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Shallow water cnoidal wave interactions

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Abstract. The nonlinear dynamics of cnoidal waves, within the context of the general N-cnoidal wave solutions of the periodic Korteweg-deVries (KdV) and Kadomtsev-Petvishvilli (KP) equations, are considered. These equations are important for describing the propagation of small-but-finite amplitude waves in shallow water; the solutions to KdV are unidirectional while those of KP are directionally spread. Herein solutions are constructed from the θ–function representation of their appropriate inverse scattering transform formulations. To this end a general theorem is employed in the construction process: *All solutions to the KdV and KP equations can be written as the linear superposition of cnoidal waves plus their nonlinear interactions.* The approach presented here is viewed as significant because it allows the exact construction of N degree-of-freedom cnoidal wave trains under rather general conditions.

1 Introduction

Recently measured laboratory-generated, shallow water, unidirectional wave trains have been analyzed using the inverse scattering transform (IST) for the Korteweg-deVries equation (Korteweg-deVries, 1895) in the so-called *hyperelliptic-function representation* (Osborne and Petti, 1993, 1994). A major result of this work was the identification of "coherent structures" in the data. The authors identified these structures as *cnoidal waves which appeared only once in a single period of the measured wave train*; the moduli of the cnoidal waves were substantially large, but significantly less than 1, $0.7 < m < 0.85$. The data investigated by them contained from 2 to 8 cnoidal waves in each of the 6 measured wave trains.

The appearance of *coherent, stable structures which are not solitons* came as a surprise to the investigators. An important issue addressed by them was whether or not structures of this type could be characterized with inverse scattering theory. Using the rather unwieldy mathematical formulation of the hyperelliptic function representation the structures were identified as interacting cnoidal waves. In this formulation, the cnoidal wave trains are constructed from a number of nonlinearly interacting hyperelliptic functions which are *phase locked* with each other.

The focus in the present paper is to address the presence of coherent structures not only in the KdV equation, but also in the KP equation, a two-dimensional generalization of KdV. To this end I consider N-component cnoidal wave trains, using an alternative, physically simpler set of basis functions for KdV and KP which is known as the *θ–function representation* (Its and Matveev, 1975; Date and Tanaka, 1976; Dubrovin et al, 1976; Flaschka and McLaughlin, 1976; McKean and Trubowitz, 1976). I discuss how one and two dimensional wave trains in this representation can be easily written explicitly as a *linear superposition of cnoidal waves plus terms which include their pair-wise nonlinear interactions* (Osborne, 1994). An important consequence of using the θ–functions is that the cnoidal waves (and therefore the coherent structures) are mathematically explicit objects; this is in stark contrast to the use of the hyperelliptic functions where phase locking must be invoked to obtain N-cnoidal wave solutions.

Since the discovery of the KdV equation (Korteweg and deVries, 1895) and the calculation of its travelling wave solution in terms of a particular Jacobian elliptic function (the cnoidal wave) an outstanding problem in mathematical physics has been to determine the *general periodic solutions of KdV in terms of an arbitrary number of cnoidal waves and their nonlinear interactions.* The theoretical issues regarding this problem are addressed elsewhere (Osborne, 1994). Here I focus on a concrete implementation of this idea to generate physically relevant examples of nonlinear cnoidal wave interactions in both one and two dimensions. The goal of the present paper is to specifically address the θ–function representation, analytically, physically and numerically, in order to allow practical application of this method to N degree-of-freedom solutions to KdV and KP.

The theoretical framework of this paper is based primarily on the periodic solution to the KdV equation in terms of the hyperelliptic-function and the θ–function representations as developed in several important papers (Its and Matveev, 1975; Dubrovin et al, 1976; Flaschka and McLaughlin, 1976; Date and Tanaka, 1976; MacKean and Trubowitz, 1976); for the KP equation the relevant references are included in Krichever and Novikov (1979). A significant series of papers are those of Boyd (Boyd, 1984a,b,c; 1990) who explored in detail the case for two cnoidal waves and their mutual interactions. A number of important monographs are available on the general theory of solitons (Karpman, 1975; Zakharov et al 1980; Ablowitz and Segur, 1981; Newell, 1985; Ablowitz and Clarkson, 1991).

As is well known the KdV equation describes small-but-finite-amplitude, long-wave motion:

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0 \qquad (1)$$

KdV governs the space-time evolution of the nonlinear field, $\eta(x,t)$, here assumed to be spatially periodic, $\eta(x,t) = \eta(x+L,t)$, for $0 \le x \le L$, L the spatial period. The coefficients of (1) are constant parameters and have values that depend upon the physical application; these include surface water waves, internal waves, Rossby waves, plasma waves, equatorial motions and bores (Long, 1964; Peregrine, 1964; Benny, 1966; Zabusky and Galvin, 1971; Maxworthy and Redekopp, 1976; Hammack and Segur, 1974, 1978). For surface water waves $c_o = \sqrt{gh}$, $\alpha = 3c_o/2h$ and $\beta = c_o h^2/6$, where h is the water depth and g is the acceleration of gravity; the associated linear dispersion relation is given by: $\omega = c_o k - \beta k^3$. While the results given herein are discussed with regard to water waves, the conclusions are completely general and can be applied to any problem for which KdV is valid. Herein the wave amplitude $u(x,t) = \lambda\eta(x,t)$ for $\lambda = \alpha/6\beta$ is used.

