**Natural Hazards
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On Benjamin-Feir instability and evolution of a nonlinear wave with finite-amplitude sidebands

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Abstract. In the past decade it became customary to relate the probability of appearance of extremely steep (the so-called freak, or rogue waves) to the value of the Benjamin-Feir Index (BFI) that represents the ratio of wave nonlinearity to the spectral width. This ratio appears naturally in the cubic Schrödinger equation that describes evolution of unidirectional narrow-banded wave field. The notion of this index stems from the Benjamin-Feir linear stability analysis of Stokes wave. The application of BFI to evaluate the evolution of wave fields, with non-vanishing amplitudes of sideband disturbances, is investigated using the Zakharov equation as the theoretical model. The present analysis considers a 3-wave system for which the exact analytical solution of the model equations is available.

1 Introduction

Extreme (rogue, or freak) waves appear locally and spontaneously in the sea. Usually waves with heights exceeding twice the significant wave height are considered to belong to this category. Parameters of those waves, and in particular the frequency of their appearance, constitute an important design parameter in marine and off-shore engineering. There are numerous indications that under certain circumstances these waves may appear more frequently than expected for a random Gaussian sea. In recent years, numerous experimental studies aimed at experimental investigation of probability of appearance of extremely high waves were carried out under controlled conditions in large experimental facilities (Onorato et al., 2006, 2009; Shemer and Sergeeva, 2009; Waseda et al., 2009; Shemer et al., 2010 and additional references therein). Both unidirectional random wave fields and those with directional spreading were considered. Random waves with an initially narrow-banded unidirectional Gaussian spectrum were studied in Shemer and Sergeeva (2009). Three series of experiments were carried out for different values of nonlinearity determined by the characteristic wave amplitude. It was found that the frequency spectrum of the wave field undergoes significant variation in the course of the wave field evolution along the tank. The initially narrow spectrum becomes wider at the early stages of the evolution, up to the distance from the wavemaker ranging from about 70 m for the highest characteristic amplitude employed, to roughly twice that length for the weakest nonlinearity. After attaining maximum, the spectrum gets narrower, its width, however, remains larger than the initial one. The values of all the statistical wave parameters appear to be strongly related to the local spectral width. The deviations of various statistical parameters from the Gaussian statistics increase with the width of the spectrum so that the probability of extremely large waves is the highest when the local spectral width attains maximum. The probability of extremely steep waves also increases with the degree of nonlinearity.

The effect of the spectral width for a constant nonlinearity ε, defined as a product of the dominant wavenumber k_0 by the r.m.s. value of the surface elevation $\sigma = \left(\overline{\eta^2}\right)^{1/2}$, $\varepsilon = k_0\sigma$, was studied, e.g., in Onorato et al. (2006) and in Shemer et al. (2010) for a unidirectional wave field, and in Onorato et al. (2009) in a large basin for a constant frequency spectrum but with varying directional spreading. These studies demonstrated that as either the frequency range or the directional spreading widen, the probability of appearance of extremely steep waves decreases.

To describe the evolution of random nonlinear wave fields and following Alber (1978) and Onorato et al. (2001), Janssen (2003) introduced the Benjamin-Feir index, BFI $= 2^{1/2}\varepsilon/(\Delta\omega/\omega_0)$, as a ratio of nonlinearity ε to the relative spectral width $\Delta\omega/\omega_0$, ω_0 being the carrier wave frequency and $\Delta\omega$ the width of the frequency spectrum; $\Delta\omega/\omega_0$ can be seen as a measure of wave dispersion.

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Janssen stressed that BFI can be expected to fully charac-
terise the evolution of a unidirectional wave field only within
the framework of the cubic Schrödinger equation (CSE) valid
for a very narrow spectrum, whereas for wider spectra this
is not the case. Nevertheless, quantitative estimates pertain-
ing to the character of random wave fields' evolution that
are based on BFI are quite common. For example, Onorato
et al. (2006) use BFI to analyse the experiments carried out
for a relatively wide JONSWAP spectrum, while Onorato et
al. (2009) even invoke BFI in their experimental study of 2-D
random waves.

