Nonlinear Processes in Geophysics (1994) 1: 234 - 248

Nonlinear Processes in Geophysics
© European Geophysical Society 1994

Hamiltonian description of wave dynamics in nonequilibrium media

N. N. Romanova

Institute of Atmospheric Physics, Russian Academy of Sciences, Pyzhevsky 3, 109017, Moscow, Russia

Received 26 June 1994 - Accepted 16 September 1994 - Communicated by V. Shrira

Abstract. We consider Hamiltonian description of weakly nonlinear wave dynamics in unstable and nonequilibrium media. We construct the appropriate canonical variables in the whole wavenumber space. The essentially new element is the construction of canonical variables in a vicinity of marginally stable points where two normal modes coalesce. The commonly used normal variables are not appropriate in this domain. The matter is that the approximation of weak nonlinearity breaks down when the dynamical system is written in terms of these variables. In this case we introduce the canonical variables based on the linear combination of modes belonging to the two different branches of dispersion curve.

As an example of one of the possible applications of presented results the evolution equations for weakly nonlinear wave packets in the marginally stable area are derived. These equations cannot be derived if we deal with the commonly used normal variables.

1 Introduction

The methods of Hamiltonian formalism are extremely fruitful for the description of weakly nonlinear wave dynamics (Zakharov, 1968). The most adequate instrument for the description of wave motions in stable media is the formalism of normal modes. It is based on the possibility to introduce in the space of wavenumbers k the special canonical variables $a_i(k)$ which are called normal variables.

The quadratic part H_2 of Hamiltonian in its expansion into a series in a small parameter of nonlinearity ϵ has the well-known simplest form in these variables. When the number of normal modes is finite, the integrand of H_2 has the form $\sum_{j=1}^{n} \omega_j \left|a_j\right|^2$, where the eigenfrequencies ω_j corresponding to the normal modes are positive for equilibrium media.

In case of instability when a pair of complex conjugated roots appears in the spectrum of eigenvalues, we can construct the transformation from the initial variables to the canonical normal variables in a similar way. However the canonical form of Poisson structure and the structure of quadratic part of Hamiltonian have then the other form. In the unstable area of wavenumbers k there are linear solutions that grow exponentially with time but the quadratic part of Hamiltonian H_2 is conserved with time due to its special form.

In the vicinity of marginally unstable point k_0 where two different modes coalesce, the matrix of linear transformation from the initial variables to the normal ones degenerates. We introduce the other canonical variables connected with the coalesced mode in the vicinity of the point k_0. The corresponding columns of the matrix of linear transformation to the new variables in this area are the linear combinations of eigenvectors corresponding to two close eigenvalues ω. The coefficients of dynamical equations written in these variables are analytical functions of wavenumber deviation $\Delta k = k - k_0$ and can be expanded into a series in Δk.

As a consequence we can obtain the evolution equations describing the dynamics of weakly nonlinear wavepackets in a marginally unstable area. The introduction of canonical variables connected with the coalesced modes demonstrates that the commonly accepted expression for wave energy density of the form $\omega \partial L/\partial\omega \left|a\right|^2$ (see Whitham, 1974) corresponding to normal mode is not applicable in the neighborhood of a marginally stable point. The well-known exppession for group velocity of the form $d\omega/dk$ fails in this area also.

2 The governing equations

We begin by considering the system of dynamical equations written in terms of Fourier transform by one-dimensional space variable:

$$\dot{\mathbf{y}}(k) = -I(k)\frac{\delta H}{\delta \mathbf{y}(-k)} \tag{1}$$

where dot indicates differentiation with respect to time. Here H is the total energy of system, $\mathbf{y}(k)$ is the $2n$-dimensional vector of dependent variables, and $\delta/\delta\mathbf{y}$ is the symbol of variational derivative; the system depends on wavenumber k parametrically. The demand for the initial variables to be real leads us to the condition

$$y(k) = y^*(-k) \tag{2}$$

where asterisk means complex conjugation. We employ the approximation of weak nonlinearity and expand the Hamiltonian H as a power series in the small parameter of nonlinearity ϵ:

$$H = H_2 + \epsilon H_3 + \cdots$$

where the quadratic part of Hamiltonian is equal to

$$H_2 = \frac{1}{2}\int (y(-k), h(k)y(k))dk \tag{3}$$

and the domain of integration is the number axis. The matrices of $2n$ order $I(k)$, $J(k) = I^{-1}(k)$ and $h(k)$ are subject to the following conditions:

$$I^*(k) = I(-k), \qquad I^*(k) = -I'(k) \tag{4a}$$

$$J^*(k) = J(-k), \qquad J^*(k) = -J'(k) \tag{4b}$$

$$h^*(k) = h(-k), \qquad h^*(k) = h'(k) \tag{4c}$$

where the prime means the transposition symbol. So, we have the dynamical system (1) determined in the phase space of $2n$-dimensional complex-valued vectors $\mathbf{y}(k)$, the wavenumber k being a continuous parameter. The matrix $I(k)$ defines the Poisson brackets in the space of smooth functionals in our phase space in the following way:

$$\begin{aligned}\{H(y), G(y)\} &= \int (\frac{\delta H}{\delta \mathbf{y}(-k)},\ I(-k)\frac{\delta G}{\delta \mathbf{y}(k)})dk = \\ &= -\int (\frac{\delta G}{\delta \mathbf{y}(k)},\ I(k)\frac{\delta H}{\delta \mathbf{y}(-k)})dk\end{aligned} \tag{5}$$

where the domain of integration is the number axis. Due to the properties (4a) the Poisson brackets (5) are subject to the standard conditions of skew symmetry. The Jacobi identity is fulfilled the matrix $I(k)$ being independent of the variables $\mathbf{y}(k)$. If the functional $G(\mathbf{y})$ is equal to $y_p(k')$ – the value of the p-th coordinate of the vector $\mathbf{y}$ in the point k', then

$$\frac{\delta G(\mathbf{y})}{\delta y_m(k)} = \delta_m^p \delta(k - k')$$

where δ_m^p is the Kronecker symbol, and $\delta(k - k')$ is the Dirac delta function. It is easy to see that if we take Hamiltonian as the second functional in (5), the dynamical equations (1) can be written in the form (Arnol'd, 1978; Dubrovin et al., 1984):

$$\dot{y}_j(k) = \{H, y_j(k)\} = \mathcal{Y}_j^H$$

where $\mathcal{Y}^H$ is the Hamiltonian vector field determined by the Poisson structure.

So the dynamical system (1) is the Hamiltonian system determined in a phase space that is in fact a complex infinite-dimensional manifold. We can also construct in this manifold the symplectic 2-form $\Omega(x, y)$ connected with the Hermitian structure (Arnol'd, 1978; Dubrovin et al., 1984). This symplectic form is determined by the inverse matrix $J(k)$:

$$\Omega(x, y) = -\int (x(-k), J(k)y(k))dk.$$

The matrix $I(k)$ is not degenerate, and the inverse matrix $J(k)$ exists. It follows from the conditions (4b) that $\Omega(x, y) = -\Omega(y, x)$. It is easy to show that the value of the Poisson brackets defined on the two functionals is equal to the value of the symplectic form on the two vector fields determined by these functionals:

$$\{H, G\} = \Omega(\mathcal{X}^H, \mathcal{X}^G)$$

A special example of the Hamiltonian system described above is the obtained by Goncharov (1986) system of equations describing the weakly nonlinear waves in the piecewise linear n-layer stratified flow. Let's take a closer look at this system.

We consider the n-layer inviscid two-dimensional parallel stably stratified flow with a piecewise constant density profile and a piecewise linear velocity profile. We shall deal with the irrotational perturbance of this flow. The associated velocity perturbations have the form $U_j = \nabla\Psi_j$, where Ψ_j is a velocity potential satisfying Laplace's equation in the j-th layer. The disturbances are assumed to decay to zero as $|z| \to \infty$. At the interfaces $z = h_j + \eta_j(x, t)$, where $\eta_j(x, t)$ is the vertical displacement of the j-th interface at $z = h_j$, the traditional kinematic and dynamical boundary conditions should be satisfied. As it was shown in (Goncharov, 1986), after the introduction of dependent variables $\eta_j(x, t)$, $\phi_j(x, t)$, where $\phi_j = \Psi_{j+1} - \Psi_j$ is the difference of velocity potentials in the two neighbouring layers (in the Boussinesq approximation), the governing equations for description of wave perturbations have the form:

$$\frac{\partial}{\partial x}\dot{\phi}_j = -\frac{\partial}{\partial x}\frac{\delta H}{\delta \eta_j} + \nu_j \frac{\delta H}{\delta \phi_j}, \qquad \dot{\eta} = \frac{\delta H}{\delta \phi_j} \tag{6}$$

where ν_j is the vorticity jump of unperturbed flow at the j-th boundary. In terms of Fourier-transform by x the system (6) has the form (1), where the vector $Y(k)$ is equal to

$$Y(k) = (\phi_1(k), \cdots, \phi_n(k),\ \eta_1(k), \cdots, \eta_n(k)). \tag{7}$$

The matrix $I(k)$ of order $2n$ has the form:

$$I(k) = \begin{pmatrix} -i\nu_1/k & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -i\nu_n/k & & & 1 \\ -1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 0 \end{pmatrix}$$

and the matrix that determines the quadratic part of Hamiltonian in equation (3) is equal to

$$h(k) = \begin{pmatrix} a_{jm}(k) & -v_{jm} \\ v_{jm} & r_{jm} \end{pmatrix},$$

$$v_{jm} = ikV(h_j)\delta^j_m,\ r_{jm} = (g\Delta\rho_j/\rho_j + \nu_j V(h_j))\,\delta^j_m.$$

where $a_{jm}(k)$ is the symmetric matrix of order n, its form depending on the boundary conditions. It is easy to see that these matrices satisfy the conditions (4). Note that the physical variables are not always the canonical ones, and this is the case.