For shallow water waves the two dimensional generalization of the KdV equation is given by the KP equation:

$$\frac{\partial}{\partial x}\left[\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx}\right] + \frac{c_o}{2}\eta_{yy} = 0 \qquad (2)$$

In the literature this equation is called KPII to distinguish it from the KPI equation, the latter of which is valid for water depths less than about a centimeter, where surface tension dominates. The remainder of this paper addresses only KPII, which generally describes two dimensional shallow water wave motion in the same depth regime as the KdV equation. The general solution to (2) depends on both x and y: $\eta(x,y,t)$. The linear dispersion relation for (2) (set $\alpha = 0$) is given by

$$\omega = c_o k_x - \beta k_x^3 + \frac{c_o}{2}\frac{k_y^2}{k_x} \qquad (3)$$

The two dimensional nature of the waves implies that they be "directionally spread" about the dominant direction of wave propagation which is taken to be the x axis. It is in this sense that KP describes waves which have a spectrum consisting not only of wave number (or frequency) but also direction. The solutions to KP are here assumed to be periodic in both x and y: $\eta(x,y,t) = \eta(x+X, y+Y, t)$, where X and Y are the respective spatial periods.

The remainder of this paper is organized as follows. Section 2 briefly discusses the single degree of freedom travelling wave solution of the KdV and KP equations; the Fourier transform for *linearized* KdV and KP are also mentioned with emphasis on the fundamental role that sinusoidal basis functions play in the solution of *linear* partial differential equations with well-defined dispersion relations. This Section asks the fundamental question: Since the KdV and KP equations are fully nonlinear, can numerical implementation of their general periodic solutions be represented in terms of N-cnoidal wave trains, just as the linearized equations can be represented in terms of a linear superposition of N sinusoidal waves? Herein I focus on the concrete implementation of a particular resolution of this problem. Section 3 discusses the inverse scattering transform in the *hyperelliptic-function representation* for the KdV equation; the fact that the hyperelliptic functions act as a set of *nonlinear (Fourier-like) basis functions* for KdV is elaborated on. Section 4 discusses the inverse scattering transform in the *θ–function representation* for both KdV and KP; this Section emphasizes the fundamental role that the θ–functions play as an *alternative* set of nonlinear (Fourier-like) basis functions for these equations, here described as "N-cnoidal wave interactions." Numerical examples of KdV and KP solutions are discussed in Section 5. The Summary and Conclusions are given in Section 6.

2 The cnoidal wave solution to the KdV and KP equations; Fourier series solutions to the linearized equations

Korteweg and deVries (1895), who discovered their equation 100 years ago, found a periodic travelling wave solution which is known as the *cnoidal wave*:

$$\eta(x,t) = \frac{4k^2}{\lambda}\sum_{n=1}^{\infty}\frac{n(-1)^n q^n}{1-q^{2n}}\cos[n(kx-\omega t)] = \qquad (4)$$

$$= 2\eta_o cn^2\{(K(m)/\pi)[kx - \omega t]; m\}$$

The modulus m of each elliptic function is given by

$$mK^2(m) = \frac{3\pi^2\eta_o}{2k^2h^3} = 4\pi^2 U; \qquad U = \frac{3\eta_o}{8k^2h^3} \tag{5}$$

where U is the Ursell number, k is the wave number and h is the water depth. The frequency is

$$\omega = c_o k\{1 + 2\eta_o/h - 2k^2h^2K^2(m)/3\pi^2\} \tag{6}$$

The series representation for the cnoidal wave, given in (4), is the *Stokes series* solution to the KdV equation (Whitham, 1974). When the modulus $m \to 0$ the cnoidal wave approaches a sine wave; when $m \to 1$ the cnoidal wave approaches a solitary wave. The travelling wave (4) thus offers a way to include nonlinearity and dispersion in a simple wave propagation model.

The cnoidal wave solution to the KP equation is given by

$$\eta(\mathbf{x},t) = \frac{4k^2}{\lambda}\sum_{n=1}^{\infty}\frac{n(-1)^n q^n}{1-q^{2n}}\cos[n(\mathbf{K}\cdot\mathbf{x} - \omega t)] = \tag{7}$$

$$= 2\eta_o cn^2\{(K(m)/\pi)[\mathbf{K}\cdot\mathbf{x} - \omega t]; m\}$$

Here

$$\mathbf{K} = (K_x, K_y); \qquad \mathbf{x} = (x, y) \tag{8}$$

where the frequency ω is defined by:

$$\omega = c_o|\mathbf{K}|\{1 + 2\eta_o/h - 2k^2h^2K^2(m)/3\pi^2\} \tag{9}$$

The cnoidal wave (7) for KP is identical to the cnoidal wave solution for the KdV equation except that the KP solution can propagate in an arbitrary direction defined by $\mathbf{K}$.

An important N degree-of-freedom solution to the *linearized* KdV equation (let $\alpha \to 0$ in (4))

$$\eta_t + c_o\eta_x + \beta\eta_{xxx} = 0 \tag{10}$$

is given by an ordinary Fourier series (see for example Osborne and Bergamasco (1985) and cited references):

$$\eta(x,t) = \sum_{j=1}^{N} c_j\cos(k_j x - \omega_j t + \phi_j) \tag{11}$$

where the commensurable wave numbers are given by

$$k_j = 2\pi j/L \tag{12}$$

and the associated frequencies have the cubic dispersion relation

$$\omega_j = c_0 k_j - \beta k_j^3 \tag{13}$$

The spatial Fourier transform of a wave train $\eta(x,t)$ consists of the set of Fourier amplitudes and phases $\{c_j, \phi_j\}$, for $1 \le j \le N$. The mean of (11) (and of all spatio-temporal solutions of linear and nonlinear wave motion discussed in the present paper) is assumed to be zero for the reasons discussed in Osborne and Bergamasco (1985). The form of (11), for uniformly distributed random phases, ϕ_j, has been used extensively for generating *linear random functions* (Osborne, 1982). This latter work recently motivated the study of *nonlinear random function* solutions to KdV using the periodic inverse scattering transform in the hyperelliptic-function representation (Osborne, 1993b).