While there is a general consensus that BFI can be useful
as a rough indicator characterising the random field evolution
pattern, application of this index as a quantitative predictive
tool raises some questions. One commonly acknowledged
problem with the BFI is a non-unique way in which both the
nonlinearity ε and the spectral width $\Delta\omega/\omega_0$ are defined. For
example, the value of ω_0 is often defined as corresponding
to the spectral peak. A more robust definition of the domi-
nant wave frequency (and, thus, via the dispersion relation, of
the dominant wavenumber k_0) in a random wave field would
invoke spectral momentum based of the free waves spectra
(see, e.g., Shemer et al., 2010). Similarly, the dimension-
less spectral width $\Delta\omega/\omega_0$ is sometimes defined according
to the energy level that corresponds to half of the spectral
peak value, whereas again, the definition based on spectral
momenta seems more appropriate. Alternative possible def-
initions of those parameters can result in quantitatively dif-
ferent values of BFI for identical wave spectra, especially for
wider and asymmetrically shaped spectra like JONSWAP.

An additional problem related to BFI is addressed here. By
its very nature, this index is closely related to the Benjamin-
Feir instability of the Stokes waves, and the coefficient pre-
ceding the ratio of nonlinearity to dispersion in the definition
of BFI reflects either the most unstable sideband disturbance
according to CSE, or the limit of the sideband linear stability
in a slightly different definition of BFI. Moreover, it is quite
common to refer to nonlinear evolution of the wave field as
stemming from the Benjamin-Feir modulational instability.

The Benjamin-Feir (BF) instability is a result of linear
analysis of a system that consists of a carrier wave with am-
plitude a_0 and wave vector k_0, and two infinitesimal side-
band disturbances with wave vectors k_1 and k_2 that satisfy
the condition

$$2k_0 = k_1 + k_2. \tag{1}$$

The linearized (with respect to the sideband disturbances
amplitudes) governing equations are used to carry out the sta-
bility analysis that delineates the boundaries of Stokes wave
instability domain and allows determining the most unstable
modes. Random wave fields have spectral "tails" extending
well beyond the wavenumber domain around the dominant
wave determined by the spectral width. The amplitude of
those spectral components, while small relative to that of

the dominant wave, is finite. It is, thus, instructive to con-
sider nonlinear evolution of a 3-wave system consisting of
a carrier wave with a given steepness, and two side distur-
bances with initially much smaller but still finite amplitudes.
The three-wave system defined by (1) can be seen as a ba-
sic "molecule" in the evolution pattern representing one of
the numerous possible nonlinear interactions in a wave field
with a given spectrum. The advantage of such a system is
that it allows an exact analytic solution of the wave evolution
problem based on the Zakharov (1968) equation (Shemer and
Stiassnie, 1985; Stiassnie and Shemer, 2005). The availabil-
ity of the analytic solution enables the parametric study of
the 3-wave system that can shed light on behaviour of much
more complicated random wave fields.

2   Theoretical background

The nonlinear evolution of each of the waves due to quartet
interactions at the 3rd order in the nonlinearity parameter ε
is described by the Zakharov equation

$$i\frac{\partial B_j}{\partial t} = \int\int\int_{-\infty}^{\infty} V_{j,l,m,n} B_l^* B_m B_n \delta$$

$$(k_j + k_l - k_m - k_n) e^{i\Delta_{j,l,m,n}t} dk_l dk_m dk_n \tag{2}$$

where the interacting quartets are defined by the δ-function
and the frequency detuning

$$\Delta_{j,l,m,n} = \omega_j + \omega_l - \omega_m - \omega_n \tag{3}$$

The interaction coefficients $V_{j,l,m,n} = V(k_j, k_l, k_m, k_n)$ are
given in Krasitskii (1994). The free-surface elevation is re-
lated to the generalized amplitude $B(k,t)$ by

$$\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|k|}{2\omega}\right)^{1/2}\left\{Be^{i(k \times x - \omega t)} + c.c\right\}dk, \tag{4}$$

where the frequency ω is related to wave vector k by the
deep-water linear dispersion relation

$$\omega^2 = g|k| \tag{5}$$

Following Stiassnie and Shemer (1984) and Shemer and
Stiassnie (1985), consider an interaction wave quartet con-
sisting of a carrier wave with the normalized wave vector
$k_0 = (1,0)$ that is taken twice, and a pair of sideband distur-
bances $k_{1,2} = (1 \pm p, \pm q)$, so that (1) is satisfied. The de-
viation from unity of the wavenumber is denoted as p_m for
the most unstable pair of disturbances and p_ℓ for the limit
of the linear instability. The linear instability analysis based
on the CSE yields $p_\ell = 2^{1/2}p_m = 2^{3/2}\varepsilon$, where the nonlin-
earity $\varepsilon = k_0a_0$ represents the carrier wave steepness. The

