Nonlinear Processes in Geophysics (1994) 1: 219 - 223

Nonlinear Processes in Geophysics

On the possibility of wave-induced chaos in a sheared, stably stratified fluid layer

W. B. Zimmermann[1] and M. G. Velarde[2]

[1] Department of Chemical Engineering, UMIST, Sackville Street, PO Box 88, Manchester M60 1QD, England
[2] Instituto Pluridisciplinar, Universidad Complutense, Paseo Juan XXIII, No. 1, Madrid 28040, Spain

Received 6 May 1994 - Accepted 26 June 1994 - Communicated by V. Shrira

Abstract. Shear flow in a stable stratification provides a waveguide for internal gravity waves. In the inviscid approximation, internal gravity waves are known to be unstable below a threshold in Richardson number. However, in a viscous fluid, at low enough Reynolds number, this threshold recedes to $Ri = 0$. Nevertheless, even the slightest viscosity strongly damps internal gravity waves when the Richardson number is small (shear forces dominate buoyant forces). In this paper we address the dynamics that approximately govern wave propagation when the Richardson number is small and the fluid is viscous. When $Ri << 1$, to a first approximation, the transport equations for thermal energy and momentum decouple. Thus, a large amplitude temperature wave then has little effect on the fluid velocity. Under such conditions in the atmosphere, a small amplitude "turbulent burst" is observed, transporting momentum rapidly and seemingly randomly. A regular perturbation scheme from a base state of a passing temperature wave and no velocity disturbance is developed here. Small thermal energy convection–momentum transport coupling is taken into account. The elements of forcing, wave dispersion, (turbulent) dissipation under strong shearing, and weak nonlinearity lead to this dynamical equation for the amplitude A of the turbulent burst in velocity:

$$A_\xi = \lambda_1 A + \lambda_2 A_{\xi\xi} + \lambda_3 A_{\xi\xi\xi} + \lambda_4 A A_\xi + b(\xi)$$

where ξ is the coordinate of the rest frame of the passing temperature wave whose horizontal profile is $b(\xi)$. The parameters λ_i are constants that depend on the Reynolds number. The above dynamical system is known to have limit cycle and chaotic attrators when the forcing is sinusoidal and wave attenuation negligible.

Correspondence to: M.G. Velarde

1 Introduction

1.1 Linear dispersive waves as harmonic oscillators

Waves manifest in a great many physical circumstances from the classical studies of optics and water surfaces to plasma and condensed matter. In nearly all of these processes, the first analysis of the wave propagation comes from assuming the amplitude of the wave is infinitesimally small. It follows that linear waves are solutions to linearized equations. This simplification, without considering complicated interactions of linear waves with boundaries, leads to only one distinguishing factor for waves of any given wavenumber–its phase velocity. If waves of different colour have different phase velocity, a wave packet disperses. If the dispersion effect is small and the waves neither grow nor decay, to a good approximation, in isotropic media the phase relation is

$$\omega = ck - k^3 \tag{1}$$

which is equivalent to a wave equation for the amplitude A of the wave

$$A_t + cA_x + A_{xxx} = 0 \tag{2}$$

The above equation is the linearized Korteweg-deVries equation and is generally applicable to many weakly dispersive wave phenomena if the amplitude is small. Particular interest focuses on waves with permanent form, since many signals are observed that propagate unchanged over large distances have permanent form. Mathematically, this means that there is a frame of reference, given by a transformation $\tau = t$ and $\xi = x - c_0 t$. with relative velocity c_0 in which the amplitude is steady. Applying this transformation to (2) and requiring $\frac{\partial A}{\partial \tau} = 0$ gives