Extending these ideas to two dimensions the linearized KP equation can be written:

$$\frac{\partial}{\partial x}[\eta_t + c_o\eta_x + \beta\eta_{xxx}] + \frac{c_o}{2}\eta_{yy} = 0 \tag{14}$$

This equation has the two dimensional Fourier series solution for MN degrees of freedom:

$$\eta(x,y,t) = $$

$$= \sum_{m=-M}^{M}\sum_{n=-N}^{N} C_{mn}\cos(\mathbf{k}_{mn}\cdot\mathbf{x} - \omega_{mn}t + \phi_{mn}) \tag{15}$$

Here $\mathbf{k}_{mn} = (k_{xm}, k_{yn})$ and $\omega_{mn} = \omega_{mn}(\mathbf{k}_{mn})$ (see (3) above).

The fact that periodic series solutions (11), (15) to the *linearized* KdV and KP equations (10), (14) can be easily represented *spectrally* as a linear superposition of sine waves raises a fundamental question: Can the general solutions to periodic KdV and KP be represented in terms of cnoidal waves and their nonlinear interactions (4), (7)? The following Sections address this issue and discuss a simple scenario for its practical resolution.

3 The hyperelliptic-function series solution of the KdV equation

The general spectral solution to the periodic KdV equation (1) may be written as a *linear superposition of nonlinearly interacting, nonlinear waves (hyperelliptic functions)*, $\mu_j(x,t)$ (Its and Matveev, 1975; Dubrovin et al, 1976; Flaschka and McLaughlin, 1976; Date and Tanaka, 1976; MacKean and Trubowitz, 1976):

$$\lambda\eta(x,t) = -E_1 + \sum_{j=1}^{N}[2\mu_j(x,t) - E_{2j} - E_{2j+1}] \tag{16}$$

where $\lambda = \alpha/6\beta$ and the E_i $(1 \le i \le 2N+1)$ are constant eigenvalues derived from Floquet theory for the time-independent Schroedinger equation (see theoretical and numerical discussions in Osborne (1993a)). Eq. (16) reduces to a linear Fourier series (11) in the limit of small amplitude motion (Osborne and Bergamasco, 1985). It is for this reason that (16) may be interpreted as a *nonlinear Fourier series*.

The spatial evolution of the μ_j is governed by the following system of coupled, nonlinear, ordinary differential equations (ODEs):

$$\frac{d\mu_j}{dx} = 2i\sigma_j R^{1/2}(\mu_j) / \prod_{\substack{k=1 \\ j\neq k}}^{N} (\mu_j - \mu_k) \qquad (17)$$

where $1 \le j \le N$ and

$$R(\mu_j) = \prod_{k=1}^{2N+1} (\mu_j - E_k) \qquad (18)$$

The numerical analysis of the hyperelliptic functions is rather complex and we refer the reader to a detailed exposition in Osborne (1993a). The time dynamics of the $\mu_j(x,t)$ are also discussed elsewhere (Osborne, 1994).

4 The θ–function solutions to KdV and KP

In addition to the hyperelliptic-function representation (16) the general solution to the KdV equation may also be written in terms of the *θ-function formulation* (Its and Matveev, 1975; Dubrovin et al, 1976; Flaschka and McLaughlin, 1976; Date and Tanaka, 1976; MacKean and Trubowitz, 1976)

$$\lambda\eta(x,t) = 2\frac{\partial^2}{\partial x^2} \ln \Theta_N(\eta_1, \eta_2, \ldots \eta_N) \qquad (19)$$

The θ–function is given by

$$\Theta_N(\eta_1, \eta_2, \ldots \eta_N) = \sum_{M_1,\ldots,M_N=-\infty}^{\infty} \exp\left[i\sum_{k=1}^{N} M_k\eta_k + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} M_j B_{jk} M_k \right] \qquad (20)$$

Here, as before, N is the number of degrees of freedom in a particular solution to the KdV equation. The θ–function phases are given by

$$\eta_k = k_k x - \omega_k t + \phi_k$$

where the wave numbers, k_k, the frequencies, ω_k, and the phases, ϕ_k, are found by the relations given in the Appendix; these relations generally depend upon algebraic geometric loop integrals whose numerical implementation is discussed elsewhere (Osborne, 1994). The period matrix **B** is constant and provides the amplitudes of the selected cnoidal waves (along the diagonal) and their mutual pair-wise nonlinear interactions (off-diagonal terms).

It is then straightforward to address the following theorem (Osborne, 1994):

Theorem I: The θ–function solution to the KdV equation (19), (20) can be written in the following form $(u(x,t) \equiv \lambda\eta(x,t))$:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\Theta_N(\vec{\eta}) = \underbrace{u_{cn}(\vec{\eta})}_{\substack{\text{Linear superposition} \\ \text{of cnoidal waves}}} + \underbrace{u_{\text{int}}(\vec{\eta})}_{\substack{\text{Nonlinear interactions} \\ \text{among the cnoidal waves}}} \qquad (21)$$

where in vector notation:

$$\Theta_N(\vec{\eta}) = \sum_{\vec{M}} e^{i\vec{M}\cdot\vec{\eta} + \frac{1}{2}\vec{M}^T\cdot\mathbf{B}\cdot\vec{M}} \qquad (22)$$

The phases of the θ–function are given by

$$\vec{\eta} = \vec{k}x - \vec{\omega}t + \vec{\phi} = [\eta_1, \eta_2, \ldots \eta_N] \qquad (23)$$

where the wave numbers, frequencies and constant phases have the form:

$$\vec{k} = [k_1, k_2, \ldots k_N];$$

$$\vec{\omega} = [\omega_1, \omega_2, \ldots \omega_N]; \qquad (24)$$

$$\vec{\phi} = [\phi_1, \phi_2, \ldots \phi_N]$$

Analytic expressions for the interaction terms $u_{int}(\vec{\eta})$ are given elsewhere (Osborne, 1994).