Janssen's (2003) comment regarding the limits of applicability of BFI is based, in part, on the fact that the linear relation between p_ℓ (or p_m) and ε breaks down when the stability analysis is performed using either the linearized Zakharov equation (Crawford et al., 1981; Stiassnie and Shemer, 1984), or full nonlinear equations (McLean et al., 1981). The dependence of p_m and p_ℓ on the carrier wave steepness ε is plotted in Fig. 1 according to calculations based on the cubic Schrödinger and Zakharov equations. As demonstrated by Stiassnie and Shemer (1984), the linear stability results based on the Zakharov equation do not differ notably from those obtained by McLean et al. It is obvious from Fig. 1 that the two models yield identical results only for vanishingly small values of ε; for steeper carrier waves, the CSE-derived values of p_ℓ and p_m are significantly higher than those based on the more accurate Zakharov model.

The main goal of this study is to analyse nonlinear interactions of a carrier wave with sidebands of initially finite amplitude. For a 3-wave system considered here, the integro-differential equation (2) degenerates into a system of three first-order nonlinear ordinary differential equations:

$$i\frac{dB_0}{dt} = (\Omega_0 - \omega_0)B_0 + 2V_{0,0,1,2}e^{i\Delta\omega t}B_0^* B_1 B_2 \quad (6a)$$

$$i\frac{dB_1}{dt} = (\Omega_1 - \omega_1)B_1 + 2V_{1,2,0,0}e^{-i\Delta\omega t}B_2^* B_0^2 \quad (6b)$$

$$i\frac{dB_2}{dt} = (\Omega_2 - \omega_2)B_2 + 2V_{2,1,0,0}e^{-i\Delta\omega t}B_1^* B_0^2 \quad (6c)$$

where the frequency detuning

$$\Delta\omega = \Delta_{0,0,1,2} = 2\omega_0 - \omega_1 - \omega_2 \quad (7)$$

and the so-called "Stokes-corrected" frequencies are:

$$\Omega_0 = \omega_0 + V_{0,0,0,0}\,|B_0|^2 + 2V_{0,1,0,1}\,|B_1|^2 + 2V_{0,2,0,2}\,|B_2|^2 \quad (8a)$$

$$\Omega_1 = \omega_1 + 2V_{1,0,1,0}\,|B_1|^2 + V_{1,1,1,1}\,|B_1|^2 + 2V_{1,2,1,2}\,|B_2|^2 \quad (8b)$$

$$\Omega_2 = \omega_2 + 2V_{2,0,2,0}\,|B_0|^2 + 2V_{2,1,2,1}\,|B_1|^2 + V_{2,2,2,2}\,|B_2|^2 \quad (8c)$$

Following Stiassnie and Shemer (2005), after some manipulations one obtains from (6)
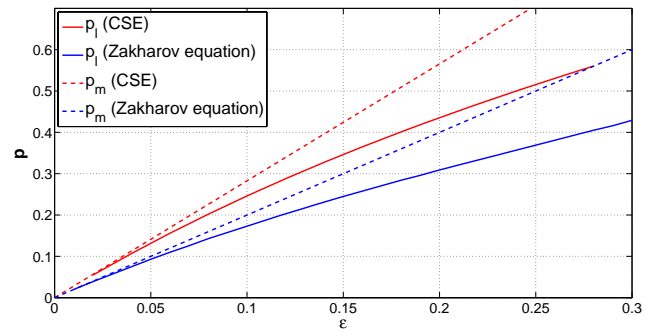
$$\frac{d}{dt}|B_0|^2 = -\frac{d}{dt}|B_1|^2 = -\frac{d}{dt}|B_2|^2$$
$$= 4V_{0,0,1,2}\text{Im}\left\{(B_0^*)^2 B_1 B_2 e^{i\Delta\omega t}\right\} \quad (9)$$

The complex amplitudes of all three waves at all times, thus, depend on a single auxiliary real function $Z(t)$ defined by

$$\frac{dZ}{dt} = \text{Im}\left\{(B_0^*)^2 B_1 B_2 e^{i\Delta\omega t}\right\}, \qquad Z(0) = 0 \quad (10)$$

