equations are obtained and boundary and initial conditions are stated. This theory is relevant to an atmospheric mixture consisting of any two species from rain, snow and/or sand. Also, this is a continuum theory for oceanic mixtures, such as water and silt, or water and oil spills, etc.

1 Introduction

In the atmosphere, many complex physical phenomena can occur. For example, simultaneously, it may be raining and snowing. Tornadoes or hurricanes may lift sands, mixing with rain or snow. In these cases, we have a mixture of two different fluids. We consider snow, sand and rain as fluids. Although ice and sand grains are solids by themselves, we must treat each species as a liquid, since there is no cohesion among the grains as in a solid. The friction between rain, snow or sand, falling through the saturated air, attributes viscosities to the species. We do not consider the air as a third species, but rather as an agent that allows the fluid assumption for the species to be plausible. Alternatively, air, and either rain, snow or sand, may be considered to constitute a binary mixture of two fluids.

In the case of oceanography, the mixture of water and silt may compose the two species of the mixture. When oil spills occur in a region of ocean, oil and water may be considered as the members of the mixture, in that region.

Most studies in geophysical fluid dynamics are based on the Navier-Stokes (N-S) equations, with the inclusion of the Coriolis acceleration. The Navier-Stokes equations assume the atmosphere as a single fluid. Moreover, in the N-S equations, the effects of couple stresses and intrinsic rotations are ignored. These effects may become important in some instances, e.g. tornadoes, hurricanes, tumbling ice or sand.

Kirwan and Chang (1976) have demonstrated the influence of couple stress in the Ekman problem, in the case of a single micropolar fluid.

Literature is extensive on the mixture theories. Some of the early references are quoted by Bowen (1976), up to 1976, and more recent ones by Eringen (1994). However, these are concerned with nonpolar materials. Twiss and Eringen (1971, 1972) gave a theory of mixtures for micromorphic materials and investigated propagation of waves in two-constituent micropolar elastic solids. Several new elastic wave branches predicted by this theory are in accordance with the phonon dispersion experiments. These branches are not predicted by the mixture models based on the classical elastic solids. Consequently, the present theory, too, is expected to exhibit some new physical phenomena that are not in the domain of a mixture theory based on the N-S equations.

1. The raison d'être of the present paper stems from these observations.

In Sect. 2, we discuss briefly, the kinematics of a mixture. Balance laws are given in Sect. 3. Section 4 is concerned with the development of the constitutive equations and the thermodynamic restrictions required on the viscosity moduli. In Sect. 5, we obtain the field equations and state the boundary and initial conditions. Here, we also give the restrictions when fluids are incompressible. In Sect. 6, as a special case, we derive the field equations of a binary mixture of classical fluids.

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2 Kinematics

We consider a binary mixture of two continuous fluent bodies B^α, $\alpha = 1, 2$. In the case of atmospheric activities, B^1 may represent the rain, and B^2 the snow or sand. In an ocean atmosphere, they may represent water and silt. We consider that, continuous bodies B^α, occupy a region of the three-dimensional space. A material point, in the αth species, at the natural state, is denoted by X^α, in Cartesian coordinates. The motion of X^α, at time t, is represented by a mapping of X^α, to a spatial point x:

$$x = \chi^\alpha(X^\alpha, t), \tag{1}$$

which is assumed to possess the unique inverse

$$X^\alpha = \chi^{-1\alpha}(x, t). \tag{2}$$

The velocity and the acceleration of X^α, at time t, are given by

$$v^\alpha = x'^\alpha = \frac{\partial \chi^\alpha(X^\alpha, t)}{\partial t},$$
$$a^\alpha = x''^\alpha = \frac{\partial^2 \chi^\alpha(X^\alpha, t)}{\partial t^2}. \tag{3}$$

As usual, the material time derivative of a function $\psi(x, t)$ is denoted by

$$\dot{\psi} = \frac{\partial \psi}{\partial t}|_X = \frac{\partial \psi(x, t)}{\partial t} + \psi_{,k} \dot{x}_k(x, t), \tag{4}$$

where and henceforth, we employ the summation convention on repeated Latin indices, and an index following a comma for a partial derivative with respect to x_k, e.g.

$$\psi_{,k} \dot{x}_k = \frac{\partial \psi}{\partial x_1} \dot{x}_1 + \frac{\partial \psi}{\partial x_2} \dot{x}_2 + \frac{\partial \psi}{\partial x_3} \dot{x}_3.$$

The material derivative of a function $\psi[\chi^\alpha(X^\alpha, t), t]$, following the motion of the constituent B^α, is defined by

$$\psi'^\alpha = \frac{\partial \psi[\chi^\alpha(X^\alpha, t), t]}{\partial t}|_{X^\alpha} = \frac{\partial \psi(x, t)}{\partial t} + \psi_{,k} x_k'^\alpha. \tag{5}$$

From Eq. (4) and Eq. (5), it follows that

$$\psi'^\alpha - \dot{\psi} = (\nabla \psi) \cdot (\dot{x}'^\alpha - \dot{x}^\alpha). \tag{6}$$

The mass density of the constituent B^α is denoted by $\rho^\alpha(x, t)$. The density $\rho(x, t)$ and the velocity $v(x, t)$ of the mixture are given by

$$\rho(x, t) = \sum_\alpha \rho^\alpha(x, t), \quad \dot{x} = v = \frac{1}{\rho} \sum_\alpha \rho^\alpha v^\alpha. \tag{7}$$

The barycentric velocity is defined by

$$\bar{v}^\alpha = v^\alpha - v \tag{8}$$

The mass concentration c^α of the constituent B^α, at (x, t) is

$$c^\alpha(x, t) = \rho^\alpha / \rho. \tag{9}$$

From Eq. 7 and Eq. 9, it follows that

$$\sum_\alpha c^\alpha = 1. \tag{10}$$

Since rain drops, snow and sand undergo translations and rotations, the mixture continua must possess rotational degrees of freedom. This suggests that we consider each species as a micropolar continuum. In a micropolar continuum, a material point is endowed with a director, so that rotational degrees of freedom arise from the rotation of the director. For micropolar fluids, this is represented by gyration vector v_k. Micropolar fluid continua possess two deformation-rate tensors (c.f. Eringen, 2001):

$$a_{kl} = v_{l,k} + \epsilon_{lkm} v_m, \qquad b_{kl} = v_{k,l}. \tag{11}$$

Consequently, each species of the mixture is endowed with two deformation tensors

$$a_{kl}^\alpha = v_{l,k}^\alpha + \epsilon_{lkm} v_m^\alpha, \qquad b_{kl}^\alpha = v_{k,l}^\alpha. \tag{12}$$

3 Balance laws

We assume that no chemical reactions take place between the two constituents of the mixture.