The θ–function solution to the KP equation has many similarities to that for the KdV equation. In fact (19), (20) still hold. However, the *phases* η_j of the θ–function (20) are found by the following expression

$$\eta_j = k_{xj}x + k_{yj}y - \omega_j t + \phi_j \qquad (25)$$

where the k_{xj}, k_{yj}, ω_j and ϕ_j are given by an algebraic geometric formulation (Osborne, 1994). We have the following

Theorem II: The θ–function solution to the KP equation can be written in the following form ($u(x,t) \equiv \lambda\eta(x,t)$):

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\Theta_N(\vec{\eta}) = \tag{26}$$

$$= \underbrace{u_{cn}(\vec{\eta})}_{\substack{\text{Linear superposition}\\ \text{of cnoidal waves}}} + \underbrace{u_{\text{int}}(\vec{\eta})}_{\substack{\text{Nonlinear interactions}\\ \text{among the cnoidal waves}}}$$

where:

$$\Theta_N(\vec{\eta}) = \sum_{\vec{M}} e^{i\vec{M}\cdot\vec{\eta}+\frac{1}{2}\vec{M}^T\cdot\mathbf{B}\cdot\vec{M}} \tag{27}$$

The phases of the θ–function are given by

$$\vec{\eta} = \vec{\mathbf{k}}_x x + \vec{\mathbf{k}}_y y - \vec{\omega}t + \vec{\phi} = [\eta_1, \eta_2, \ldots \eta_N] \tag{28}$$

where the wave numbers, frequencies and constant phases have the vector form:

$$\vec{\mathbf{k}}_x = [k_{x1}, k_{x2}, \ldots k_{xN}]$$
$$\vec{\mathbf{k}}_y = [k_{y_1}, k_{y_2}, \ldots k_{y_N}] \tag{29}$$

$$\vec{\omega} = [\omega_1, \omega_2, \ldots \omega_N]$$
$$\vec{\phi} = [\phi_1, \phi_2, \ldots \phi_N] \tag{30}$$

The interaction or period matrix **B** in (27) is a constant matrix which describes the amplitudes and nonlinear interaction coefficients of the cnoidal waves; the diagonal elements define the cnoidal wave amplitudes and the off-diagonal elements define the nonlinear interactions among the cnoidal waves. The precise analytical expression for u_{int} is given in Osborne (1994). Details for the numerical implementation of the above theorems are also given in the latter reference together with algorithms for the fast numerical implementation of the approach.

5. Numerical examples

I now apply the above theoretical results to numerical simulations. In particular I verify the applicability of the θ–function formulation as an effective method for computing exact solutions to the KdV and KP equations. To this end I first consider a solution to KdV in the *hyperelliptic function representation* (Fig. 1). Fig. 1(a) shows the solution, while

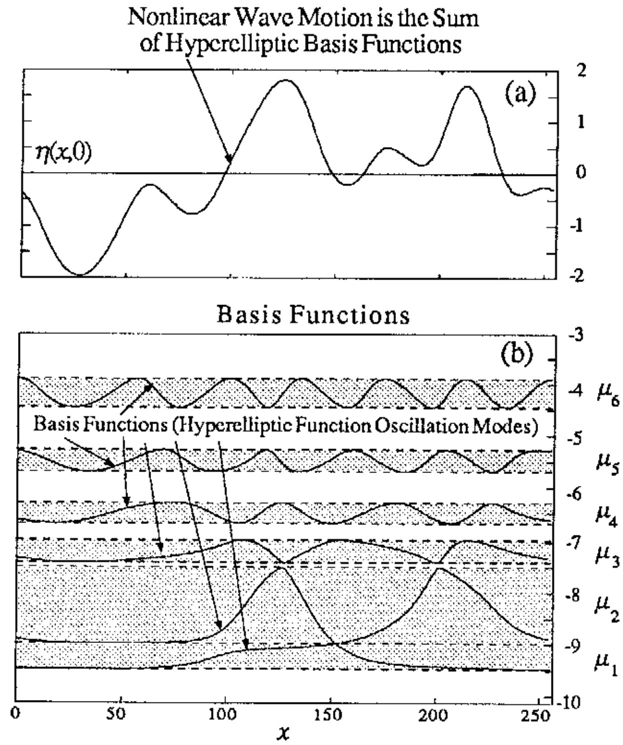

Fig. 1. Construction of KdV wave trains in the hyperelliptic function representation, see Eq. (16). The solution to KdV (a) is found by linearly superposing the hyperelliptic basis functions shown in (b).

Fig. 1(b) gives the six hyperelliptic functions (N=6) used to construct the solution. Each degree of freedom is indicated in the figure by $\mu_j(x,0)$, $j=1-6$. For larger values of j (~5,6) the functions tend to have more regular shapes (they are more linear than their counterparts) and for smaller values of j (~1,2) the functions are more irregular (they are more nonlinear). This latter interpretation arises because the longer, low wave number components feel the influence of the bottom and are hence more nonlinear. In fact, the two μ_j for j=1,2, taken together, form two solitons in the spectrum, while the other modes (j=3-6) are radiation degrees of freedom. This latter observation recognizes that solitons, in the hyperelliptic function representation, are constructed by the linear superposition of the hyperelliptic modes which are phase locked with each other. An additional observation is that the μ_j are quite irregular in shape due to the presence of nonlinear interactions with the other components. Finally it is worth pointing out that no single μ_j is a solution to the KdV equation, only their linear superposition (16) constitutes a solution. Further details on the spectral structure of this particular approach to solving the KdV equation are given elsewhere (Osborne, 1993a).