It follows from (9) and (10) that

$$|B_0|^2 - |\beta_0|^2 = -|B_1|^2 + |\beta_1|^2 = -|B_2|^2 + |\beta_2|^2 = 4V_{0,0,1,2}Z \quad (11)$$

where the initial complex amplitudes $\beta_j = B_j(0)$, $j = 0, 1, 2$. From (6) and (10) one can show that

$$\frac{d}{dt}\text{Re}\left\{(B_0^*)^2 B_1 B_2 e^{i\Delta\omega t}\right\} = -\Omega\frac{dZ}{dt} \quad (12)$$

where

$$\Omega = 2\Omega_0 - \Omega_1 - \Omega_2. \quad (13)$$

In view of (8) and (11), Ω is a linear function of Z that can be presented as

$$\Omega = \Omega_a + \Omega_b Z. \quad (14)$$

Integrating (12) from $t = 0$ gives

$$\text{Re}\left\{(B_0^*)^2 B_1 B_2 e^{i\Delta\omega t}\right\} = \text{Re}\left\{(\beta_0^*)^2 \beta_1 \beta_2\right\} - \int_0^Z \Omega dZ \quad (15)$$

Using (10) and (15) yields

$$\left(\frac{dZ}{dt}\right)^2 = |B_0|^4\,|B_1|^2\,|B_2|^2$$
$$- \left[\text{Re}\left\{(\beta_0^*)^2 \beta_1 \beta_2\right\} - \Omega_0 Z - \frac{\Omega_1}{2}Z^2\right]^2 = P_4(Z) \quad (16)$$

The coefficients of the 4th order polynomial $P_4(Z)$ are fully determined by the initial conditions. It follows from (16) that

$$t = \int_0^Z dZ/\sqrt{P_4(Z)} \quad (17)$$

It is possible to express the integral (17) in terms elliptic functions (Byrd and Friedman, 1971). For wave parameters considered in this study, the coefficient of the leading term Z^4 in the polynomial is positive in most cases, all roots of $P_4(Z)$ are real and can be arranged so that $Z_4 > Z_3 > 0 > Z_2 > Z_1$. The following derivation is carried out for this case, nevertheless, using appropriate expressions in Byrd and

Fig. 1. The dependence of values of p corresponding to the limit of linear instability (p_ℓ) and to the linearly most unstable disturbance (p_m) on the carrier wave steepness according to the NLS and Zakharov equations.

Friedman it can easily be modified to cases when the roots of the polynomial $P_4(Z)$ are arranged differently. For the case considered, the value of Z can vary between Z_2 and Z_3, so that the solution (17) is periodic with the period given by

$$T = 2 \int_{Z_2}^{Z_3} dZ \bigg/ \sqrt{P_4(Z)} \qquad (18)$$

Shemer and Stiassnie (1985) and Stiassnie and Shemer (2005) presented the following explicit expression for (17):

$$Z = \frac{Z_4(Z_3 - Z_2)\, sn^2 u - Z_3(Z_4 - Z_2)}{(Z_3 - Z_2)\, sn^2 u - (Z_4 - Z_2)} \qquad (19)$$

where sn is the Jacobi elliptic function of argument

$$u = sn^{-1}(\delta, \kappa) - a_o^{1/2} t / \gamma \qquad (20)$$

and modulus

$$\kappa = \sqrt{\frac{(Z_3 - Z_2)(Z_4 - Z_1)}{(Z_4 - Z_2)(Z_3 - Z_1)}} \qquad (21)$$

In (20),

$$\gamma = \frac{2}{\sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}}, \qquad \delta = \sqrt{\frac{Z_3(Z_4 - Z_2)}{Z_4(Z_3 - Z_2)}} \qquad (22)$$

The modulation period (18) can be rewritten as

$$T = \frac{2\gamma}{a_o^{1/2}} K(\kappa), \qquad (23)$$

where K is the complete elliptic integral of the first kind. Note, that wave complex amplitudes B depend on Z and are, therefore, periodic; it also follows from (9) and (11) that the solution does not depend on the individual initial phases of the participating harmonics, but rather on a single initial phase given by

$$\theta = 2\arg\beta_0 - \arg\beta_1 - \arg\beta_2. \qquad (24)$$

Thus, the solution, in terms of the Jacobi elliptic functions, allows computing the variation of the amplitude of each of the three wave components as a function of their initial amplitudes and of a single phase combination θ.