A second special case of the system (1) is the described by Zakharov (1974) canonical system of the second order that has the following form in terms of Fourier transform:

$$\dot{p} = \frac{\delta H}{\delta q^*},\ \dot{q} = -\frac{\delta H}{\delta p^*}.$$

The first term in the expansion of the Hamiltonian has the form (3), where h is the second order matrix satisfying to conditions (4c).

The canonical systems constitute the particular case of the type (1) systems.

3 The construction of normal variables

Our aim is the construction for the considered systems of the special type of canonical variables that are called the normal variables, everywhere over the region k, in both stable and unstable domains. Note that when we do the linear transformation to the new dependent variables $\mathbf{a}$, $\mathbf{y} = \mathbf{Z}\mathbf{a}$, where $\mathbf{Z}$ is a transformation matrix, then evolution equation for $\mathbf{a}$ has the same form (1) but with the matrices J and h interchanged into

$$\tilde{J} = \mathbf{Z}'(-k)J(k)\mathbf{Z}(k),\ \tilde{h} = \mathbf{Z}'(-k)h(k)\mathbf{Z}(k). \quad (8)$$

Let $\mathcal{Z}$ be an eigenvector of the following linear system of algebraic equations correspomding to linearized system (1):

$$\big(h(k) - i\omega(k)J(k)\big)\,\mathcal{Z}(k) = 0 \quad (9)$$

where eigenvalue ω is a solution of a dispersion equation

$$\det\|h(k) - i\omega(k)J(k)\| = 0. \quad (10)$$

It follows from (4), (9), (10) that $\mathcal{Z}(k) = \mathcal{Z}^*(-k)$ for both the real and complex values of ω.

The space of eigenvectors consists of $2n$ linearly independent vectors $\mathcal{Z}_j$. We can naturally unite them into pairs at any point k. There are three possible variants. If the two eigenvectors $\mathcal{Z}_{j1}$, $\mathcal{Z}_{j2}$ correspond to the different real values of ω_{j1}, ω_{j2}, we unite them into a pair if the values of expressions $(\mathcal{Z}^*_{j1}(k), J(k)\mathcal{Z}_{j1}(k))$ and $(\mathcal{Z}^*_{j2}(k), J(k)\mathcal{Z}_{j2}(k))$ have opposite signs. Due to the properties (4) these values are purely imaginary for real eigenvalues. In the unstable case when a pair of complex conjugated eigenvalues ω_{j1}, ω_{j2} appears in the spectrum of eigenvalues, we unite the corresponding eigenvectors into a pair. A pair of eigenvectors corresponding to one double root ω in the spectrum of eigenvalues are the eigenvector and the adjoint eigenvector. Linear transformation from the vector of initial variables to the normal variables $a_{j1,2}$ is

$$\mathbf{Y}(k,t) = \sum_{j=1}^{n}(\mathcal{Z}_{j1}(k)a_{j1}(k,t) + \mathcal{Z}_{j2}(k)a_{j2}(k,t)) \quad (11)$$

where

$$a_{j1,2}(k,t) = a^*_{j1,2}(-k,t). \quad (12)$$

The other form in which we can write down the transformation of initial variables to the normal ones can be constructed if we introduce the vector $Z_j(k)$ defined as

$$Z_j(k) = \begin{cases} \mathcal{Z}_{j1}(k), & k > 0 \\ \mathcal{Z}_{j2}(k), & k < 0, \end{cases} \quad (13)$$

and the function

$$a_j(k) = \begin{cases} a_{j1}(k), & k > 0, \\ a_{j2}(k), & k < 0. \end{cases} \quad (14)$$

It follows from the properties of the eigenvectors and from (13), (14) that

$$Z^*_j(-k) = \begin{cases} \mathcal{Z}_{j2},\ k > 0, \\ \mathcal{Z}_{j1},\ k < 0, \end{cases} \quad a^*_j(-k) = \begin{cases} a_{j2},\ k > 0, \\ a_{j1},\ k < 0. \end{cases}$$

In these variables expression (11) has the form

$$\mathbf{Y}(k,t) = \sum_{j=1}^{n}(Z_j(k)a_j(k,t) + Z^*_j(-k)a^*_j(-k,t)). \quad (15)$$

So, instead of $2n$ variables defined on the semi axis $k > 0$, we are looking for the n functions a_j defined over the whole axis. In these new variables the transformed matrix $\tilde{J}(k)$ determining the symplectic structure is

$$\tilde{J}(k) = \begin{pmatrix} B_1 & & & & \\ & \ddots & & 0 & \\ & & B_j & & \\ & 0 & & \ddots & \\ & & & & B_n \end{pmatrix} \quad (16)$$

where B_j is the second order matrix with coefficients

$$\begin{aligned} b_{11}^{(j)} &= \quad (Z_j(-k), J(k)Z_j(k)), \\ b_{21}^{(j)} &= \quad (Z_j^*(k), J(k)Z_j(k)), \\ b_{12}^{(j)} &= (Z_j(-k), J(k)Z_j^*(-k)), \\ b_{22}^{(j)} &= \quad (Z_j^*(k), J(k)Z_j^*(-k)). \end{aligned} \tag{17}$$