In order to contrast the hyperelliptic functions with the θ–functions I show in Fig. 2 a typical solution to KdV in

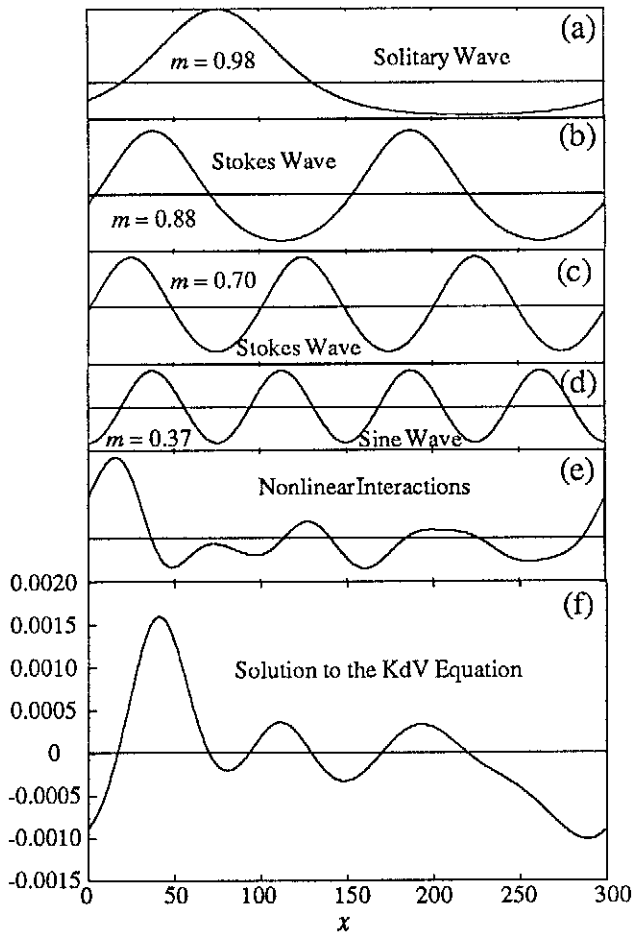

Fig. 2. Construction of KdV wave trains in the θ–function representation. The basis functions are ordinary cnoidal waves as shown in (a)-(d); each cnoidal wave has its associated modulus m, as shown in the figure. The nonlinear interactions among the cnoidal waves are given in (e). The linear superposition of the cnoidal waves plus the nonlinear interactions (sum of (a)-(e)) give the solution to KdV as shown in (f).

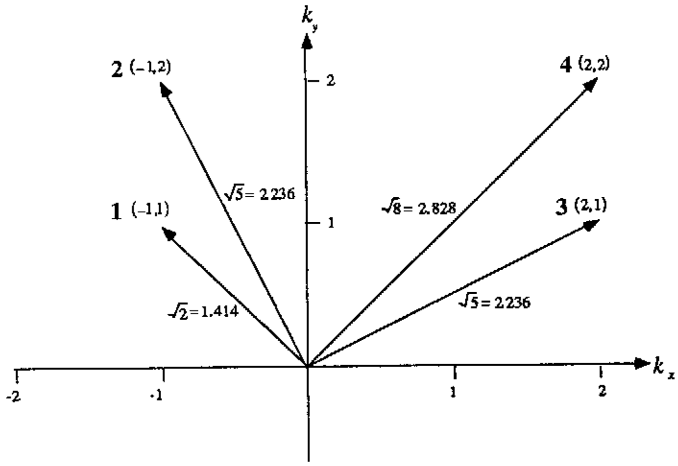

Fig. 3. Wave-vector diagram for an example solution of the KP equation. Here there are four wave vectors with their associated values and directions as plotted on the axes (k_x, k_y).

the θ–function representation as constructed by Theorem I. In this approach one computes a sum of cnoidal waves plus nonlinear interactions. In the present case there are four degrees of freedom, N=4; these are the cnoidal waves in Fig. 2(a)-(d). Each cnoidal wave is of course a solution to the KdV equation and in the present case their associated moduli, respectively, are given by m=0.98 (a soliton), 0.88 (a Stokes wave), 0.70 (a Stokes wave) and 0.37 (a sine wave). The spatial variation in the amplitude due to pair-wise nonlinear interactions is given in Fig. 2(e); this latter consists of an irregular wave train which has a root-mean-square amplitude of about one third that of the total wave train shown in Fig. 2(f). This solution to KdV has been constructed by summing panels (a)-(e) of Fig. 2 as required by Theorem I. One observation about the θ–functions is that they differ substantially from the hyperelliptic functions. Here is a list of some of the differences: (1) The μ_j are quite irregular and generally are not solitons, while the θ–functions are very regular, e.g. cnoidal waves, and individually can be solitons. (2) The μ_j are not generally solutions to the KdV equation, while each θ–function (cnoidal wave) is a solution. (3) The θ–functions are a more physical basis set (as constructed from cnoidal waves) than are the hyperelliptic functions.

I now look at the KP equation and the requisite numerically-generated solutions. A first observation is that there are no hyperelliptic function solutions for KP (Osborne, 1994); in the leap from one spatial dimension (KdV) to two spatial dimensions (KP) one finds that the hyperelliptic functions no longer exist. On the other hand the θ–functions remain a quite natural representation for KP, even in two spatial dimensions. A second observation is that the nonlinear interactions for KP can be quite different than those for the KdV equation, suggesting that computation of the **B** matrix is highly non trivial. This occurs because of the different directional properties of interactions among the travelling wave solutions of KP (Miles 1977). Waves which propagate in nearly the same direction interact 'strongly' while those propagating in very different directions interact 'weakly.' This physical result leads to a quite different structure in the algebraic geometric relationship between the Cauchy initial condition $\eta(x,0)$ and the spectral inverse problem in the θ–function formulation (Osborne, 1994). Consequently the nonlinear interactions as characterized in the period matrix **B** for KP can be quite different than those for the KdV equation. The physical consequences of this statement cannot be over emphasized, as the following numerical example indicates.