The balance laws for each constituent B^α, consist of: conservation of mass, balance of momentum, balance of moment of momentum, and conservation of energy.

Conservation of Mass

$$\frac{\partial \rho^\alpha}{\partial t} + \nabla \cdot (\rho^\alpha v^\alpha) = 0 \quad \text{or} \quad \rho^\alpha \mid \det F^\alpha \mid = \rho_0^\alpha, \tag{13}$$

where ρ_0^α is the mass density in the natural state, and F^α is the deformation gradient, defined by

$$F^\alpha = \nabla \chi^\alpha(X^\alpha, t) \quad \text{or} \quad F_{kK}^\alpha = \frac{\partial x_k}{\partial X_K^\alpha}. \tag{14}$$

Summing Eq. (13) over α, we obtain the equation of conservation of mass for the mixture:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0. \tag{15}$$

Another useful form of Eq. (13) follows upon using Eq. (7) and Eq. (8):

$$\rho \dot{c}^\alpha + \nabla \cdot (\rho^\alpha \bar{v}^\alpha) = 0. \tag{16}$$

Balance of Momentum

$$\rho^\alpha x''^\alpha = \nabla \cdot t^\alpha + \rho^\alpha f^\alpha + \hat{p}^\alpha, \tag{17}$$

$$\sum_\alpha \hat{p}^\alpha = 0, \tag{18}$$

where, t_{kl}^α, f_k^α and $\hat{p}_k^\alpha$ are, respectively, the stress tensor, the body force density, and the force exerted on the αth constituent from the other constituent.

Balance of Moment of Momentum

$$\rho^\alpha j^\alpha v_l'^\alpha = m_{kl,k}^\alpha + \epsilon_{lmn} t_{mn}^\alpha + \rho^\alpha l_l^\alpha + \hat{m}_l^\alpha, \tag{19}$$

$$\sum_{\alpha} \hat{m}_l^{\alpha} = 0, \tag{20}$$

where, $\rho^{\alpha} j^{\alpha}$, v_l^{α}, m_{kl}^{α}, $\rho^{\alpha} l_l^{\alpha}$ and $\hat{m}_l$ are, respectively, microinertia density, microrotation-rate (gyration), body couple density, and the couple exerted on αth constituent from the other constituent. ϵ_{lmn} is the alternating tensor. Note that, when j^{α}, m_{kl}^{α} and l_l^{α} vanish, Eq. (19) reduces to the classical form

$$\epsilon_{lmn} t_{mn}^{\alpha} + \hat{m}_l^{\alpha} = 0. \tag{21}$$

If we sum this over α, we arrive at the symmetry of the mixture stress tensor $t_{kl} = t_{lk}$. Thus, Eq. (19) expresses the balance of spin inertia.

Conservation of Energy

$$\rho^{\alpha} \acute{\epsilon}^{\alpha} = t_{kl}^{\alpha} a_{kl}^{\alpha} + m_{kl}^{\alpha} b_{lk}^{\alpha} + q_{k,k}^{\alpha} + \rho^{\alpha} h^{\alpha} + \hat{\epsilon}^{\alpha}, \tag{22}$$

Subject to

$$\sum_{\alpha} (\hat{\epsilon}^{\alpha} + \hat{\boldsymbol{p}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha} + \hat{\boldsymbol{m}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha}) = 0, \tag{23}$$

where

$$\bar{\boldsymbol{v}}^{\alpha} = \boldsymbol{v}^{\alpha} - \boldsymbol{v}, \quad \bar{\boldsymbol{v}}^{\alpha} = \boldsymbol{v}^{\alpha} - \boldsymbol{v}. \tag{24}$$

Here, ϵ^{α}, $\boldsymbol{q}^{\alpha}$ and h^{α} are, respectively, the internal energy density, the heat vector, and the energy source of αth species. $\hat{\epsilon}^{\alpha}$ denotes the transfer of energy to α th species from the other species. For the derivation of these equations, we refer the reader to Twist and Eringen (1994), Eringen (2001), Eringen and Ingram (1965) and Bowen (1976).

We consider that both species have the same temperature. Certainly this is not true for the rain and ice mixture, except perhaps at the melting point. But, for a mixture consisting of rain and sand, it can be valid. In this case, we need only a single energy equation, namely that of the mixture. To this end, we sum Eq. (22) over α, use Eq. (23) and

$$\boldsymbol{q} = \sum_{\alpha} (\boldsymbol{q}^{\alpha} - \rho^{\alpha} \epsilon^{\alpha} \bar{\boldsymbol{v}}^{\alpha}), \qquad \rho h = \sum_{\alpha} \rho^{\alpha} h^{\alpha}. \tag{25}$$

Using the expression

$$\sum_{\alpha} \rho^{\alpha} \acute{\psi}^{\alpha} = \rho \dot{\psi} + \sum_{\alpha} \boldsymbol{\nabla} \cdot (\rho^{\alpha} \psi^{\alpha} \bar{\boldsymbol{v}}^{\alpha}),$$

$$\rho \psi = \sum_{\alpha} \rho^{\alpha} \psi^{\alpha}, \tag{26}$$

we obtain

$$\rho \dot{\epsilon} = \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho h$$
$$+ \sum_{\alpha} (t_{kl}^{\alpha} a_{kl}^{\alpha} + m_{kl}^{\alpha} b_{lk}^{\alpha} - \hat{\boldsymbol{p}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha} - \hat{\boldsymbol{m}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha}). \tag{27}$$

Using Eq. (18) and Eq. (20), we have

$$\sum_{\alpha=1}^{2} \hat{\boldsymbol{p}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha} = \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12}, \quad \sum_{\alpha=1}^{2} \hat{\boldsymbol{m}}^{\alpha} \cdot \bar{\boldsymbol{v}}^{\alpha} = \hat{\boldsymbol{m}}^1 \cdot \bar{\boldsymbol{v}}^{12}, \tag{28}$$

where

$$\bar{\boldsymbol{v}}^{12} = \boldsymbol{v}^1 - \boldsymbol{v}^2, \quad \bar{\boldsymbol{v}}^{12} = \boldsymbol{v}^1 - \boldsymbol{v}^2. \tag{29}$$

With this, Eq. (27) becomes

$$\rho \dot{\epsilon} = \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho h$$
$$+ \sum_{\alpha} (t_{kl}^{\alpha} a_{kl}^{\alpha} + m_{kl}^{\alpha} b_{lk}^{\alpha}) - \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12} - \hat{\boldsymbol{m}}^1 \cdot \bar{\boldsymbol{v}}^{12}. \tag{30}$$