Below we omit the index j. The other coefficients in the transformed matrix (16) are equal to zero due to orthogonality of eigenvectors corresponding to different eigenvalues, if they are not complex conjugated. It follows from (1) that the equations for variables $a(k)$ and $a^*(-k)$ corresponding to any pair of eigenvectors are

$$B(k)\begin{pmatrix} \dot a(k) \\ \dot a^*(-k) \end{pmatrix} = -\begin{pmatrix} \delta H/\delta a(-k) \\ \delta H/\delta a^*(k) \end{pmatrix}. \tag{18}$$

We can write down (17) in terms of eigenvectors $\mathcal{Z}$:

$$\begin{aligned} b_{11} &= \begin{cases} (\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_1(k)), & k>0, \\ (\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_2(k)), & k<0, \end{cases} \\ b_{21} &= \begin{cases} (\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_1(k)), & k>0, \\ (\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_2(k)), & k<0, \end{cases} \\ b_{12} &= \begin{cases} (\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_2(k)), & k>0, \\ (\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_1(k)), & k<0, \end{cases} \\ b_{22} &= \begin{cases} (\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_2(k)), & k>0, \\ (\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_1(k)), & k<0. \end{cases} \end{aligned} \tag{19}$$

These coefficients are different depending on the three different situations referred above: (a) a pair of real eigenvalues: (b) a pair of complex conjugated eigenvalues; (c) a double eigenvalue.

In case (a), the coefficients b_{11} and b_{22} are equal to zero owing to the orthogonality of eigenvectors $\mathcal{Z}_1$ and $\mathcal{Z}_2$. It can easily be shown that due to the properties (4) the expression $(\mathcal{Z}^*(k), J(k)\mathcal{Z}(k))$ is purely imaginary, its sign changing with the change in the sign of k. Any eigenvector $\mathcal{Z}$ is determined correct to arbitrary constant C choosing which we can write the coefficients of matrix B equal to: $b_{21} = -i$, $b_{12} = i$. Let $L(\omega, k)$ be the right-hand side of the dispersion equation. Then the expression $\partial L/\partial\omega$ is the wave-action density. It can be easily shown (Whitham, 1974; Voronovich, 1979), that $\partial L/\partial\omega = i(\tilde{\mathcal{Z}}^*(k), J(k)\tilde{\mathcal{Z}}(k))$, where $\tilde{\mathcal{Z}}(k)$ is the eigenvector normalized in some definite way. The arbitrary eigenvector has the form $\mathcal{Z} = C\tilde{\mathcal{Z}}$, and it follows that the constant $C = 1/\sqrt{\partial L/\partial\omega}$, $\partial L/\partial\omega$ being positive due to our choice of eigenvector pairs. As a result the equation (18) for a pair of real eigenvalues has the form (Zakharov, 1968):

$$\dot a(k) = -i\frac{\delta H}{\delta a^*(k)}. \tag{20}$$

In case (b) when a pair of eigenvectors $\mathcal{Z}_1, \mathcal{Z}_2$ corresponds to the two complex conjugated roots, it follows from (4) that the expression $(\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_2(k))$ is not equal to zero. However the coefficients b_{12}, b_{21} in the matrix B are equal to zero due to the relations $(\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_1(k)) = (\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_2(k)) = 0$ following from (4) in case of complex conjugated eigenvalues. It is easy to show that in this case

$$b_{11}(-k) = -b_{11}(k),\ b_{22}(-k) = -\ b_{22}(k),\ b_{11}(-k) = b_{22}^*(k).$$

By the appropriate choice of arbitrary constants we can make the coefficients b_{11}, b_{22} equal to

$$b_{11}(k) = b_{22}(k) = i\operatorname{sgn} k$$

and as a consequence we obtain from (18) the following equation corresponding to the complex conjugated pair of eigenvalues:

$$\dot a(k) = i\operatorname{sgn} k\frac{\delta H}{\delta a(-k)}. \tag{21}$$

We can also pose the sign of the right-hand side of (21) to be minus.

So, the canonically conjugated pair in the case of instability is $a(k)$ and $i\operatorname{sgn} k a(-k)$, i.e. the canonical Poisson structure has the other form in this case, as it was shown in (Goncharov et al., 1993).

Now we regard the third case (c), when a pair of variables $a(k)$ and $a^*(-k)$ corresponds to one two-fold root ω. Let $\mathcal{Z}_1$ be the eigenvector, and $\mathcal{Z}_2$ be the adjoint eigenvector. Then we can write $\mathcal{Z}_1 = C_1\tilde{\mathcal{Z}}$, and $\mathcal{Z}_2 = C_1\tilde{\mathcal{Z}}_\omega + C_2\tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}}$ is the eigenvector normalized in a definite way. Note that the wave-action density $\partial L/\partial\omega = 0$ at the point of root coalescence, and the quantity $(\mathcal{Z}_1^*(k), J(k)\mathcal{Z}_1(k)) = 0$ at this point. If the connection $C_2 = -C_1 L_{\omega\omega\omega}/(2L_{\omega\omega})$ between the arbitrary constants holds, then $(\mathcal{Z}_2(k), J(k)\mathcal{Z}_2(k)) = 0$.