3 Results

In all cases considered, the initial sideband amplitudes are defined relative to that of the carrier wave, so that $a_{1,2} = c_{1,2} a_0$. The coefficients $c_{1,2}$ mostly do not exceed 0.4, so that the power of sidebands remains well below 20%. Therefore, these sidebands are beyond the energy-containing part of the spectrum and can be seen as belonging to spectral tails that practically do not contribute to the effective width of the
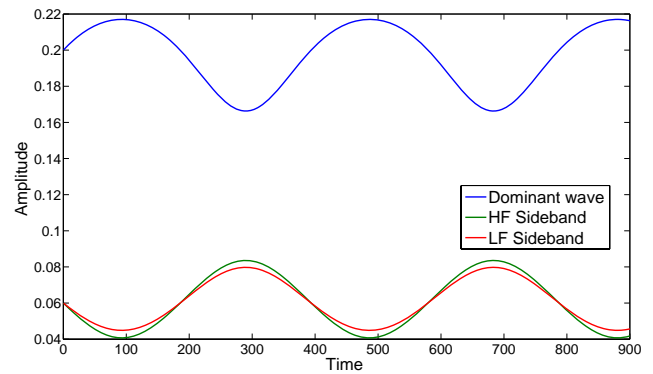
Fig. 2. Nonlinear evolution of a 3-wave system: $p = 0.2$; $q = 0$; $\theta = 0.7\pi$; $\varepsilon = 0.2$; $c_1 = c_2 = 0.3$.

spectrum in any conceivable definition of this parameter. In deep water the most unstable infinitesimal disturbances propagate in the direction of the carrier wave (McLean et al., 1981; Stiassnie and Shemer, 1984), so that the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are parallel to the carrier wave vector $\boldsymbol{k}_0$ and $q = 0$. Unidirectional wave systems are, thus, considered first.

The results of the previous section indicate that a three-wave system undergoes periodic Fermi-Pasta Ulam recurrence for an arbitrary value of the sideband wavenumber deviation p. A typical example of the temporal evolution of a unidirectional 3-wave system is presented in Fig. 2 for the carrier wave steepness $\varepsilon = 0.2$ and identical initial relative amplitudes of sidebands $c_1 = c_2 = 0.2$. The sidebands selected in this example belong to the BF instability domain, see Fig. 1. The dimensionless modulation period T that characterises the exchange of energy between the carrier wave and both sidebands can be computed using (23). For the conditions of Fig. 2, the dimensionless $T = 393.8$, or about 62 periods of the carrier wave. This is in agreement with the so called "dynamic" temporal evolution scale of the order of ε^{-2} appropriate for 4-wave nonlinear interactions.

As follows from (11), the amplitude of periodic modulation of each one of the 3 components is determined by a single real variable Z; the range of variation in Z, ΔZ, for given initial conditions defines the extent of modulation and can, thus, be seen as a measure of the intensity of nonlinear interactions at a slow time scale. The domain of variation ΔZ is determined solely by the values of the roots of the polynomial $P_4(Z)$.

The effect of the initial amplitude of sidebands is investigated in Fig. 3 for a unidirectional wave system with the carrier wave amplitude $\varepsilon = 0.2$, as a function of the sideband wavenumber determined by the value of p. In Fig. 3a, both sidebands are assumed to have identical initial amplitude, whereas in Fig. 3b and c, the initial amplitudes of either the low frequency sideband with $\boldsymbol{k}_3 = (1 - p, 0)$, or the high frequency sideband with $\boldsymbol{k}_2 = (1 + p, 0)$, respectively, are set as zero. In those cases, the initial amplitude of the initially

non-zero sideband is increased by a factor of $2^{1/2}$ compared to the corresponding values in Fig. 3b, to keep the total power of all 3 components identical in Fig. 3. The range of variation of p extends in Fig. 3 well beyond the linear stability limits computed using either the NLS or the Zakharov equations, as plotted in Fig. 1.