We introduce Helmholtz's free energy ψ by

$$\epsilon = \psi + \theta \eta, \tag{31}$$

where, $\theta > 0$ (inf $\theta = 0$) is the absolute temperature, and η is the entropy density. With this, the energy balance law Eq. (30) takes the form

$$\rho \dot{\psi} + \rho \theta \dot{\eta} + \rho \dot{\theta} \eta - \boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho h - \sum_{\alpha} (t_{kl}^{\alpha} a_{kl}^{\alpha} + m_{kl}^{\alpha} b_{lk}^{\alpha})$$
$$+ \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12} + \hat{\boldsymbol{m}}^1 \cdot \bar{\boldsymbol{v}}^{12} = 0. \tag{32}$$

To the balance laws, we adjoin the second law of thermodynamics,

$$\rho \dot{\eta} - \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{q}}{\theta}\right) - \frac{\rho h}{\theta} \geq 0. \tag{33}$$

By means of Eq. (32), we eliminate $\rho h / \theta$ from Eq. (33), to obtain

$$-\rho (\dot{\psi} + \eta \dot{\theta}) + \sum_{\alpha} (t_{kl}^{\alpha} a_{kl}^{\alpha} + m_{kl}^{\alpha} b_{lk}^{\alpha}) - \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12}$$
$$- \hat{\boldsymbol{m}}^1 \cdot \bar{\boldsymbol{v}}^{12} + \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} \theta}{\theta} \geq 0. \tag{34}$$

This is the generalized Clausius-Duhem inequality. It is fundamental to the development of the constitutive equations.

4 Constitutive equations

According to the causality hypothesis, Eringen (1980), the independent variables are

$$Y = (\theta, \rho^{\alpha}, a_{kl}^{\alpha}, b_{kl}^{\alpha}, \bar{\boldsymbol{v}}^{12}, \bar{\boldsymbol{v}}^{12}, \theta_{,k}), \qquad \alpha = 1, 2 \tag{35}$$

where,

$$\bar{\boldsymbol{v}}^{12} = \boldsymbol{v}^1 - \boldsymbol{v}^2, \quad \bar{\boldsymbol{v}}^{12} = \boldsymbol{v}^1 - \boldsymbol{v}^2. \tag{36}$$

Dependent variables

$$Z = (\psi, \eta, t_{kl}^{\alpha}, m_{kl}^{\alpha}, -\hat{\boldsymbol{p}}^1, -\hat{\boldsymbol{m}}^1, \boldsymbol{q}) \tag{37}$$

are considered to be functions of Y.

We observe that, all independent variables are frame-independent, except for $\bar{\boldsymbol{v}}^{12}$ and $\bar{\boldsymbol{v}}^{12}$. Generally, these latter quantities are not admissible to form a constitutive equation in nonlinear continuum mechanics. However, it appears in classical Darcy's law, and is well accepted in porous media theories. The admissibility of this term, as a linear approximation in continuum mechanics, was noted by Wilmanski (2003). The same argument readily applies to the variable $\bar{\boldsymbol{v}}^{12}$.

Static variables are θ and ρ^α, so that

$$\psi = \psi(\theta, \rho^1, \rho^2). \tag{38}$$

We calculate

$$\dot\psi = \frac{\partial \psi}{\partial \theta}\dot\theta - \sum_\alpha \frac{\partial \psi}{\partial \rho^\alpha}(\rho^\alpha v^\alpha_{k,k}), \tag{39}$$

where, we used Eq. (13) to replace $\dot\rho^\alpha$. Substituting Eq. (39) into Eq. (34), we have

$$-\rho\left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot\theta$$
$$+ \sum_\alpha \left[\left(t^\alpha_{kl} + \rho\rho^\alpha \frac{\partial \psi}{\partial \rho^\alpha}\delta_{kl}\right)a^\alpha_{kl} + m^\alpha_{kl}b^\alpha_{lk}\right]$$
$$- \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12} - \hat{\boldsymbol{m}}^1 \cdot \tilde{\boldsymbol{v}}^{12} + \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla}\theta}{\theta} \geq 0. \tag{40}$$

This inequality cannot be maintained in one sign, for all independent variations of $\dot\theta$ unless

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{41}$$

and

$$\sum_\alpha (_D t^\alpha_{kl}a^\alpha_{kl} + m^\alpha_{kl}b^\alpha_{lk}) - \hat{\boldsymbol{p}}^1 \cdot \bar{\boldsymbol{v}}^{12} - \hat{\boldsymbol{m}}^1 \cdot \bar{\boldsymbol{v}}^{12}$$
$$+ \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla}\theta}{\theta} \geq 0, \tag{42}$$

where,

$$_D t^\alpha_{kl} = t^\alpha_{kl} + \pi^\alpha \delta_{kl}, \quad \pi^\alpha = \rho\rho^\alpha \frac{\partial \psi}{\partial \rho^\alpha}. \tag{43}$$

Here, π^α is the partial pressure in αth species.

We recognize the thermodynamic forces $\boldsymbol{Y}$ by

$$\boldsymbol{Y} = (a^\alpha_{kl}, b^\alpha_{lk}, \bar{v}^{12}_k, \tilde{v}^{12}_k, \theta_{,k}/\theta), \tag{44}$$

and thermodynamic fluxes $\boldsymbol{J}$, by

$$\boldsymbol{J} = (_D t^\alpha_{kl}, m^\alpha_{kl}, -\hat{p}^1_k, -\hat{m}^1_k, q_k). \tag{45}$$

The inequality Eq. (42) may be abbreviated by

$$\Gamma = \boldsymbol{Y} \cdot \boldsymbol{J} \geq 0. \tag{46}$$

The dissipation (dynamic) part of constitutive equations may be expressed in the form

$$\boldsymbol{J} = \mathfrak{J}(\boldsymbol{Y}; \theta). \tag{47}$$

Edelen (1993) gave a general solution of Eq. (46), in the form

$$\mathfrak{J}(\boldsymbol{Y}; \theta) = \boldsymbol{\nabla}_Y \Phi(\boldsymbol{Y}; \theta) + \boldsymbol{W}(\boldsymbol{Y}; \theta), \tag{48}$$

where, $\boldsymbol{W}$ is restricted by

$$\boldsymbol{W} \cdot \boldsymbol{Y} = 0, \tag{49}$$

and the dissipation potential, $\Phi(\boldsymbol{Y}; \theta)$ is given by

$$\Phi(\boldsymbol{Y}; \theta) = \int_0^1 \boldsymbol{Y} \cdot \mathfrak{J}(\tau\boldsymbol{Y}, \theta)\frac{d\tau}{\tau}. \tag{50}$$