So, the coefficients $b_{12} = b_{21} = 0$. For the other two coefficients the equations $b_{11}(-k) = -b_{11}(k)$, $b_{22}(k) = -b_{11}^*(k)$ hold. It can be shown that for positive values of k the term $b_{11}(k)$ is equal to

$$b_{11}(k) = -(\mathcal{Z}_2^*(k), J(k)\mathcal{Z}_1(k)) =$$

$$= -C_1C_1^*(\tilde{\mathcal{Z}}_\omega^*, J\tilde{\mathcal{Z}}) = i\,|C_1|^2\,\frac{1}{2}\frac{\partial^2 L}{\partial\omega^2}.$$

The sign of $b_{11}(k)$ depends on the sign of $\partial^2 L/\partial\omega^2$. If it is positive when k is positive, we take C_1 equal to $1/\sqrt{\frac{1}{2}\partial^2 L/\partial\omega^2}$, and get $b_{11}(k) = i\operatorname{sgn} k$. As a result we obtain the canonical equations in the same form (21) as for the unstable case. In the opposite case the coefficients b_{11}, b_{22} change sign, and the right-hand side of (21) has the opposite sign.

4 The expression for wave energy density in normal variables.

We have to obtain the transformed matrix $\tilde h$ determining the quadratic part of Hamiltonian (5) in the normal variables according to the formula (8).

It is easy to show that the transformation to normal variables leads to the following expression for $\widetilde{h}(k)$:

$$\tilde{h}(k) = \begin{pmatrix} h_1 & & & & \\ & \ddots & & 0 & \\ & & h_j & & \\ & 0 & & \ddots & \\ & & & & h_n \end{pmatrix}$$

where h_j are the second order matrices with coefficients

$$\begin{aligned} h_{11}^{(j)} &= (Z_j(-k), h(k)Z_j(k)), \\ h_{21}^{(j)} &= (Z_j^*(k), h(k)Z_j(k)), \\ h_{12}^{(j)} &= (Z_j(-k), h(k)Z_j^*(-k)), \\ h_{22}^{(j)} &= (Z_j^*(k), h(k)Z_j^*(-k)). \end{aligned}$$

Omitting the index j we write down the part of quadratic term of energy H_2 that corresponds to the pair of new variables $a(k)$, $a^*(-k)$ for three different cases. In case (a) when normal variable $a(k)$ corresponds to a pair of real eigenvalues, the energy of this stable mode has the form

$$E = \int \omega(k) a(k) a^*(k) dk$$

where

$$\omega(k) = \begin{cases} \omega_1(k), & k > 0, \\ \omega_2(k), & k < 0, \end{cases}$$

In case (b) when we have the complex conjugated pair of roots ω_1 and $\omega_2 = \omega_1^*$,

$$E = \frac{1}{2} \int (-\omega(k) \operatorname{sgn} k a(k) a(-k) + \text{c.c.}) dk$$

where the area of integration is that of instability.

In case (c), when the two eigenvalues coalesce, the energy density is

$$\begin{aligned} E_k = \pm\frac{1}{2}\omega(k) \operatorname{sgn} k (a(k)a(-k) + \text{c.c.}) + \\ + \theta(-k) a(k) a^*(k), \end{aligned} \tag{22}$$

where + and − correspond to the two points bordering the area of instability, and $\theta(k)$ is the Heaviside function.

5 Canonical variables in the vicinity of marginally stable points.

As we can see from the previous sections, the method of normal modes is not appropriate for description of weakly nonlinear waves in the vicinity of a marginally stable point. The expansion of Hamiltonian written in terms of normal variables in the small parameter of nonlinearity ϵ fails in the vicinity of this point. The reason is the normalization of the eigenvectors by the quantity $\sqrt{\partial L/\partial \omega}$, that is small in this area. In the area $k \cong k_0 + \Delta k$, where Δk is of the order $\epsilon^{1/4}$, the following terms in the expansion of Hamiltonian are comparable with the first one, and should be taken into account.

The coefficients of dynamical system written in terms of normal variables are not analytical functions of k in the vicinity of k_0 because of the singularity of $\omega(k)$ in this area. The usual statement (Ostrovskiy et al., 1986) that wave energy density in the vicinity of marginally stable point tends to zero (the waves of zero energy) means only the fact that the notion of normal mode connected with the pair of dispersion equation roots does not hold in the domain where these two roots are close to each other. In this case we should deal with the coalesced mode.

In case when two eigenvalues ω_{j1} and ω_{j2} are close to each other, we take in the matrix of transformation instead of eigenvector the following vector $Z_j(k)$ (omitting index j below):

$$Z(k) = \begin{cases} (\mathcal{Z}_1(k) + \mathcal{Z}_2(k))/2, & k > 0, \\ \alpha(k)(\mathcal{Z}_1(k) - \mathcal{Z}_2(k)), & k < 0, \end{cases}$$

$$Z^*(-k) = \begin{cases} -\alpha(k)(\mathcal{Z}_1(k) - \mathcal{Z}_2(k)), & k > 0, \\ (\mathcal{Z}_1(k) + \mathcal{Z}_2(k))/2, & k < 0, \end{cases}$$

where $\alpha(k) = 1/(\omega_1(k) - \omega_2(k))$. These are the linear combinations of the two eigenvectors corresponding to the two close eigenfrequencies.