The four wave-number components considered in the example solution for the KP equation are shown in Fig. 3. Note that the first two components (labelled 1 and 2 in the

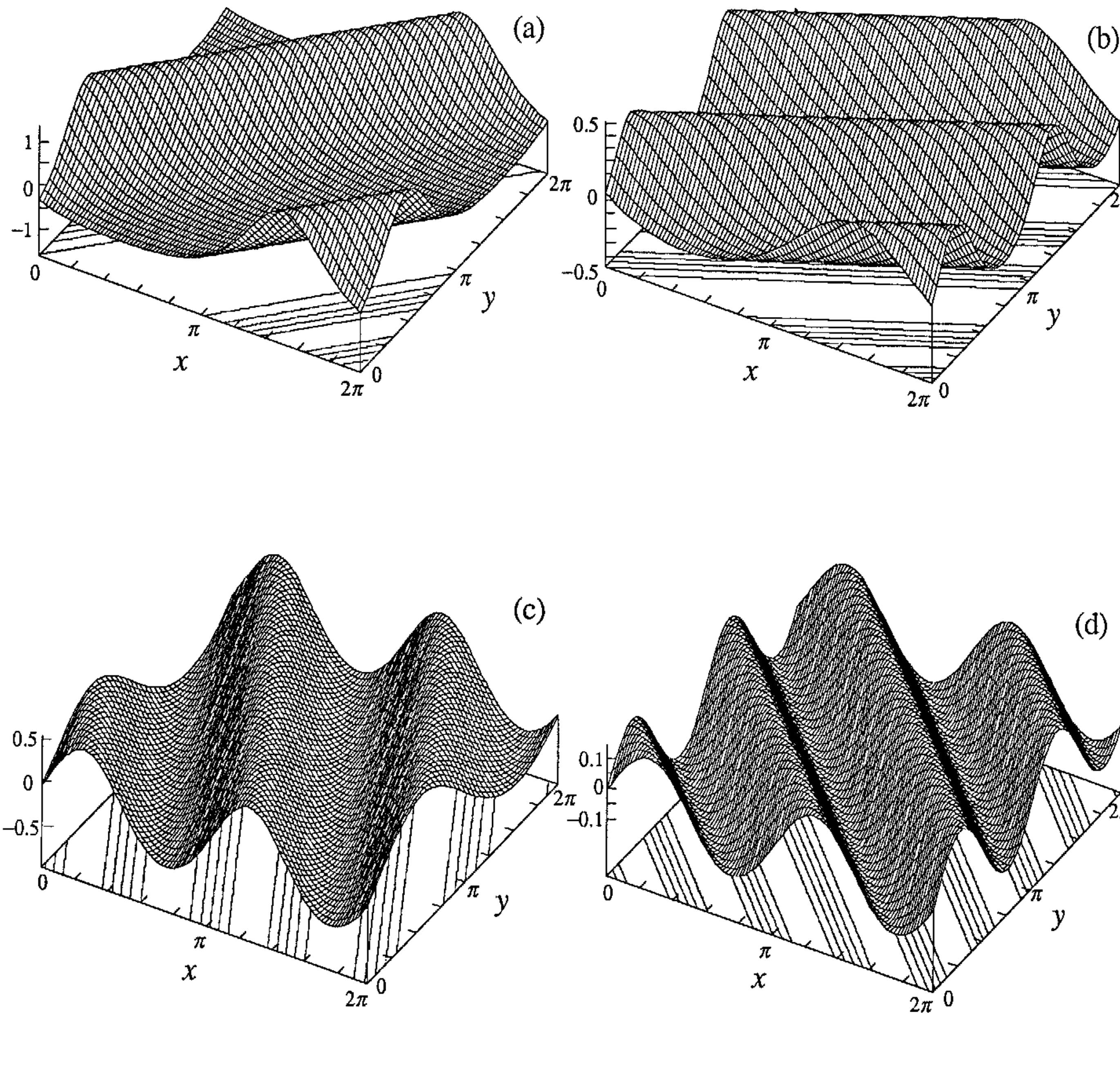

Fig. 4. Four cnoidal waves in the example solution of the KP equation. The wave moduli are: (a) $m = 0.98$, (b) $m = 0.88$, (c) $m = 0.70$ and (d) $m = 0.37$. These cnoidal waves are identical to those given in the example of Fig. 2 for a solution of the KdV equation. Their directions, however, have been distributed according to Fig. 3. Furthermore, the nonlinear interactions are more complex for the KP equation than for the KdV equation.