For the case of identical sideband amplitudes considered in Fig. 3a, the intensity of nonlinear interactions characterised by ΔZ increases monotonically with the amplitude of sidebands. In the whole range of values of p considered, the values of ΔZ vary continuously with p and do not vanish, although they decrease significantly for wider-spaced sidebands. For higher initial sideband amplitudes, significant nonlinear interactions exist even well beyond the BF stability domain. The linear stability limit (corresponding to $p \approx 0.44$ for the BF stability analysis based on the Zakharov equation) does not play any particular role. Moreover, the modulation depth does not attain maximum for the most unstable disturbance. The exact value of p where the intensity of nonlinear interactions attains maximum depends on the initial amplitude of the sidebands.

The evolution of an asymmetric wave system in which only one sideband exists initially does not differ qualitatively from that observed for the case of initially symmetric sideband disturbances. However, notable quantitative differences exist. For sideband wavenumbers adjacent to that of the carrier wave, the modulation intensity in Fig. 3b and c is smaller by a factor close to 2 as compared to that in Fig. 3a. The values of ΔZ then decay faster with increase in p, although decrease in the amplitude of modulations is less steep in Fig. 3c where the low-frequency sideband initially has non-zero amplitude, as compared to Fig. 3b. In Fig. 3b and c the most intensive nonlinear interactions are obtained for sidebands that are not too far from the carrier wave, rather than for the values of p corresponding to the most unstable disturbances in the linear BF analysis.

The results of Fig. 3 can be supported by findings of Shemer and Chamesse (2000) who studied experimentally the BF instability of gravity-capillary waves on the basis of the Zakharov equation. In the experiments a carrier wave was generated by a plane wavemaker, while one of the sidebands (with a frequency either higher or lower than that of the carrier) was excited with a small but finite amplitude by an independent auxiliary conical wavemaker; its frequency was varied in a range exceeding the instability domain. The appearance in the measured spectrum of the sideband that has not been excited directly was interpreted as a manifestation of the BF instability. In the measured spectra, however, this sideband was often visible even when the auxiliary wavemaker was operated at frequencies outside of the instability domain. The results of Fig. 3 are in general agreement with those observations.

The examples presented in Figs. 2 and 3 were computed for an arbitrarily selected initial phase angle combination $\theta = 0.7\pi$. The effect of θ on the intensity of nonlinear in-
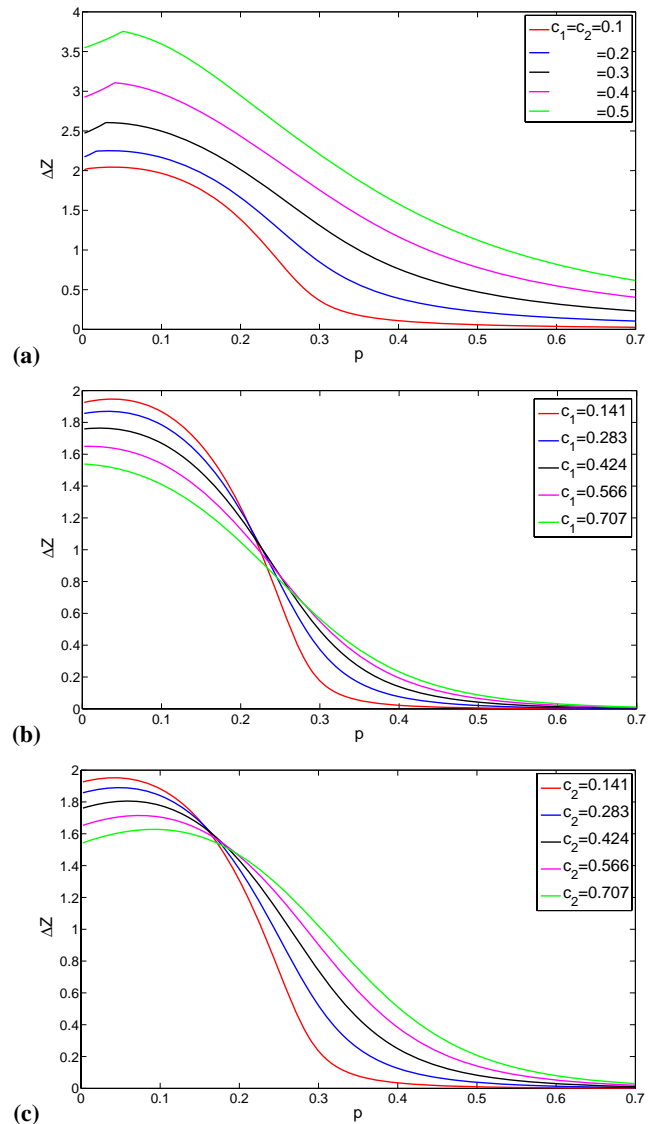
Fig. 3. Effect of the relative initial sideband amplitude on the modulation depth for $\varepsilon = 0.2$ and $\theta = 0.7\pi$: **(a)** $c_1 = c_2$; **(b)** initial low frequency sideband amplitude $c_3 = 0$; **(c)** initial high frequency sideband amplitude $c_2 = 0$.