This transformation is valid in the vicinity of the marginally stable point k_0, where $\omega_1 = \omega_2$, in both stable and unstable areas adjacent to the point k_0.

The properties of function $\alpha(k)$ are the following ones. In the stable area adjacent to the marginally stable point it is a real quantity ($\alpha(k) = \alpha^*(k)$), and $\alpha(k) = -\alpha(-k)$. In the domain of instability, when ω_1 and ω_2 are complex conjugated, it is purely imaginary, and $\alpha(-k) = \alpha(k)$.

The coefficients of corresponding matrix B are determined by the equations (17), if the vector $Z(k)$ is determined by eq.(23). In the domain of stability they are

$$b_{11} = b_{22} = -(\alpha(k)/2) \operatorname{sgn} k((\mathcal{Z}_1^*, J\mathcal{Z}_1) - (\mathcal{Z}_2^*, J\mathcal{Z}_2)),$$

$$b_{12} = \begin{cases} \alpha^2(k)[(\mathcal{Z}_1^*, J\mathcal{Z}_1) + (\mathcal{Z}_2^*, J\mathcal{Z}_2)], & k > 0, \\ (1/4)[(\mathcal{Z}_1^*, J\mathcal{Z}_1) + (\mathcal{Z}_2^*, J\mathcal{Z}_2)], & k < 0, \end{cases}$$

$$b_{21} = \begin{cases} (1/4)[(\mathcal{Z}_1^*, J\mathcal{Z}_1) + (\mathcal{Z}_2^*, J\mathcal{Z}_2)], & k > 0, \\ \alpha^2(k)[(\mathcal{Z}_1^*, J\mathcal{Z}_1) + (\mathcal{Z}_2^*, J\mathcal{Z}_2)], & k < 0. \end{cases}$$

It follows from these equations that $b_{12}(-k) = b_{21}(k)$. It is necessary to remind that the quantities $(\mathcal{Z}_1^*, J\mathcal{Z}_1)$ and $(\mathcal{Z}_2^*, J\mathcal{Z}_2)$ are purely imaginary and have the opposite signs. Normalizing the eigenvectors $\mathcal{Z}_j$ we can obtain the equations

$$\begin{aligned} (\mathcal{Z}_1^*, J\mathcal{Z}_1) = -(\mathcal{Z}_2^*, J\mathcal{Z}_2), \\ -\alpha(k) \operatorname{sgn} k (\mathcal{Z}_1^*, J\mathcal{Z}_1) = i \operatorname{sgn} k \end{aligned}$$

if the quantity $i(\alpha(k))(\mathcal{Z}_1^*, J\mathcal{Z}_1) > 0$. As a result the matrix B corresponding to the two close eigenvectors is equal to

$$B = \begin{pmatrix} i\,\mathrm{sgn}\,k & 0 \\ 0 & i\,\mathrm{sgn}\,k \end{pmatrix}$$

and dynamical equation in new variables $a(k)$, $a^*(-k)$ has the same form (21) as in the case of instability.

Now we consider the unstable area of wavenumbers adjacent to the marginally stable point. In this area $(\mathcal{Z}_1^*, J\mathcal{Z}_1) = (\mathcal{Z}_2^*, J\mathcal{Z}_2) = 0$, and the expression $\lambda(k) = (\mathcal{Z}_1^*, J\mathcal{Z}_2) \neq 0$, and $(\mathcal{Z}_2^*, J\mathcal{Z}_1) = -\lambda^*(k)$. As a consequence the coefficients of matrix B in the domain of instability are equal to:

$$b_{11}(k) = b_{22}(k) = \frac{\alpha(k)}{2}\,\mathrm{sgn}\,k\,[\lambda(k) + \lambda^*(k)],$$

$$b_{12}(k) = \begin{cases} \alpha^2(k)[\lambda(k) - \lambda^*(k)], & k > 0, \\ (1/4)[\lambda(k) - \lambda^*(k)], & k < 0, \end{cases}$$

$$b_{21}(k) = \begin{cases} (1/4)[\lambda(k) - \lambda^*(k)], & k > 0, \\ \alpha^2(k)[\lambda(k) - \lambda^*(k)], & k < 0. \end{cases}$$

Normalizing the eigenvectors we can do λ to be real, and as a consequence $b_{11} = b_{22} = \mathrm{sgn}\,k\alpha(k)\lambda(k)$, $b_{12} = b_{21} = 0$, and, varying the arbitrary constants, we obtain $b_{11} = b_{22} = i\,\mathrm{sgn}\,k$. So, one can see that the equation (21) is valid in both stable and unstable areas of k adjacent to the marginal point k_0.