$$(c - c_0) A_\xi + A_{\xi\xi\xi} = 0 \tag{3}$$

Upon one integration with respect to ξ, we arrive at

$$A_{\xi\xi} + (c - c_0) A = 0 \tag{4}$$

The above is the dynamical equation governing a Hooke's Law spring–the classical harmonic oscillator. The usual harmonic oscillators oscillate in time, but because the wave has permanent form, it oscillates in the space-time coordinate ξ. Viewed in this reference frame, to a first approximation, isotropic dispersive waves are harmonic oscillators. If the waves are weakly nonlinear and weakly dispersive, we can generalize the above analogy. It should be noted that Equation (2) does not support localised solutions (solitary waves). Some new physical mechanism must be incorporated balancing dispersion to have solitary waves. Throughout this paper, we will treat a specific waveguide that manifests nonlinear dispersive waves–a sheared, stably stratified fluid layer–and gain insight to the possible attractors of the wave motion by exploiting the analogy between waves and oscillators that comes from the search for the reference frame in which waves have permanent form.

1.2 Nonlinear dispersive waves as nonlinear oscillators

There has been substantial activity in understanding the low dimensional dynamics of solitary waves propagating internally in a waveguide provided by a stably stratified fluid system with shear. Since the most prominent applications are meteorological and oceanic, the traditional view is to idealize the system as inviscid. Shallow layer approximations lead to an amplitude equation for density and stratification of the form

$$A_\tau + (c_0 - c)A_\xi + \beta A A_\xi + \alpha A_{\xi\xi\xi} = 0 \tag{5}$$

α and β are constants which depend on the background conditions of vertical stratification and wind shear. Equation (5) is the KdV equation, after a Galilean transformation to the frame of reference $\xi = x - ct$. If $c = c_0$, then if the nonlinearity exactly cancels the dispersion, then $A_\tau = 0$ and the wave has permanent form. It is equivalent to require c to be the reference frame where $A_\tau = 0$ and hereafter this is adopted. Equation (5) is a restatement of the weak nonlinear analysis of Benjamin (1966) and Benney (1966) that arbitrary but stable density stratification and shearing is unstable to the propagation of large amplitude solitary waves of permanent form. A prescription for the phase velocity, c_0, and the vertical structure of the streamfunction and density were given in terms of eigensolutions to a specific boundary value problem, and the coefficients α and β were given by quadratures of the eigenfunction density and shear profiles. Maslowe and Redekopp (1980) computed α and β for the specific case of linear shear and density stratification, where α and β depend only on the gradient Richardson number, the *depth of waveguide-wavelength* ratio $m = (h/\lambda)$ and the *amplitude-depth of waveguide* ratio (ε). Integration of Equation (5) leads to

$$A_{\xi\xi} + \frac{c_0 - c}{\alpha} A + \frac{\beta}{2\alpha} A^2 = 0 \tag{6}$$

which is the dynamical equation of a harmonic oscillator, if $\beta = 0$, corresponding to linear dispersive waves as in the previous section, and acting as a simple nonlinear oscillator with a phase space of dimension two if $\beta \neq 0$. Equation (6) supports steady periodic solutions corresponding to cnoidal waves and also homoclinic solutions corresponding to the famous $sech^2\xi$ solitary waves.

Recently, Zimmerman and Velarde (1994) have reconsidered internal solitary wave evolution in a waveguide of a stably stratified and sheared fluid layer, paying particular attention to weak viscous effects. They found the attenuation of nonlinear waves from friction and thermal conduction losses is magnified as the Richardson number approaches 1/4, and additionally, the diffusion of wave energy is important as well. The equation governing the evolution of solitary waves takes the form

$$A_\xi + \gamma_0 A + \gamma_2 A_{\xi\xi} + \alpha A_{\xi\xi\xi} + \beta A A_\xi = 0 \tag{7}$$

where γ_0 and γ_2 depend on the Prandtl number, Reynolds number and Ri. Equation (7) is truncated to $O(m^2)$ and $O(\varepsilon)$.

For an arbitrary shear flow with nonuniform vorticity profile, we expect diffusion of wave energy to dominate attenuation by viscous forces. Hence $\gamma_0 << \gamma_2$. If we neglect γ_0, then upon integrating, Equation (7) becomes

$$A_{\xi\xi} + \frac{\gamma_2}{\alpha} A_\xi + \frac{1}{\alpha} A + \frac{\beta}{2\alpha} A^2 = 0 \tag{8}$$

This is the equation of a damped linear oscillator if $\beta = 0$, but if $\beta \neq 0$, Equation (8) corresponds to a damped nonlinear oscillator with a phase space dimension two. If γ_2 is diffusive, then solutions to Equation (8) have a simple phase space portrait–spirals to fixed points.