teractions is studied in Fig. 4 for the carrier wave amplitude $\varepsilon = 0.2$ and a set of identical initial sideband amplitudes. In Fig. 4a, the sidebands that are relatively close to the carrier wave, $p = 0.2$, are considered, while in Fig. 4b $p = 0.45$, beyond the limits of the BF linear instability. In Fig. 4a, the phase dependence of ΔZ increases strongly with the initial sideband amplitude. For phase combination angles sufficiently close to zero, the modulation intensity may become even stronger for initially smaller sideband amplitudes. For widely-spaced sidebands in Fig. 4b, the phase dependence of ΔZ weakens notably; the values of ΔZ increase with the initial amplitude of sidebands for all initial phases θ. Note that
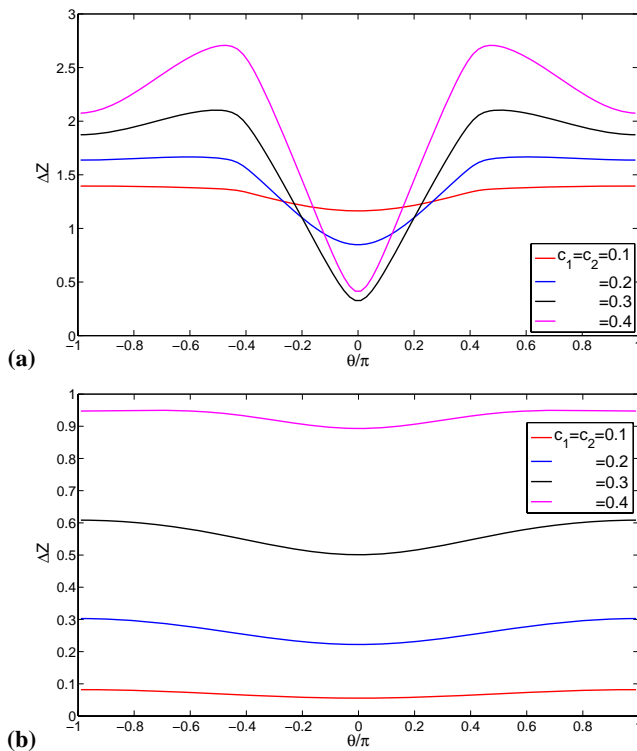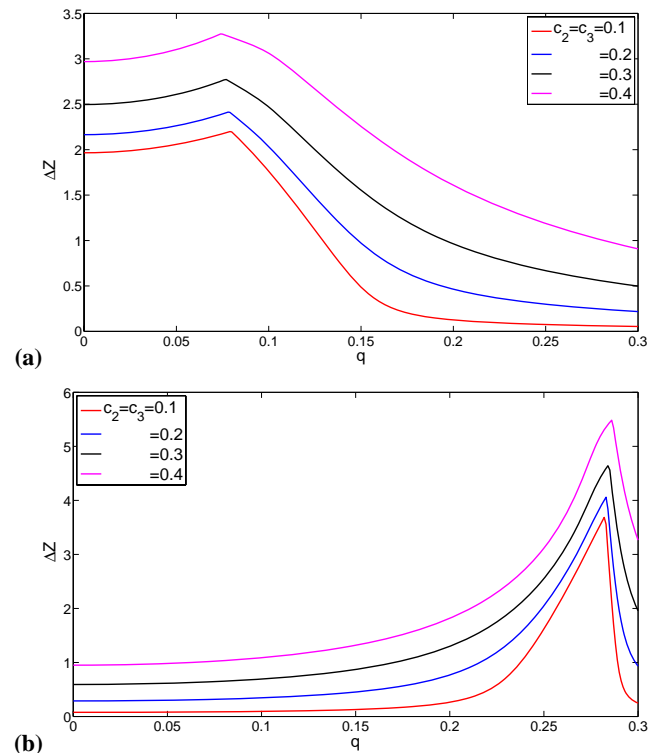
Fig. 4. Modulation intensity for the carrier wave steepness $\varepsilon = 0.2$ as a function of the initial phase angle θ for: **(a)** $p=0.2$; **(b)** $p = 0.45$.