Now we have to express the quadratic part of Hamiltonian in the new variables corresponding to the two close eigenvalues. The components of matrix h_j in the new variables are equal to (omitting calculations):

$$h_{11}(k) = h_{22}(k) = \Omega(k) = -\,\mathrm{sgn}\,k(\omega_1 + \omega_2)/2,$$

$$h_{12}(k) = \tilde{\Omega}(k) = \begin{cases} 1, & k > 0, \\ (\omega_1 - \omega_2)^2/4, & k < 0, \end{cases}$$

$$h_{21}(k) = \tilde{\Omega}(-k) = \begin{cases} (\omega_1 - \omega_2)^2/4, & k > 0, \\ 1, & k < 0, \end{cases}$$

and, as a consequence, the expression for the part of energy corresponding to the coalesced mode in new variables has the form:

$$E = \int\limits_{\Delta k} \left[\frac{1}{2}(\Omega(k)(a(k)a(-k) + \text{c.c.}) + \tilde{\Omega}(k)a^*(k)a(k)\right] dk \tag{23}$$

where Δk is the interval of proximity of roots. The width of this interval depends on the parameter of nonlinearity ϵ, namely, $\Delta k \approx \epsilon^4 k_0$.

It is quite natural that at the point k_0, where $\omega_1 = \omega_2$, we obtain for the coefficients of energy matrix h the following values:

$$h_{11}(k) = -\omega\,\mathrm{sgn}\,k, \quad h_{22}(k) = -\omega\,\mathrm{sgn}\,k,$$
$$h_{12}(k) = h_{21}(-k) = \theta(k),$$

The expression for energy density in this point coincides with (23).

6 Evolution equations in the neighbourhood of marginally stable points.

The introduced canonical variables are useful in many applications. In particular the amplitude equation for a weakly nonlinear wave-packet in the neighbourhood of a marginally stable point k_0 can be obtained using the canonical equation (21). It is convenient to write down this equation in terms of variables $a_1(k)$ and $a_2(k)$. We have in mind that now

$$a(k) = \begin{cases} a_1(k), & k > 0, \\ a_2(k), & k < 0, \end{cases} \quad a^*(-k) = \begin{cases} -a_2(k), & k > 0, \\ a_1(k), & k < 0. \end{cases}$$

It follows from (21) that the linearized dynamical equations in these variables are

$$\begin{aligned} \dot{a}_1(k) &= -i\,[(\omega_1 + \omega_2)a_1/2 + a_2], \\ \dot{a}_2(k) &= -i\left[(\omega_1 + \omega_2)a_2/2 + (\omega_1 - \omega_2)^2 a_1/4\right]. \end{aligned} \tag{24}$$

We will now obtain the evolution equation for the wave-packet with wavenumbers adjacent to the point of coalescence of modes k_0 from both of sides. It means that we take into account both stable and unstable parts of spectrum. We regard the case when k_0 is not the critical point of instability, i.e. $\partial L/\partial k \neq 0$ in this point. When the relation $\partial L/\partial k = 0$ holds, evolution equation differs from that given below.

It is necessary to mention that if the function $\omega(k)$ is singular in the vicinity of k_0, the functions $(\omega_1 + \omega_2)/2$ and $(\omega_1 - \omega_2)^2/4$ are analytical functions in the vicinity of k_0, and can be expanded into into a series in powers of $\Delta k = k - k_0$.

For simplicity we shall obtain the evolution equation for a wave packet in case $n = 1$, when the dispersion equation has the form (Zakharov, 1974):

$$\det\begin{vmatrix} A(k) & B^*(k) + i\omega \\ B(k) - i\omega & C(k) \end{vmatrix} = 0,$$

and the solutions of it are

$$\omega_{1,2} = B_{\mathrm{im}}(k) \pm \sqrt{A(k)C(k) - B_{\mathrm{re}}^2(k)}$$

and, as follows,

$$(\omega_1 + \omega_2)/2 = B_{\mathrm{im}}(k),$$

$$(\omega_1 - \omega_2)^2/4 = A(k)C(k) - B_{\mathrm{re}}^2(k) \equiv \phi(k), \quad \phi(k_0) = 0$$

Both the functions are analytical in the vicinity of marginally stable point k_0, and can be expanded into a series by $\Delta k = k - k_0$:

$$B_{\mathrm{im}}(k) = B_{\mathrm{im}}(k_0) + B'_{\mathrm{im}}(k_0)\Delta k + \ldots = \Omega(k),$$
$$\phi(k) = \phi'(k_0)\Delta k + \phi''(k_0)\Delta k^2 + \ldots$$

We substitute them into the equations (24). Then we eliminate the variable a_2 and set

$$a_1 = A(\tau, k)\exp(-i\Omega(k_0)t),$$

where the amplitude $A(\tau, k)$ depends on the slow time τ. After this we do the inverse Fourier transform and as a result we obtain the linear part of the evolution equation for amplitude $A(\tau, x)$ of wave packet in the vicinity of marginally stable wavenumber k_0. It is evident that the main nonlinear term is of the third order in A, and we obtain the evolution equation in the form:

$$\left(\frac{\partial}{\partial\tau}-\Omega'\frac{\partial}{\partial x}\right)^2 A - i\phi'\frac{\partial A}{\partial x} + \frac{1}{2}\phi''\frac{\partial^2 A}{\partial x^2} = iT\,|A|^2 A. \quad (25)$$