So far, the view of nonlinear internal waves as nonlinear oscillators in a steady frame of reference has only brought us simple dynamics in two-dimensional phase spaces. If we jump to the simplest three-dimensional phase space we have the additional possibility of chaotic dynamics. Consider for a moment the *ad hoc* addition of a sinusoidal forcing term to Equation (8):

$$A_{\xi\xi} + \frac{\gamma_2}{\alpha} A_\xi + \frac{1}{\alpha} A + \frac{\beta}{2\alpha} A^2 = B \sin(\Omega\xi) \tag{9}$$

This is a damped, forced nonlinear oscillator. The specific choice of the nonlinearity gives us the so called Helmholtz-Thompson oscillator (see del Rio et al. (1992)) which is known to have chaotic attractors.

From a dynamical systems viewpoint, Equation (8) barely differs from Equation (6) – the additional term representing damping is a minor but ever present effect in the best known oscillator in mechanical systems - the pendulum. The *ad hoc* addition of the sinusoidal forcing term in Equation (9), however, requires explanation and justification in terms of fluid physics. We interpret the forcing terms as representing a transport process that translates with the same phase velocity as the response wave with amplitude A. It is possible that such a process could be caused by a geometrical change to the bounding surfaces of the waveguide. The premise of Benney (1966), Benjamin (1966) and Zimmerman and Velarde (1994) is that the bounding surface of the waveguide comprises planes parallel to the direction of wave propagation and perpendicular to the gradient of background density. If these surfaces were corrugated sinusoidally, the disturbance would be a sinusoidal forcing, but not one that is then steady in the frame of reference of the wave. We are left with the conclusion that a force taking the form of the forcing term in Equation (9) must be a bulk transport process.

Indeed, we have two scalar fields of the bulk at our disposal – disturbance streamfunction and density. For convenience we will assume that the density stratification arises from differences in potential temperature with height. The assumption leading to Equations (5) and (7) is that to leading order, the stream function and temperature fields are intimately coupled according to

$$\psi(x,y,t) = A(x-ct)\psi^{(0)}(y) + O(\varepsilon, m^2)$$

$$T(x,y,t) = A(x-ct)T^{(0)}(y) + O(\varepsilon, m^2) \qquad (10)$$

Ri was taken to be a free parameter. Approximate solutions to the inviscid transport equations exist for $1/4 < \mathrm{Ri} < \infty$ with the ansatz (10). With finite Reynolds number, the lower bound recedes from $\mathrm{Ri} = 1/4$ and the waveform of the solution is altered to include an oscillatory head. For one of these two scalar fields to provide the forcing function for the oscillatory response of the other, according to Equation (9), the coupling of Equation (10) must be broken.

Davey and Reid (1977) studied the numerical solutions to the fully viscous transport equations in the linear approximation. In addition to finding classical internal gravity waves in the inviscid limit for $\mathrm{Ri} > 1/4$ they also studied the lower Richardson number modal structure, $0 < \mathrm{Ri} < 1/4$, where internal waves of the form Equation (10) grow rapidly and break up in the inviscid approximation. For the special case of $\mathrm{Ri} = 0$, they noticed the mathematically obvious fact that the linearized equations for energy and momentum transport decouple. That is, there exist solutions to the temperature equation that are travelling waves when the disturbance streamfunction vanishes identically. These are known as the temperature modes. Further, when the disturbance temperature vanishes identically, there exist solutions to the momentum equation (the Orr-Sommerfield equation). These are known as the velocity modes. Whenever Ri is small but nonvanishing, Davey and Reid demonstrated that a perturbation scheme in Ri about a base state that is either a velocity mode or a temperature mode gives slightly coupled temperature and streamfunction fields which are nearly pure temperature modes or nearly pure velocity modes. This classification scheme works quite well for the exact numerical solutions to the linearised transport equations throughout the interval $0 \leq \mathrm{Ri} \leq 1/4$.