Fig. 5. Modulation intensity for the carrier wave steepness $\varepsilon = 0.2$ and the initial phase angle $\theta = 0.7\pi$ as a function of q for: **(a)** $p = 0.2$; **(b)** $p = 0.45$.

for $c_1 = c_2 = 0.4$, the modulation amplitude in the vicinity of $\theta = 0$ is higher for $p = 0.45$ in Fig. 4b as compared to that for $p = 0.2$ in Fig. 4a.

Finally, the effect of directional spreading is examined in Fig. 5 for $\theta = 0.7\pi$, other conditions being as in Fig. 4. Note that according to the exact linear stability analysis of finite-amplitude waves by McLean et al. (1981), the instability limits are $0 \le q \le 0.125$ for $p = 0.2$, and $0 \le q \le 0.25$ for $p = 0.45$. As evident from Fig. 4a, for sidebands with finite initial amplitude the nonlinear interactions are quite strong well beyond those limits. Even more unexpected may be the results of Fig. 4b that indicate that the most significant nonlinear interactions occur for wave vectors that correspond to linearly stable disturbances. These interactions result in sideband amplitudes that, for certain phases of the Fermi-Pasta-Ulam recurrence process, can exceed that of the carrier, even for initially relatively small values of $c_{1,2}$.

4 Conclusions

The notion of BFI was coined considering a 3-wave system consisting of a carrier and initially vanishingly small sideband disturbances. The 3-wave approach intrinsic to the BFI concept is expanded here to include sidebands that, while sufficiently small not to affect notably the spectral width, have nevertheless finite initial amplitudes. The Zakharov

equation is used as the theoretical model. Contrary to envelope equations (the CSE and its modifications), it does not pose any restrictions on the spectral width. Advantage is taken of the availability of an exact solution of the resulting system of equations in terms of Jacoby elliptic functions. The analytical solution demonstrates that the 3-wave system undergoes periodic Fermi-Pasta-Ulam recurrence. The intensity of nonlinear interactions can be characterised by a single real parameter ΔZ.

The present parametric study suggests that conclusions based on the linear stability analysis in general, and in particular when the CSE is used, are largely irrelevant for a 3-wave system in which initial sideband amplitudes are small but finite. The Benjamin-Feir analysis that considers the stability of a finite amplitude Stokes wave to infinitesimal sideband disturbances, therefore, cannot be seen as meaningful representation of the evolution of a nonlinear wave field. The most significant nonlinear interactions are not necessarily related to the linearly most unstable disturbances. The domains of the BF instability determined either by approximate models, such as narrow-banded CSE and the Zakharov equation, or by an application of fully nonlinear water-wave equations, cannot serve as indicators of the intensity of nonlinear interactions of a carrier wave with finite amplitude sidebands.

The present results indicate that for a unidirectional wave field and initial disturbances with non-vanishing amplitudes, the values of sidebands' wavenumber deviation from that of the carrier p and the carrier wave steepness ε do not uniquely define the nonlinear evolution pattern of the whole wave system. This pattern appears to be crucially dependent also on the initial sideband amplitudes and often on their phases as well.

When sideband disturbances are allowed to propagate in directions different from that of the carrier wave, as is the case in laboratory studies in wave basins, or in field experiments, the spectral width, calculated in most experimental investigations from the measured frequency spectrum, becomes even less dominant in governing the nonlinear evolution process, which becomes crucially dependent not just on the sidebands' initial amplitude but on their directions as well.

The results of the present exercise with a simple deterministic nonlinear wave system can be useful in analysing much more complicated random wave fields. It should be kept in mind, however, that statistics of random water waves in laboratory and in sea is determined by a large number of interacting triads and quartets, each having random combinations of phases. Thus, extreme care should be exercised in the application of the present findings to a wider context of statistics of random nonlinear wave fields.

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