Retaining the terms of main order in linear part, namely, proportional to $\partial A/\partial x$ and $\partial^2 A/\partial\tau^2$, we obtain from (25) the evolution equation in the form of the nonlinear Schrödinger equation, but with the roles of x and τ interchanged. It is clear that the linear part of evolution equation (25) can be written heuristically by the way suggested by Whitham (1974). If the spatial and temporal modulations of amplitude $A(\tau, k)$ in wave-packet are small compared with the characteristic wavelength and wave period, the linear evolution equation for A is

$$L(\omega_0 + i\partial/\partial t, k_0 + i\partial/\partial x)A(x,t) = 0.$$

Expanding L as a series in partial derivatives we obtain the linear part of equation (25). Nevertheless the proposed method is extremely useful, because the calculation of the nonlinear coefficients using Hamiltonian approach is essentially simplified comparing with calculation based on the primitive equations.

Using the equations (21) we can derive the evolution equation for nonlinear wave packet in the domain of instability when the two modes are weakly coupled. Let the dispersion equation have the form

$$(\omega - \Omega_1(k))(\omega - \Omega_2(k)) = \pm\varepsilon^2(k),$$

where $\varepsilon(k)$ is a small quantity. It is evident that so long as $\Omega_1(k)$ and $\Omega_2(k)$ are well apart, we can consider these waves separately. If there is a point k_0, where $\Omega_1(k_0) = \Omega_2(k_0) = \omega_0$, i.e. the two dispersion curves regarded separately intersect, we have in the vicinity of this point the following expansions:

$$\frac{\omega_1+\omega_2}{2} \cong \omega_0 + (V_1+V_2)\Delta k/2$$

$$\frac{(\omega_1-\omega_2)^2}{4} \cong \frac{1}{4}(V_1-V_2)^2\Delta k^2 \pm \varepsilon^2(k)$$

where $\Delta k = k - k_0$, $V_{1,2}$ are the group velocities of waves considered without coupling, i.e. $V_i = \Omega'_{ik}$. Substituting these expressions into equations (23), eliminating central wave frequency, doing the inverse Fourier transform and taking into account the main nonlinear terms, we get the following equation:

$$\left(\frac{\partial}{\partial\tau} + V_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial\tau} + V_2\frac{\partial}{\partial x}\right)A \pm \varepsilon^2(k)A = T\,|A|^2 A.$$

It is a well-known evolution equation at the point of linear inviscid instability in the absence of the mean flow variation (Dodd et al., 1982).

7 Conclusion

For a broad class of Hamiltonian systems describing the weakly nonlinear wave dynamics for finite number of modes the general method of canonical variables construction is proposed. Both in the stable and unstable area the canonical variables are the standard normal variables connected with the notion of normal modes. In the vicinity of marginally stable points when we have the two-fold roots in the spectrum of eigenvalues corresponding to linearized system of equations, the new canonical variables connected with the notions of coalesced mode are constructed. The equations written in these variables can be used to obtain the evolution equations in this domain.

Acknowledgements. The author is grateful to Dr. V. I. Shrira for useful discussions. The work was partially supported by Russian Fund of Fundamental Research under grant No. 94-05-16200 and by INTAS under contract No. 93-1373.

References

Arnold, V.I., *Mathematical Methods of Classical Mechanics.* Springer, Berlin, 1978.

Dodd, R.K., Eilbeck, J.C., Gibbon, G.D., and Morris, H.C., *Solitons and nonlinear wave equations.* Academic Press Inc., 1982.

Dubrovin, B.N., Novikov, S.P., and Fomenko, A.T., *Modern Geometry, Vol. 1.* Springer, Berlin, 1984.

Goncharov, V.P., Nonlinear waves in flows that are vorticity-homogeneous in a layerwise manner, *Izvestiya Academy of Sciences USSR, Atmospheric and Oceanic Physics, 22,* 5, 355–361, 1986.

Goncharov, V.P. and Pavlov, V.P., *The problems of hydrodynamics in Hamiltonian description* (in Russian). Moscow State University, Moscow, 1993.

Ignatov, A.M., The interaction of waves in active conservative media (in Russian), *JETPh. 87,* 5, 1652–1659, 1984 .

Ostrovskiy, L.A., Rybak, S.A., and Tsimring, L.Sh., Negative energy waves in hydrodynamics (in Russian), *Uspekhi Fiz. Nauk, 150,* 3, 417–437, 1986.

Voronovich, A.G., Hamiltonian formalism for internal waves, *Izvestiya Academy of Sciences USSR, Atmospheric and Oceanic Physics, 15,* 1, 52–64, 1979.

Whitham, G.B., *Linear and nonlinear waves.* Wiley-Interscience Publication, 1974.

Zakharov, V.E., Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech.Tech. Phys., 2,* 190–194, 1968.

Zakharov, V.E., Hamiltonian formalism for waves in nonlinear dispersive media (in Russian), *Izv. Vyssh. Uchebn. Zaved. Radiophysika, 17,* 4, 431–453, 1974.