Equation (49) indicates that $\boldsymbol{W}$ do not contribute to the dissipation of energy. In fact, for the linear constitutive equations, $\boldsymbol{W}$ vanish, and we obtain

$$_D\boldsymbol{t}^\alpha = \frac{\partial \Phi}{\partial \boldsymbol{a}^\alpha}, \quad \boldsymbol{m}^\alpha = \frac{\partial \Phi}{\partial \boldsymbol{b}^{\alpha T}}, \quad \hat{\boldsymbol{p}}^1 = -\hat{\boldsymbol{p}}^2 = -\frac{\partial \Phi}{\partial \bar{\boldsymbol{v}}^{12}},$$
$$\hat{\boldsymbol{m}}^1 = -\hat{\boldsymbol{m}}^2 = -\frac{\partial \Phi}{\partial \bar{\boldsymbol{v}}^{12}}, \quad \boldsymbol{q} = \frac{\partial \Phi}{\partial(\boldsymbol{\nabla}\theta/\theta)}, \tag{51}$$

where, a superscript T denotes the transpose, e.g. $b^{\alpha T}_{kl} = b^\alpha_{lk}$.

The dissipation potential Φ depends on the invariants of $\boldsymbol{Y}$ and θ. We distinguish pseudo-tensors $\boldsymbol{b}^\alpha$ and $\bar{\boldsymbol{v}}^{12}$ from the absolute tensors $\boldsymbol{a}^\alpha$, $\bar{\boldsymbol{v}}^{12}$ and $\boldsymbol{\nabla}\theta/\theta$. For first degree constitutive equations, we need the second degree invariants only. These can be read from a table, Eringen (1980 p. 577):

$$tr\boldsymbol{a}^\alpha, \quad tr(\boldsymbol{a}^\alpha)^2, \quad tr(\boldsymbol{a}^\alpha\boldsymbol{a}^{\alpha T}), \quad tr(\boldsymbol{a}^1\boldsymbol{a}^{2T}), \quad (tr\boldsymbol{b}^\alpha)^2,$$
$$tr\boldsymbol{b}^\alpha, \quad tr(\boldsymbol{b}^\alpha\boldsymbol{b}^{\alpha T}) \quad tr(\boldsymbol{b}^1\boldsymbol{b}^{2T}), \quad \bar{\boldsymbol{v}}^{12} \cdot \bar{\boldsymbol{v}}^{12},$$
$$\bar{\boldsymbol{v}}^{12} \cdot \bar{\boldsymbol{v}}^{12}, \quad \bar{\boldsymbol{v}}^{12} \cdot \boldsymbol{\nabla}\theta/\theta, \quad K\frac{\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\theta}{\theta^2}, \quad \alpha = 1, 2 \tag{52}$$

The dissipation function Φ may be expressed as

$$\Phi = \Phi_1 + \Phi_2, \tag{53}$$

where,

$$2\Phi_1 = \sum_\alpha \left[\lambda_\alpha(tr\boldsymbol{a}^\alpha)^2 + \mu_\alpha\, tr(\boldsymbol{a}^\alpha)^2 + (\mu_\alpha + \kappa_\alpha)\, tr(\boldsymbol{a}^\alpha\boldsymbol{a}^{\alpha T})\right] + 2\sigma tr(\boldsymbol{a}^1\boldsymbol{a}^{2T}) + \xi\bar{\boldsymbol{v}}^{12} \cdot \bar{\boldsymbol{v}}^{12} + 2\varsigma\bar{\boldsymbol{v}}^{12} \cdot \frac{\boldsymbol{\nabla}\theta}{\theta}$$
$$+ K\frac{\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\theta}{\theta^2},$$
$$2\Phi_2 = \sum_\alpha \left[\alpha_\alpha\,(tr\boldsymbol{b}^\alpha)^2 + \beta_\alpha\, tr(\boldsymbol{b}^\alpha)^2 + \gamma_\alpha\, tr(\boldsymbol{b}^\alpha\boldsymbol{b}^{\alpha T})\right] + 2\tau\, tr(\boldsymbol{b}^1\boldsymbol{b}^{2T}) + \varpi\bar{\boldsymbol{v}}^{12} \cdot \bar{\boldsymbol{v}}^{12}, \tag{54}$$

where, $\lambda_\alpha, \mu_\alpha, \kappa_\alpha, \alpha_\alpha, \beta_\alpha, \gamma_\alpha, \xi, \varpi, \varsigma, \sigma$, τ and K are material moduli.

Substituting Eq. (54) into Eq. (51), we obtain the constitutive equations

$$
\begin{aligned}
_D t^\alpha &= \lambda_\alpha \, tr \boldsymbol{a}^\alpha \mathbf{1} + \mu_\alpha \boldsymbol{a}^{\alpha T} + (\mu_\alpha + \kappa_\alpha) \boldsymbol{a}^\alpha + \sigma(\boldsymbol{a}^1 \delta_{\alpha 2} + \boldsymbol{a}^2 \delta_{\alpha 1}), \\
\boldsymbol{m}^\alpha &= \alpha_\alpha \, tr \boldsymbol{b}^\alpha \mathbf{1} + \boldsymbol{\beta}_\alpha \boldsymbol{b}^\alpha + \gamma_\alpha \boldsymbol{b}^{\alpha T} + \tau(\boldsymbol{b}^{1T} \delta_{\alpha 1} + \boldsymbol{b}^{2T} \delta_{\alpha 1}), \qquad \alpha = 1, 2 \\
-\hat{\boldsymbol{p}}^1 &= \hat{\boldsymbol{p}}^2 = \xi \bar{\boldsymbol{v}}^{12} + \varsigma \nabla \theta / \theta, \\
-\hat{\boldsymbol{m}}^1 &= \hat{\boldsymbol{m}}^2 = \varpi \bar{\boldsymbol{v}}^{12}, \\
\boldsymbol{q} &= \varsigma \bar{\boldsymbol{v}}^{12} + K \nabla \theta / \theta.
\end{aligned}
\tag{55}
$$

The dissipation potentials Φ_1 and Φ_2 must be nonnegative for all independent variations of the independent variable set. For $\boldsymbol{a}^\alpha = \boldsymbol{0}$, $2\Phi_1$ reduces to

$$
\xi \bar{\boldsymbol{v}}^{12} \cdot \bar{\boldsymbol{v}}^{12} + 2\varsigma \bar{\boldsymbol{v}}^{12} \cdot \frac{\nabla \theta}{\theta} + K \frac{\nabla \theta \cdot \nabla \theta}{\theta^2} \geq 0.
$$