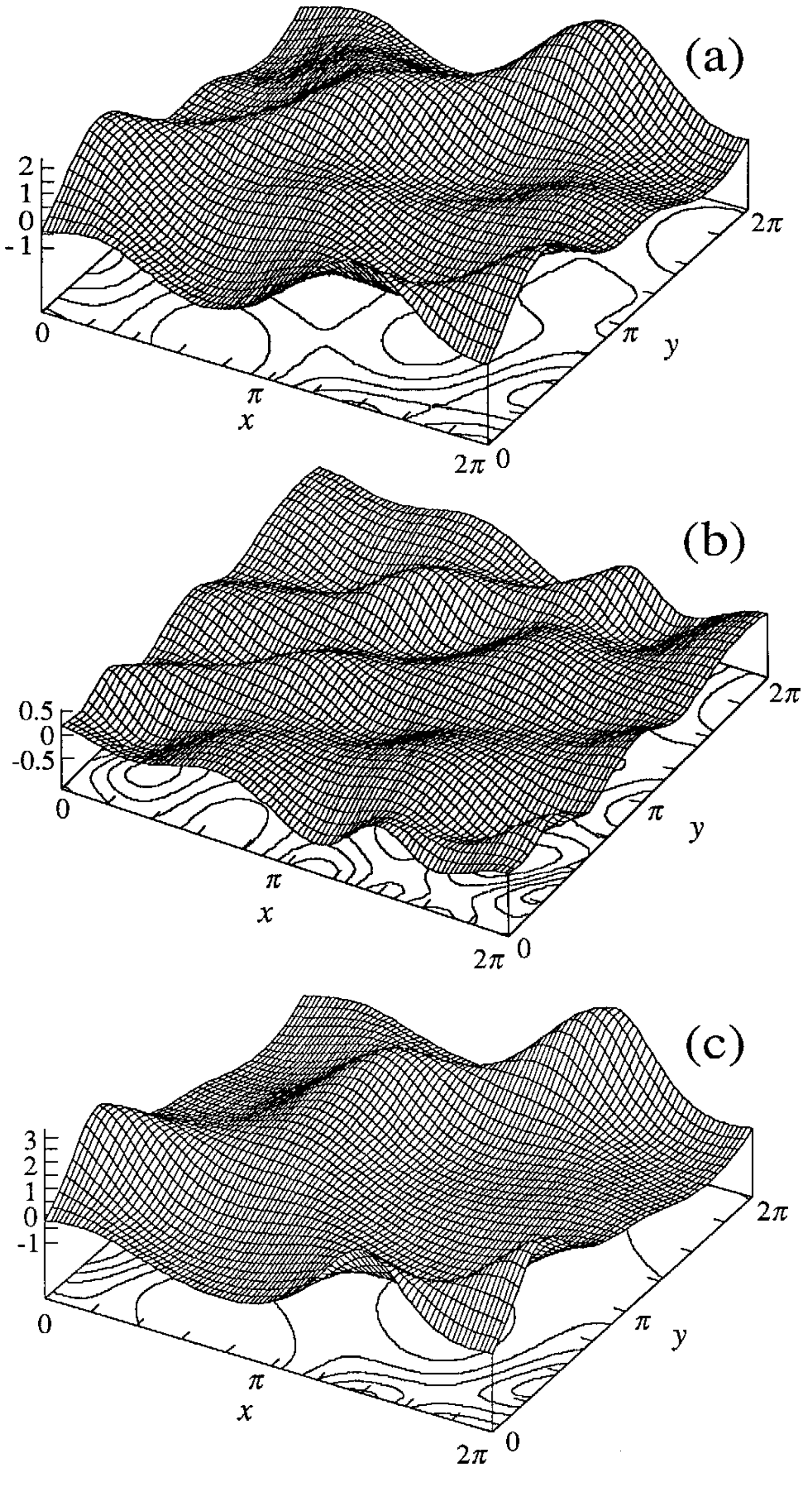

Fig. 5. Example solution to the KP equation based upon the wave vectors shown in Fig 3. In panel (a) is the linear superposition of the cnoidal waves given in Fig. 4. The nonlinear interactions are shown in panel (b). The solutions to KP is the sum of (a) and (b) and is given in (c).

figure) point in nearly the same direction, while the second two (labelled 3 and 4) point in a different relative direction. However, the two pairs are at near right angles with one another. Therefore the interaction pairs may be classified in the following obvious notation: (1,2) (strong interaction), (3,4) (strong), (1,3) (weak), (1,4) (weak), (2,3) (weak) and (2,4) (weak). This mixture of pair-wise interactions is of course non trivial to construct mathematically in terms of the period matrix **B**. These results are beyond the scope of the present paper and the details have been left to the literature (Osborne, 1994).

It is quite instructive to see how the θ–function solution works for this four degree-of-freedom case. Theorem II has been used to construct these numerical results. In Fig. 4 are the four degrees of freedom (the amplitudes and moduli have been taken to be the same as in the previous example given for the KdV equation in Fig. 2). Fig. 4(a) is essentially a solitary wave, Figs. 4(b), (c) are Stokes waves and Fig. 4(d) is a sine wave. These four components have been linearly summed in Fig. 5(a). The nonlinear interaction terms (including both strong and weak interactions) are shown in Fig. 5(b). Finally the solution to the KP equation is given in Fig. 5(c); this has been computed as the linear superposition of panels (a), (b) of Fig. 5. Note that the nonlinear interaction contribution (Fig. 5(b)) is roughly one third of the total root-mean-square wave amplitude; clearly the nonlinear interactions are an important aspect of this formulation.

Another interesting result, from the point of view of this author, is the fact that the sum of cnoidal waves in Fig. 5(a) seems to be rather irregular when contrasted with the solution to KP as shown in Fig. 5(c). The nonlinear interactions have in effect smoothed out the resultant solution to the KP equation. A further interesting observation is that the solutions of KdV and KP, as discussed from the points of view of Theorems I and II, can be viewed in terms of 'particles' (i.e. cnoidal waves) and 'gluons' (nonlinear interactions). The particle contribution consists of a wide variety of nonlinear behavior, i.e. linear (sine waves), moderately nonlinear (Stokes waves) and strongly nonlinear (solitons) wave forms. The nonlinear interaction contribution requires explicit knowledge of the cnoidal waves themselves in order to determine the 'gluon flux.'

6 Summary and Conclusions

This paper emphasizes the important role of the θ–function representation for constructing shallow water wave trains in both one and two spatial dimensions. The fact that the θ–functions are uniquely related to the inverse problem for the Cauchy initial condition for the periodic KdV and KP equations is of course an important consideration, since the mathematical machinery for constructing N degree-of-freedom solutions is thereby guaranteed. The main thrust of the present paper is the numerical construction of particular *low* degree-of-freedom solutions which shed new light on the physics of shallow water wave trains. A major future effort relates to the development of fast numerical codes for the implementation of this approach to the experimental analysis of laboratory and oceanic wave trains. Further refinement in the physical accessibility of periodic inverse scattering theory through enhanced numerical capability and physical understanding, particularly in an experimental context, is underway.