The observation by Davey and Reid that at high shearing rates there is decoupling, or rather weak coupling, between temperature and streamfunction enables us to satisfy the conditions necessary for Equation (9) to characterise the approximate dynamics of the wave evolution. Firstly, the forcing term may be due to a passing temperature wave with phase velocity c_0. With strong shearing, the buoyant forces associated with the temperature wave passing will be weak and make negligible contribution to the momentum equation at leading order in Ri. Thus, we can assume that the response in streamfunction will be small as O(Ri). Secondly, we need tangible dissipation effects for Equation (9) to be the description of the dynamics. In laminar flows, viscosity and thermal conductivity provide the dissipative mechanisms. In oceanic and atmospheric flows, however, the dimensionless measure of dissipation by viscous forces –the inverse Reynolds number– is very small and unlikely to be important in the bulk, but rather only in boundary regions. A much more plausible source of dissipation is turbulent dissipation. The usual model of eddy viscosity (ν_T) gives a large contribution to the momentum balance in turbulent shear flows:

$$-\overline{u_1' u_3'} \sim \nu_T \frac{\partial U}{\partial x_3}$$

where $\mathbf{u}'$ is the disturbance velocity and U is the mean shear flow. Further, the stronger the shearing, the larger ν_T. It follows that with high shear (low Ri) we achieve quasi-decoupling of the passing temperature wave and strong turbulent dissipation in meteorological flows – the prerequisites for Equation (9) to describe the dynamics. Indeed, in Section 2 we will put this claim on the firm mathematical footing of perturbation theory along the lines already outlined.

2 Waves in a stably stratified shear layer

2.1 Basic Equations and Scaling

Consider a Boussinesq fluid occupying a shallow fluid layer bounded below by a plate and heated from above. The temperatures are maintained constant on each planar surface. The depth of the layer is uniform, h, with gravity acting antiparallel to the gradient of the temperature so that the stratification is stable. We require that the basic flow, temperature profile, density stratification and constant pressure are only functions of the vertical: $\hat{u}(y)$, $\hat{T}(y)$, $\hat{\rho}(y)$ and $\hat{p}(y)$. Velocity components in the (x,y) directions are denoted (u,v). In order to simplify the problem, we will take $\hat{u} = y$ and $\hat{\rho} = 1 - \beta y$ (constant density gradient). We have scaled as follows –we expect long waves of characteristic horizontal scale λ, so that the co-ordinates (x,y) are scaled by (λ,h); the velocities are scaled with (U,mU), density with ρ_0, pressure with $\rho_0 gh$, temperature with $\beta_* h$, where β_* is the characteristic temperature gradient, and time with the horizontal convection time λ/U. The velocity scale U is constructed from the characteristic Couette shear rate γ, i.e. $U = \gamma h$. For this two-dimensional flow, the Boussinesq approximation admits a dimensionless disturbance streamfunction, $\psi(x,y,t)$. Neglecting molecular transport processes of viscosity and thermal conductivity, but retaining turbulent eddy viscosity in the model, we consider these Boussinesq equations for the disturbance heat and momentum transport.

$$\frac{\partial T}{\partial t} + yT_x - \psi_x + \varepsilon(\psi_y T_x - \psi_x T_y) = 0 \tag{11}$$

$$\begin{aligned} &[\psi_{yt} + y\psi_{yx} - \psi_x]_y + \mathrm{Ri}T_x = \\ &\varepsilon[\psi\psi_{xy} - \psi_{yy}\psi_x]_y - m^2[\psi_{xt} + y\psi_{xx}]_x \\ &-\varepsilon m^2[\psi_y\psi_{xx} - \psi_x\psi_{xy}]_x \\ &+\delta\psi_{yyyy} + 2\delta m^2\psi_{xxyy} + \delta m^4\psi_{xxxx} \end{aligned} \tag{12}$$