This will be nonnegative, if and only if

$$
\xi \geq 0, \qquad \xi K - \varsigma^2 \geq 0. \tag{56}
$$

From Eq. (54) (with $\boldsymbol{b}^\alpha = \boldsymbol{0}$), $\Phi \geq 0$ leads to

$$
\varpi \geq 0. \tag{57}
$$

For $\bar{\boldsymbol{v}}^{12} = \bar{\boldsymbol{v}}^{12} = \boldsymbol{0}$, Φ_1 and Φ_2 acquire similar quadratic forms. Thus, we need to study only one of them. We express Φ_2, in terms of symmetric and antisymmetric tensors

$$
\boldsymbol{c}^\alpha = \frac{\boldsymbol{b}^\alpha + \boldsymbol{b}^{\alpha T}}{2}, \qquad \boldsymbol{d}^\alpha = \frac{\boldsymbol{b}^\alpha - \boldsymbol{b}^{\alpha T}}{2}. \tag{58}
$$

Then, we have

$$
\boldsymbol{b}^\alpha = \boldsymbol{c}^\alpha + \boldsymbol{d}^\alpha, \qquad \boldsymbol{b}^{\alpha T} = \boldsymbol{c}^\alpha - \boldsymbol{d}^\alpha. \tag{59}
$$

With this, $2\Phi_2$ may be expressed in terms of the principal values $\boldsymbol{c}^\alpha$, i.e.

$$
\Phi_2 = \Phi_{21} + \Phi_{22}, \tag{60}
$$

where,

$$
\begin{aligned}
2\Phi_{21} &= A_{ij} c_j, \\
\Phi_{22} &= (\gamma_1 - \beta_1)(d_1^2 + d_2^2 + d_3^2) + (\gamma_2 - \beta_2)(d_4^2 + d_5^2 + d_6^2) + 4\tau(d_1 d_4 + d_2 d_5 + d_3 d_6),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&A_1 = A_{11} = A_{22} = A_{33} = \alpha_1 + \beta_1 + \gamma_1, \quad A_2 = A_{44} + A_{55} + A_{66} = \alpha_2 + \beta_2 + \gamma_2, \\
&A_{12} = A_{23} = A_{31} = \alpha_1, \quad A_{14} = A_{25} = A_{36} = \tau, \quad A_{45} = A_{36} = A_{64} = \alpha_2, \\
&c_{11}^1 = c_1, \quad c_{22}^1 = c_2, \quad c_{33}^1 = c_3, \quad c_{11}^2 = c_4, \quad c_{22}^2 = c_5, \quad c_{33}^2 = c_6 \\
&d_{12}^1 = d_1, \quad d_{23}^1 = d_2, \quad d_{31}^1 = d_3, \quad d_{12}^2 = d_4, \quad d_{23}^2 = d_5, \quad d_{31}^2 = d_6.
\end{aligned}
\tag{62}
$$

From $\Phi_{22} \geq 0$, expressed by Eq. (61), it follows that

$$
\gamma_1 - \beta_1 \geq 0, \qquad (\gamma_1 - \beta_1)(\gamma_2 - \beta_2) - 4\tau^2 \geq 0. \tag{63}
$$

Φ_{21} will be nonnegative, when the eigenvalues are nonnegative. Alternatively, all sub-determinants based on the diagonals of the following matrix are nonnegative:

$$
\begin{array}{cccccc}
A_1 & \alpha_1 & \alpha_1 & \tau & 0 & 0 \\
\alpha_1 & A_1 & \alpha_1 & 0 & \tau & 0 \\
\alpha_1 & \alpha_1 & A_1 & 0 & 0 & \tau \\
\tau & 0 & 0 & A_2 & \alpha_2 & \alpha_2 \\
0 & \tau & 0 & \alpha_2 & A_2 & \alpha_2 \\
0 & 0 & \tau & \alpha_2 & \alpha_2 & A_2
\end{array}
\tag{64}
$$

Hence, we obtain

$$
\begin{aligned}
&\alpha_1 + \beta_1 + \gamma_1 \geq 0, \quad \beta_1 + \gamma_1 \geq 0, \quad 2\alpha_1 + \beta_1 + \gamma_1 \geq 0, \\
&(3\alpha_1 + \beta_1 + \gamma_1)(3\alpha_2 + \beta_2 + \gamma_2) - \tau^2 \geq 0, \\
&(\beta_1 + \gamma_1)(3\alpha_1 + \beta_1 + \gamma_1)(\alpha_2 + \beta_2 + \gamma_2) \\
&\qquad - (2\alpha_1 + \beta_1 + \gamma_1)\tau^2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
&(\beta_1 + \gamma_1)^2(3\alpha_1 + \beta_1 + \gamma_1)(\beta_2 + \gamma_2)(2\alpha_2 + \beta_2 + \gamma_2) \\
&\qquad - 2(\beta_1 + \gamma_1)[(\beta_1 + \gamma_1)(\alpha_2 + \beta_2 + \gamma_2) \\
&\qquad + \alpha_1(3\alpha_2 + 2\beta_2 + 2\gamma_2)]\tau^2 \\
&\qquad + (\alpha_1 + \beta_1 + \gamma_1)\tau^4 \geq 0.
\end{aligned}
\tag{65}
$$

In the case of $\Phi_1 \geq 0$, in Eq. (65), we replace α_α by λ_α, β_α by μ_α, γ_α by $\mu_\alpha + \kappa_\alpha$, and τ by σ.

With the inequalities Eqs. (56), (57), (63), (65), and these replacements, we will have $\Phi \geq 0$.