Appendix - Computation of wave numbers, frequencies, phases and interaction matrix of the θ–function representation

This Appendix summarizes determination of the wave numbers, k_j, frequencies, ω_j, phases, ϕ_j, and the period matrix, **B**, in the θ–function solution to KdV (Its and Matveev, 1975; Dubrovin et al, 1976); Flaschka and McLaughlin, 1976; Date and Tanaka, 1976; MacKean and Trubowitz, 1976). While some knowledge of algebraic geometry is required for the derivation of the following formulas, the results can be implemented numerically with only rudimentary understanding of algebraic geometric methods (Osborne, 1994).

The θ–function is 2π periodic in each of the N phases η_j

$$\Theta[(\eta_1+2\pi),(\eta_2+2\pi),\ldots,(\eta_N+2\pi)]= \\ =\Theta[\eta_1,\eta_2,\ldots,\eta_N] \tag{A.1}$$

Normalized holomorphic differentials on the Riemann surface Γ (see $R(E)$ below) are then introduced

$$d\Omega_n(E)=\sum_{m=1}^{N} C_{nm}\frac{E^{m-1}dE}{R^{1/2}(E)} \tag{A.2}$$

where $R(E)$ is given by

$$R(E)=\prod_{k=1}^{2N+1}(E-E_k)$$

and the following normalization condition is assumed to hold:

$$\oint_{\alpha_j} d\Omega_n(E)=2\pi i\delta_{nj} \tag{A.3}$$

These are the "α_j-cycles" or contour integrals around the 'open bands' (E_{2j}, E_{2j+1}) in the Floquet spectrum. Combining (A.2) and (A.3) yields:

$$\sum_{m=1}^{N} C_{nm} J_{mj} = 2\pi i \delta_{nj}, \tag{A.4}$$

$$J_{mj} = \oint_{\alpha_j} \frac{E^{m-1} dE}{R^{1/2}(E)}$$

which in matrix notation is

$$\mathbf{C} = 2\pi i \mathbf{J}^{-1} \tag{A.5}$$

The normalization coefficients C_{nm} in (A.2) are then given by Osborne (1993a):

$$C_{jm} = 2\pi i \left[\oint_{\alpha_j} \frac{E^{m-1} dE}{\sqrt{\prod_{k=1}^{2N+1} (E - E_k)}} \right]^{-1} = \tag{A.6}$$

$$= \pi i \left[\int_{E_{2j}}^{E_{2j+1}} \frac{E^{m-1} dE}{\sqrt{\prod_{k=1}^{2N+1} (E - E_k)}} \right]^{-1}$$

The *phases* η_j of the θ-function (20) are found by the following Abelian integrals

$$\eta_j(P_1, P_2, \ldots, P_j) = \tag{A.7}$$

$$= -i \sum_{m=1}^{N} \int_{E_{2m}}^{P_m(x,t)} d\Omega_j(E) = K_j x - \omega_j t + \phi_j$$

where $P_m(x,t) = [\mu_m(x,t), \sigma_m]$ for $1 \le m \le j$. Eq. (A.7) may be interpreted as a *linearization of the hyperelliptic function representation of the flow*, i.e. integration over the holomorphic differentials (A.2) from the lower band edge E_{2j} to the hyperelliptic functions $\mu_j(x,t)$ in effect linearizes the μ_j. This leads to the linear θ-function inverse problem for KdV described in Section 4. Equations (A.2) and (A.7) are an *Abel transform pair*. Generally speaking the phase of the hyperelliptic functions η_j (A.7) depends upon the main spectrum $(E_i, 1 \le i \le 2N+1)$ and the space-time evolution of the auxiliary spectrum $[\mu_j(x,t), \sigma_j]$, $1 \le j \le N$.

It then follows that the wave numbers K_j, and frequencies ω_j are given by

$$K_j = 2C_{N,j} \tag{A.8}$$

$$\omega_j = 8C_{N-1,j} + 4C_{N,j} \sum_{i=1}^{2N+1} E_i$$

Both K_j and ω_j are real constants since the C_{jm} and the E_k are real constants. The usual nonlinear dispersion law for a single degree of freedom may easily be obtained from (A.8). The K_j are commensurable wave numbers in the cycle integral basis considered here, while the frequencies ω_j are generally incommensurable.

The phases ϕ_j are found by fixing $x = 0$, $t = 0$ in (A.7) to get:

$$\phi_j = -i \sum_{m=1}^{N} \int_{E_{2m}}^{P_m(0,0)} d\Omega_j(E) = \tag{A.9}$$

$$= -i \sum_{m=1}^{N} C_{jm} \int_{E_{2m}}^{\mu_m(0,0)} \frac{E^{m-1} dE}{R^{1/2}(E)} = -i \sum_{m=1}^{N} C_{jm} \Phi_m$$

where

$$\Phi_m = \int_{E_{2m}}^{\mu_m(0,0)} \frac{E^{m-1} dE}{R^{1/2}(E)} \tag{A.10}$$

Thus the *constant phases* ϕ_j of the hyperelliptic functions depend upon the *starting values of the hyperelliptic functions* $\mu_j(0,0)$ *and the Riemann sheet indices* σ_j.

The *period matrix* is given by:

$$B_{nj} = \oint_{\beta_j} d\Omega_n(E) = \tag{A.11}$$

$$= \sum_{m=1}^{N} C_{nm} \oint_{\beta_j} \frac{E^{m-1} dE}{R^{1/2}(E)} = \sum_{m=1}^{N} C_{nm} \mathrm{B}_{mj}$$

where

$$B_{mj} = 2\int_{E_1}^{E_{2j}} \frac{E^{m-1}dE}{R^{1/2}(E)} \tag{A.12}$$

The integrals are over the "β-cycles" of the theory (for a discussion with regard to the numerical analysis see Osborne (1994)). Fast algorithms for computing θ–functions for KdV, together with a generalization to the KP equation, are also discussed in the latter reference.

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