Pressure has been eliminated by taking the curl of the Navier-Stokes equations. For boundary conditions, we require the temperature be maintained constant at the upper and lower surfaces and no slip on the boundaries:

$$\begin{aligned} &\psi\mid_{y=0} = \psi\mid_{y=1} = 0 \\ &\psi_y\mid_{y=0} = \psi_y\mid_{y=1} = 0 \\ &T\mid_{y=0} = T\mid_{y=1} = 0 \end{aligned} \tag{13}$$

The parameter $\varepsilon << 1$ measures the *wave amplitude - depth* ratio and is formally small so that the disturbed streamlines differ little from Couette shear and the disturbed temperature differs little from a linear profile. The dimensionless parameters that arise are $\delta = 1/(m^2 Re)$ where the Reynolds number is based on λ and either kinematic viscosity ν or turbulent viscosity ν_T, $\beta = \alpha\beta_+ h$ (the stratification parameter) and the Richardson number $\mathrm{Ri} = \alpha\beta_* gh^2/U^2$. The Boussinesq approximation sets $\beta = 0$, so that the velocity field is solenoidal, but retains the buoyant force where it is the driving force in the Navier-Stokes equations through the non-vanishing Ri.

2.2 Perturbation Scheme

Wave solutions that are steady and of permanent form in the Galilean reference frame $\xi = x - ct$ are sought, so we rewrite (12) as

$$\begin{aligned} &[(y-c)\psi_{y\xi} - \psi_\xi]_y + \mathrm{Ri}T_\xi = \varepsilon[\psi\psi_{\xi y} - \psi_{yy}\psi_\xi]_y \\ &-m^2(y-c)\psi_{\xi\xi\xi} - \varepsilon m^2[\psi_y\psi_{\xi\xi} - \psi_\xi\psi_{\xi y}]_\xi \\ &+\delta\psi_{yyyy} + 2\delta m^2\psi_{\xi\xi yy} + \delta m^4\psi_{\xi\xi\xi\xi} \end{aligned} \tag{14}$$

If $Ri = 0$, the system of equations (11) and (14) is identically satisfied by a temperature wave disturbance and vanishing disturbance streamfunction:

$$\begin{aligned} &T(x,y,t) = b(\xi)\,\theta(y) \\ &\psi(x,y,t) = 0 \end{aligned} \tag{15}$$

If the temperature disturbance takes the form above, however, the streamfunction is not constrained to vanish when $Ri > 0$. Rather, ignoring dispersive and nonlinear effects in (14), the limit $\varepsilon = m^2 = 0$, the disturbance streamfunction is estimated by

$$M[\psi] = -Rib_\xi\theta \tag{16}$$

where M is the linear operator of (19) below. It follows that $\psi = O(Ri)$, unless the operator L has eigenfunctions, i.e. solutions to the homogeneous equation $L[\psi_h] = 0$. With the assumption that these eigenfunctions exist, it is appropriate to seek solutions that are wave-like in the horizontal and stationary in the vertical, namely

$$\psi(x,y,t) = A(\xi)\,\phi(y) + O(\varepsilon, \mathrm{Ri}, m^2) \tag{17}$$

Since it is only the streamfunction response to the temperature wave $b\theta$ that is sought, the corrections to the temperature field from the convection via the disturbance velocity field are neglected. Motivated by (16) and (17), the formal scaling of $Ri \sim \varepsilon \sim m^2$ is adopted, and without loss of generality $Ri = \varepsilon = m^2$, since corrections from weak buoyant forces, weak dispersion, and weak nonlinearity are likely if the mechanisms they represent are to contribute to the waveform. Truncating the momentum equation (14) at $O(1)$ yields the following boundary value problem

$$-\delta\phi_{yyyy}A + [(y-c)\phi_y - \phi]_y A_\xi = 0 \tag{18}$$

Equation (18) is a parabolic partial differential equation which separates if $A_\xi = \lambda_1 A$ and we arrive at the following boundary value problem for $\psi^{(0)}$:

$$\begin{aligned} &-\delta\phi_{yyyy} + \lambda_1[(y-c)\phi_y - \phi]_y = 0 \\ &\phi\mid_{y=0} = \phi\mid_{y=1} = \phi_y\mid_{y=0} = \phi_y \mid y=1 = 0 \end{aligned} \tag{19}$$

which can be solved for eigenvalues $\lambda^{(n)}(c)/\delta$ and eigenfunctions ϕ_n, where n is an integer. We then realise that the separation condition is only correct to $O(Ri)$ and include corrections

$$A_\xi = \lambda_1 A + \lambda_2 RiA_{\xi\xi} + \lambda_3 RiA_{\xi\xi\xi} + \lambda_4 \mathrm{Ri}AA_\xi - \mathrm{Ri}b_\xi \tag{20}$$

and correct Equation (17) as well according to

$$\begin{aligned} \psi(x,y,t) = &A(\xi)\phi(y) + \mathrm{Ri}D^{(1)}(\xi)\psi^{(1)}(y) \\ &+\mathrm{Ri}D^{(2)}(\xi)\psi^{(2)}(y) + \mathrm{Ri}D^{(3)}(\xi)\psi^{(3)}(y) \end{aligned} \tag{21}$$

and we find the hierarchy separates if $D_\xi^{(1)} = AA_\xi$, $D_\xi^{(2)} = A_{\xi\xi}$ and $D_\xi^{(3)} = A_{\xi\xi\xi}$. With these requirements, we arrive to the following inhomogeneous boundary value problems for the functions $\psi^{(i)}$:
O(Ri) problem 1:

$$L\psi^{(1)} = -\lambda_4[(y-c)\phi_y - \phi]_y + [\phi_{yy}\psi - (\phi)^2]_y$$

$$\psi^{(1)}|_{y=0} = \psi^{(1)}|_{y=1} = \psi^{(1)}_y|_{y=0} = \psi^{(1)}_y|_{y=1}0 \tag{22}$$

$O(\mathrm{Ri})$ problems 2 and 3:

$$\begin{aligned} L\psi^{(2)} &= -\lambda_2[(y-c)\phi - \phi]_y + 2\delta\phi_{yy} \\ L\psi^{(3)} &= -\lambda_3[(y-c)\phi - \phi]_y + (y-c)\phi \\ \psi^{(2)}|_{y=0} &= \psi^{(2)}|_{y=1} = \psi^{(3)}|_{y=0} = \psi^{(3)}|_{y=1} = 0 \\ \psi^{(2)}_y|_{y=0} &= \psi^{(2)}_y|_{y=1} = \psi^{(3)}_y|_{y=0} = \psi^{(3)}_y|_{y=1} = 0 \end{aligned} \tag{23}$$

L is the linear operator of (19). Application of the Fredholm Alternative Theorem gives a constraint uniquely determining each of the coefficients α_i – that the inhomogeneous terms in Equations (22 - 23) be compatible with the boundary conditions.

If we take ξ-dependence of the passing temperature wave to be sinusoidal, $b(\xi) = B' \sin(\Omega\xi)$, then Equation (20) is a simple nonlinear oscillator with a phase space of dimension four, similar to 9. Equation (19) suggests that $\lambda_1 \sim \delta$ for the two terms to be balanced. Thus if dissipation is weak, it may be appropriate to neglect the $\lambda_1 A$ term in Equation (20). In this case, Equation (20) may be integrated to yield

$$A_{\xi\xi} + \frac{\lambda_2}{\lambda_3}A_\xi - \frac{1}{\lambda_3}A + \frac{\lambda_4}{\alpha\lambda_3}A^2 = -\mathrm{Ri}B' sin(\Omega\xi) \tag{24}$$