For simplicity and practical reasons, often two-dimensional problems are treated. In this case, Eqs. (56), (57) and (63) are not changed. But the matrix (64) is reduced to

$$
\begin{array}{cccc}
A_1 & \alpha_1 & \tau & 0 \\
\alpha_1 & A_1 & 0 & \tau \\
\tau & 0 & A_2 & \alpha_2 \\
0 & \tau & \alpha_2 & A_2
\end{array}
\tag{66}
$$

In this case, we obtain the inequalities

$$\alpha_1 + \beta_1 + \gamma_1 \geq 0, \quad \beta_1 + \gamma_1 \geq 0,$$
$$2\alpha_1 + \beta_1 + \gamma_1 \geq 0, \quad (\beta_1 + \gamma_1)(\beta_2 + \gamma_2) - \tau^2 \geq 0,$$
$$2\alpha_1 + \beta_1 + \gamma_1)(2\alpha_2 + \beta_2 + \gamma_2) - \tau^2 \geq 0. \tag{67}$$

5 Field equations

Substituting the constitutive Eqs. (41), (43) and (55) into balance laws (17), (19) and (32), and assuming that all material moduli are constants, we obtain the field equations

$$\frac{\partial \rho^1}{\partial t} + \nabla \cdot (\rho^1 \boldsymbol{v}^1) = 0, \tag{68}$$

$$\frac{\partial \rho^2}{\partial t} + \nabla \cdot (\rho^2 \boldsymbol{v}^2) = 0, \tag{69}$$

$$-\nabla \pi^1 + (\lambda_1 + \mu_1)\nabla\nabla \cdot \boldsymbol{v}^1 + (\mu_1 + \kappa_1)\nabla^2 \boldsymbol{v}^1 + \kappa_1 \nabla \times \boldsymbol{v}^1$$
$$+ \sigma(\nabla^2 \boldsymbol{v}^2 + \nabla \times \boldsymbol{v}^2) - \xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) - \varsigma \frac{\nabla\theta}{\theta} + \rho^1(\boldsymbol{f}^1 - \dot{\boldsymbol{v}}^1) = \boldsymbol{0}, \tag{70}$$

$$-\nabla \pi^2 + (\lambda_2 + \mu_2)\nabla\nabla \cdot \boldsymbol{v}^2 + (\mu_2 + \kappa_2)\nabla^2 \boldsymbol{v}^2 + \kappa_2 \nabla \times \boldsymbol{v}^2$$
$$+ \sigma(\nabla^2 \boldsymbol{v}^1 + \nabla \times \boldsymbol{v}^1) + \xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \varsigma \frac{\nabla\theta}{\theta} + \rho^2(\boldsymbol{f}^2 - \dot{\boldsymbol{v}}^2) = \boldsymbol{0}, \tag{71}$$

$$(\alpha_1 + \beta_1)\nabla\nabla \cdot \boldsymbol{v}^1 + \gamma_1 \nabla^2 \boldsymbol{v}^1 + \tau\nabla\nabla \cdot \boldsymbol{v}^2 + \kappa_1(\nabla \times \boldsymbol{v}^1 - 2\boldsymbol{v}^1)$$
$$+ \sigma(\nabla \times \boldsymbol{v}^2 - 2\boldsymbol{v}^2) - \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \rho^1(\boldsymbol{l}^1 - j^1\dot{\boldsymbol{v}}^1) = \boldsymbol{0} \tag{72}$$

$$(\alpha_2 + \beta_2)\nabla\nabla \cdot \boldsymbol{v}^2 + \gamma_2 \nabla^2 \boldsymbol{v}^2 + \tau\nabla\nabla \cdot \boldsymbol{v}^1 + \kappa_2(\nabla \times \boldsymbol{v}^2 - 2\boldsymbol{v}^2)$$
$$+ \sigma(\nabla \times \boldsymbol{v}^1 - 2\boldsymbol{v}^1) + \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \rho^2(\boldsymbol{l}^2 - j^1\dot{\boldsymbol{v}}^2) = \boldsymbol{0} \tag{73}$$

$$\rho\theta\frac{\partial^2\psi}{\partial\theta^2}\dot\theta - (\rho)^2\theta\frac{\partial^2\psi}{\partial\theta\partial\rho}\nabla \cdot \boldsymbol{v} + \sum_\alpha (_D t^\alpha_{kl}a^\alpha_{kl} + m^\alpha_{kl}b^\alpha_{lk}) + \left[\xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \varsigma\frac{\nabla\theta}{\theta}\right] \cdot$$
$$(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2) + \nabla \cdot \left[\varsigma(\boldsymbol{v}^1 - \boldsymbol{v}^2) + K\frac{\nabla\theta}{\theta}\right] + \rho h = \boldsymbol{0}. \tag{74}$$

For small variations of temperature from a constant ambient temperature T_0, we can substitute

$$\theta = T_0 + T, \qquad |T| \ll T_0, \tag{75}$$

so that, we may replace $\nabla\theta/\theta$ by $\nabla T/T_0$, and take

$$-\rho\theta\frac{\partial^2\psi}{\partial\theta^2} \simeq -\rho_0 T_0 \left(\frac{\partial^2\psi}{\partial T^2}\right)_0 = c_v, \tag{76}$$

$$(\rho)^2\theta\frac{\partial^2\psi}{\partial\theta\partial\rho} \simeq \rho_0 T_0 \left(\frac{\partial^2\psi}{\partial T\partial\rho}\right)_0 = \delta. \tag{77}$$

In the case of atmospheric fluid dynamics, we need to take into account the Coriolis acceleration. This is accomplished by replacing $\dot{\boldsymbol{v}}^1, \dot{\boldsymbol{v}}^2, \dot{\boldsymbol{v}}^1$ and $\dot{\boldsymbol{v}}^2$ as follows

$$\dot{\boldsymbol{v}}^1 \rightarrow \dot{\boldsymbol{v}}^1 - 2\boldsymbol{\Omega} \times \boldsymbol{v}^1,$$
$$\dot{\boldsymbol{v}}^2 \rightarrow \dot{\boldsymbol{v}}^2 - 2\boldsymbol{\Omega} \times \boldsymbol{v}^2,$$
$$\dot{\boldsymbol{v}}^1 \rightarrow \dot{\boldsymbol{v}}^1 - \boldsymbol{\Omega} \times \boldsymbol{v}^1,$$
$$\dot{\boldsymbol{v}}^2 \rightarrow \dot{\boldsymbol{v}}^2 - \boldsymbol{\Omega} \times \boldsymbol{v}^2, \tag{78}$$

where, $\boldsymbol{\Omega}$ is the rotation vector of Earth.