The λ_i are defined by comparison with Equation (20). This is precisely the form of Equation (20), proposed *ad hoc* from a dynamical systems approach to wave propagation. We now propose that Equation (refeq:corr), a forced KdV-Burgers equation, models the streamfunction response to a passing temperature wave in a strongly sheared fluid layer. If the passing temperature wave is sinusoidal, and if dissipation from turbulence is weak (which is usually the case in a stably stratified medium since the stratification tends to suppress turbulent three-dimensional modes), then the Helmholtz-Thompson nonlinear oscillator of Equation (9) and Equation (24) is a good approximation of the streamfunction dynamics. Indeed, if this proposition is valid, we would find regimes of temperature wave induced chaos, since the Helmholtz-Thompson nonlinear oscillator has chaotic attractor solutions. Further, the embedded dimension of the chaotic attractors would be three or four depending on the importance of dissipation.

3 Conclusions

Stably stratified shear flows provide a waveguide for internal solitary waves that are classical KdV solitons when the Richardson number is uniformly high throughout the layer. The solitons propagate due to an intimate coupling of the disturbance temperature and velocity fields (see Equation (10)). In contrast, when the Richardson number is uniformly small throughout the layer, the leading order behaviour of the velocity field is also small, $O(Ri)$ (see Equation (14)). The temperature field can be any passing wave. In this case, dissipative mechanisms (eddy viscosity or kinematic viscosity) must be important even at high *Re* as the velocity gradients are large in the background shear flow. Thus, a fluid mechanical model of the momentum and temperature transport leads to (20) as a first solvability condition– a wave equation for an attenuated, forced, nonlinear, dispersive and diffusive wave. If wave attenuation can be neglected, in the frame of reference of the wave, the dynamics are equivalent to a damped, forced nonlinear oscillator (24), which is known to have regimes with chaotic asymptotic attractors [Thompson (1989)]. It is our fervent hope that this work will spur the analysis of turbulent bursts such as those observed over an Antarctic ice shelf [Rees and Rottman (1994)] by phase portrait reconstruction to demonstrate that the dynamics are indeed those of low dimensional chaos.

Future work will consider the parameter regimes where (20) has chaotic attractors and the temporal stability of this spatial chaos. It is reasonable to assume that if the dynamics of the time reduced equations (stationary in the moving frame) have low dimensionality and are chaotic, inclusion of one more phase dimension (time) cannot increase the dimensionality of the attractor by more than one dimension. Nevertheless, the dynamics of spatial-temporal chaos may be essentially different in quality than for the stationary chaotic wave discussed here.

Acknowledgements. W.Z. would like to thank the National Science Foundation and North Atlantic Treaty Organization for a postdoctoral fellowship. We acknowledge support from NATO CRG 940242. This research has been sponsored in part by DGICYT (Spain) under Grant PB90-264.

References

T.B. Benjamin. "Internal waves of finite amplitude and permanent form.". *J. Fluid Mech.*, 25:241–270, 1966.

D.J. Benney. "Long nonlinear waves in fluid flows.". *J. Math. Phys.*, 45:52–63, 1966.

S.A. Maslowe and L.G. Redekopp. "Long nonlinear waves in stratified shear flows.". *J. Fluid Mech.*, 101(2):321–348, 1980.

W.B. Zimmerman and M.G. Velarde *"Internal solitary wave in a sheared, stably stratified fluid layer."*. Fluid Physics M.G. Velarde and C.I. Christov, editors. World Scientific, pp. 341-352, 1994.

E. del Río, A. Rodríguez-Lozano, and M.G. Velarde. "A prototype Helmholtz-Thompson nonlinear oscillator.". *Rev. Sci. Instrum.*, 69(3):4208–4212, 1992.

A. Davey and W.H. Reid. "On the stability of stratified viscous plane Couette flow: Part 1. constant buoyancy frequency.". *J. Fluid Mech.*, 80(3):509–525, 1977.

J. M. T. Thompson. "Chaotic phenomena triggering the escape from a potential well ". *Proc Roy Soc A*, 421:195–225, 1989.

J.M. Rees and J.W. Rottman. "Analysis of solitary disturbances over an antarctic ice shelf.". *aBoundary-Layer Met.*, 69:285–310, 1994.