Field Eqs. (68) to (74) are expressed in rectangular coordinates. We expressed them in terms of vector operators, which are valid in orthogonal curvilinear coordinates.

$$\frac{\partial \rho^1}{\partial t} + \nabla \cdot (\rho^1 \boldsymbol{v}^1) = 0, \tag{79}$$

$$\frac{\partial \rho^2}{\partial t} + \nabla \cdot (\rho^2 \boldsymbol{v}^2) = 0, \tag{80}$$

$$-\nabla \pi^1 + (\lambda_1 + 2\mu_1 + \kappa_1)\nabla\nabla \cdot \boldsymbol{v}^1 - (\mu_1 + \kappa_1)\nabla \times \nabla \times \boldsymbol{v}^1 + \kappa_1\nabla \times \boldsymbol{v}^1 + \sigma(\nabla\nabla \cdot \boldsymbol{v}^2$$
$$- \nabla \times \nabla \times \boldsymbol{v}^2 + \nabla \times \boldsymbol{v}^2) - \xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) - \frac{\varsigma}{T_0}\nabla T + \rho 1(\boldsymbol{f}^1 - \dot{\boldsymbol{v}}^1 + 2\boldsymbol{\Omega} \times \boldsymbol{v}^1) = \boldsymbol{0}, \tag{81}$$

$$-\nabla \pi^2 + (\lambda_2 + 2\mu_2 + \kappa_2)\nabla\nabla \cdot \boldsymbol{v}^2 - (\mu_2 + \kappa_2)\nabla \times \nabla \times \boldsymbol{v}^2 + \kappa_2\nabla \times \boldsymbol{v}^2 + \sigma(\nabla\nabla \cdot \boldsymbol{v}^1$$
$$- \nabla \times \nabla \times \boldsymbol{v}^1 + \nabla \times \boldsymbol{v}^1) + \xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \frac{\varsigma}{T_0}\nabla T + \rho^2(\boldsymbol{f}^2 - \dot{\boldsymbol{v}}^2 + 2\boldsymbol{\Omega} \times \boldsymbol{v}^2) = \boldsymbol{0}, \tag{82}$$

$$(\alpha_1 + \beta_1 + \gamma_1)\nabla\nabla \cdot \boldsymbol{v}^1 - \gamma_1\nabla \times \nabla \times \boldsymbol{v}^1 + \tau\nabla\nabla \cdot \boldsymbol{v}^2 + \kappa_1(\nabla \times \boldsymbol{v}^1 - 2\boldsymbol{v}^1)$$
$$+ \sigma(\nabla \times \boldsymbol{v}^2 - 2\boldsymbol{v}^2) - \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \rho^1(\boldsymbol{l}^1 - j^1\dot{\boldsymbol{v}}^1 + j^1\boldsymbol{\Omega} \times \boldsymbol{v}^1) = \boldsymbol{0}, \tag{83}$$

$$(\alpha_2 + \beta_2 + \gamma_2)\nabla\nabla \cdot \boldsymbol{v}^2 - \gamma_2\nabla \times \nabla \times \boldsymbol{v}^2 + \tau\nabla\nabla \cdot \boldsymbol{v}^1 + \kappa_2(\nabla \times \boldsymbol{v}^2 - 2\boldsymbol{v}^2)$$
$$+ \sigma(\nabla \times \boldsymbol{v}^1 - 2\boldsymbol{v}^1) + \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \rho^2(\boldsymbol{l}^2 - j^2\dot{\boldsymbol{v}}^2 + j^2\boldsymbol{\Omega} \times \boldsymbol{v}^2) = \boldsymbol{0}, \tag{84}$$

$$-c_v\dot{T} - \delta\nabla \cdot (\boldsymbol{v}^1 + \boldsymbol{v}^2) + \sum_\alpha ({}_D\boldsymbol{t}^\alpha \cdot \boldsymbol{a}^{\alpha T} + \boldsymbol{m}^\alpha \cdot \boldsymbol{b}^\alpha) + \left[\xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \varsigma\frac{\nabla T}{T_0}\right] \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)$$

$$+ \varpi(\boldsymbol{v}^1 - \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2) + \nabla \cdot \left[\varsigma(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \frac{K}{T_0}\nabla T\right] + \rho h = 0. \tag{85}$$

In Eq. (85), ${}_D\boldsymbol{t}^\alpha$, $\boldsymbol{a}^\alpha$, $\boldsymbol{m}^\alpha$, $\boldsymbol{b}^\alpha$, $\boldsymbol{v}^\alpha$ and $\boldsymbol{v}^\alpha$ denote the physical components of tensors.

Field Eqs. (79) to (85) constitute a system of fifteen partial differential equations to determine the fifteen unknown functions $\rho_1, \rho_2, \boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^1, \boldsymbol{v}^2$ and T, given the external loads $\boldsymbol{f}^1, \boldsymbol{f}^2, \boldsymbol{l}^1, \boldsymbol{l}^2$ and h. Note that partial pressures π^1, π^2 and c_v and δ, given by Eqs. (43), (76) and (77), are determined when the the free energy function ψ is specified. Thus, the system (79) to (85) is closed. Under appropriate boundary and initial conditions, the field equations may be solved to determine the unknown functions.

Boundary Conditions

Let $\bar{V}$ denote a regular region of the Euclidean space, occupied by the mixture, whose boundary is ∂V. The interior of $\bar{V}$ is denoted by V, and the exterior unit normal to ∂V is denoted by $\boldsymbol{n}$. Let S_i ($i=1, 2..., 6$) denote subsets of ∂V, such that

$$\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \partial V,$$
$$S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = 0. \tag{86}$$

A mixed set of boundary conditions on the surfaces, at time $T^+=[0, \infty)$, may be expressed as:

$$\boldsymbol{v}^\alpha = \boldsymbol{v}_0^\alpha \quad \text{on} \quad \bar{S}_1 \times T^+, \quad t_{kl}^\alpha n_k^\alpha = t_{0l}^\alpha \quad \text{on } S_2 \times T^+,$$
$$\boldsymbol{v}^\alpha = \boldsymbol{v}_0^\alpha \quad \text{on} \quad \bar{S}_3 \times T^+, \quad m_{kl}^\alpha n_k^\alpha = m_{0l}^\alpha \text{ on } S_4 \times T^+,$$
$$T = T_0 \quad \text{on} \quad \bar{S}_5 \times T^+, \quad q_k n_k^\alpha = q_0^\alpha \quad \text{on } S_6 \times T^+, \tag{87}$$

where, $\boldsymbol{v}_0^\alpha, \boldsymbol{v}_0^\alpha, t_{0l}^\alpha, m_{0l}^\alpha, T_0$ and q_0^α are prescribed. When any one of the surfaces is absent, the accompanying boundary conditions are extended to cover ∂V. For example, when $S_2=0$, $\bar{S}_1$, becomes ∂V, and when $S_4=0$, $\bar{S}_3$ becomes ∂V, so that

$$\boldsymbol{v}^\alpha = \boldsymbol{v}_0^\alpha, \quad \boldsymbol{v}^\alpha = \boldsymbol{v}_0^\alpha \quad \text{on} \quad \partial V. \tag{88}$$

These conditions imply *strict adherence* on the boundary.

Initial Conditions

The initial conditions usually consist of Cauchy data, expressed by

$$\rho(\boldsymbol{x}, 0) = \rho^0(\boldsymbol{x}), \quad \boldsymbol{v}^\alpha(\boldsymbol{x}, 0) = \boldsymbol{v}^{\alpha 0}(\boldsymbol{x}),$$
$$\boldsymbol{v}^\alpha(\boldsymbol{x}, 0) = \boldsymbol{v}^{\alpha 0}(\boldsymbol{x}), \quad T(\boldsymbol{x}, 0) = T^0(\boldsymbol{x}) \text{ in } V, \tag{89}$$

where, quantities carrying a superscript $(^0)$ are prescribed throughout the body, at time $t=0$.

Incompressible fluids

In incompressible fluids, $\rho^\alpha = \rho_0^\alpha = const.$ and we have

$$\nabla \cdot \boldsymbol{v}^\alpha = 0, \qquad \nabla \cdot \boldsymbol{v}^\alpha = 0. \tag{90}$$

The pressures π^α are now replaced by an unknown pressure $p^\alpha(\boldsymbol{x}, t)$ and the field equations are simplified by using (5.23).

6 Mixtures of two Newtonian fluids

When the constituent fluids can be approximated by Newtonian fluids, basic equations are greatly simplified. In this case, the field equations follow, by setting $\kappa_1 = \kappa_2 = \boldsymbol{v}^1 = \boldsymbol{v}^2 = 0$, and ignoring the remainder of the Eqs. (83) and (84), since these equations arise from the nonsymmetry of the stress tensor and the existence of the couple stress.

$$\frac{\partial \rho^1}{\partial t} + \nabla \cdot (\rho^1 \boldsymbol{v}^1) = 0, \tag{91}$$

$$\frac{\partial \rho^2}{\partial t} + \nabla \cdot (\rho^2 \boldsymbol{v}^2) = 0, \tag{92}$$

$$-\nabla \pi^1 + (\lambda_1 + 2\mu_1)\nabla\nabla \cdot \boldsymbol{v}^1 - \mu_1 \nabla \times \nabla \times \boldsymbol{v}^1 + \sigma(\nabla\nabla \cdot \boldsymbol{v}^2 - \nabla \times \nabla \times \boldsymbol{v}^2)$$
$$-\xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) - \frac{\varsigma}{T_0}\nabla T + \rho^1(\boldsymbol{f}^1 - \dot{\boldsymbol{v}}^1 + 2\boldsymbol{\Omega} \times \boldsymbol{v}^1) = \boldsymbol{0}, \tag{93}$$

$$-\nabla \pi^2 + (\lambda_2 + 2\mu_2)\nabla\nabla \cdot \boldsymbol{v}^2 - \mu_2 \nabla \times \nabla \times \boldsymbol{v}^2 + \sigma(\nabla\nabla \cdot \boldsymbol{v}^1 - \nabla \times \nabla \times \boldsymbol{v}^1)$$
$$+\xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \frac{\varsigma}{T_0}\nabla T + \rho^2(\boldsymbol{f}^2 - \dot{\boldsymbol{v}}^2 + 2\boldsymbol{\Omega} \times \boldsymbol{v}^2) = \boldsymbol{0}, \tag{94}$$

$$-c_v\dot{T} - \delta\nabla \cdot (\boldsymbol{v}^1 + \boldsymbol{v}^2) +_D \boldsymbol{t}^1 \cdot \boldsymbol{d}^{1T} +_D \boldsymbol{t}^2 \cdot \boldsymbol{d}^{2T} + \left[\xi(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \varsigma\frac{\nabla T}{T_0}\right] \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)$$
$$+ \nabla \cdot \left[\varsigma(\boldsymbol{v}^1 - \boldsymbol{v}^2) + \frac{K}{T_0}\nabla T\right] + \rho h = 0, \tag{95}$$

where,

$$\boldsymbol{t}^\alpha = -\pi^\alpha \boldsymbol{1} + \lambda_\alpha \nabla \cdot \boldsymbol{v}^\alpha + 2\mu_\alpha \boldsymbol{d}^\alpha, \tag{96}$$

$$d_{kl}^\alpha = \frac{1}{2}(v_{k,l}^\alpha + v_{l,k}^\alpha). \tag{97}$$

Boundary Conditions

The boundary conditions follow from Eq. (87), by dropping conditions on $\boldsymbol{v}^\alpha$, and m_{kl}^α.

$$\boldsymbol{v}^\alpha = \boldsymbol{v}_0^\alpha \quad \text{on} \quad \bar{S}_1 \times T^+, \quad t_{kl}^\alpha n_k^\alpha = t_{0l}^\alpha \quad \text{on} \quad S_2 \times T^+,$$
$$T = T_0 \quad \text{on} \quad \bar{S}_3 \times T^+, \quad q_k n_k^\alpha = q_0^\alpha \quad \text{on} \quad S_4 \times T^+, \tag{98}$$

where, $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \partial V$, $S_1 \cap S_2 = S_3 \cap S_4 = 0$.

Initial Conditions

$$\rho(\boldsymbol{x}, 0) = \rho^0(\boldsymbol{x}), \qquad \boldsymbol{v}^\alpha(\boldsymbol{x}, 0) = \boldsymbol{v}^{\alpha 0}(\boldsymbol{x}),$$
$$T(\boldsymbol{x}, 0) = T^0(\boldsymbol{x}) \quad \text{in} \quad V. \tag{99}$$

7 Conclusion

We have introduced a continuum theory of atmospheric fluids. Any binary combinations of rain, snow, sand and ice are considered a mixture of two micropolar fluids at one average temperature. The balance laws are given, and thermodynamically admissible constitutive equations are constructed for a mixture of two micropolar fluids. Thermodynamic restrictions on the material moduli are investigated. Field equations are obtained and boundary and initial conditions are stated. As a special case of the micropolar mixture, we obtained the field equations of a mixture of two Newtonian fluids.

The field equations can be used to predict atmospheric phenomena. These include the predictions of densities, velocities, intrinsic rotations, diffusions of species and the mean temperature of atmospheric mixtures. The theory is applicable to oceanic boundary-initial value problems as well.

From the field Eqs. (81) to (85), it is clear that strong couplings exist among the velocities and gyrations of species. Consequently, the gyrations will alter the velocities and vice versa. The constituent fluids display couple stress and a non-symmetric stress tensor.

As compared to the binary mixtures of Newtonian fluids (Eqs. 93 to 95), here, we encounter an additional physical phenomena, diffusion due to the difference of gyrations. This effect also guides the conduction of heat in the mixture.

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