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Multifractal behaviour of river networks

S. G. De Bartolo, S. Gabriele and R. Gaudio

Consiglio Nazionale delle Ricerche (C.N.R.), Istituto di Ricerca per la Protezione Idrogeologica nell'Italia Meridionale e Insulare (I.R.P.I.), Via Cavour, 87030 Rende (CS), Italy
e-mail for corresponding author: gabriele@gndci.cs.cnr.it

Abstract

A numerical multifractal analysis was performed for five river networks extracted from Calabrian natural basins represented on 1:25000 topographic sheets. The spectrum of generalised fractal dimensions, D(q), and the sequence of mass exponents, $\tau(q)$, were obtained using an efficient generalised box-counting algorithm. The multi-fractal spectrum, $f(\alpha)$, was deduced with a Legendre transform.

Results show that the nature of the river networks analysed is multifractal, with support dimensions, D(0), ranging between 1.76 and 1.89. The importance of the specific number of digitised points is underlined, in order to accurately define, the geometry of river networks through a direct generalised box-counting measure that is not influenced by their topology.

The algorithm was also applied to a square portion of the Trionto river network to investigate border effects. Results confirm the multifractal behaviour, but with D(0) = 2. Finally, some open mathematical problems related to the assessment of the box-counting dimension are discussed.

Keywords: River networks; measures; multifractal spectrum

Introduction

The multifractal theory was introduced by Mandelbrot (1972, 1974) and was further developed by Frisch and Parisi (1985) for application to fully developed turbulence phenomena. Similar developments were performed by Halsey *et al.* (1986) for the analysis of 'strange attractors' in the theory of dynamic systems.

Multifractal formalism finds its origin in the theory of measures. Multifractal measures, in general, deal with the study of the distribution of a quantity over a geometric support, e.g. an ordinary plane, a surface, a volume or a fractal itself. In particular, multifractals are "infinite sets of exponents, which describe the (power law) scaling of all the moments of a distribution of some quantities which are defined on a fractal structure. In many cases, specific members of these families of exponents coincide with the fractal dimensionalities of geometrical substructures of the underlying fractal." (Aharony, 1989, p. 1).

Recent studies show that the field of applicability of the multifractal theory can be extended to river basins (Ijjasz-Vasquez *et al.*, 1992; Rinaldo *et al.*, 1992; Rinaldo *et al.*, 1993; Rigon *et al.*, 1993). For example, the spatial distribution and the scaling properties of some important hydrological variables, such as contributing areas, slopes, dissipation energy, the channel initiation function and the width function, can be characterised through the formalism of the multifractal spectrum, $f(\alpha)$, introduced by Halsey *et al.* (1986). In particular, fluvial networks may be considered intricate spatial self-organised structures (Rodriguez-Iturbe and Rinaldo, 1997), presenting typical multifractal characteristics (De Bartolo *et al.*, 1995) analogous to Diffusion Limited Aggregation (DLA) processes (Witten and Sander, 1981). In such processes Coniglio and Zannetti (1989) showed that multifractality is connected to multiscaling properties.

Some authors, following Mandelbrot's (1977) observations inherent to the nature of river geometry, proposed relations useful to compute the fractal dimension, as a function of topological parameters. La Barbera and Rosso (1987, 1989) derived an expression for the fractal dimension, D, of river networks as a function of the Hortonian bifurcation ratio, R_B, and length ratio, R_L, assuming that ratios are constant in the basin and independent of the observation scale:

$$D = \log R_B / \log R_L \tag{1}$$

They computed the fractal dimension of several river networks and obtained values ranging between 1.5 and 2, with an average between 1.6 and 1.7.

Tarboton *et al.* (1988) assessed the box-counting fractal dimension, D_B, that came out at about 2, indicating that river networks are space filling. Tarboton *et al.* (1990) attributed the differences with respect to values obtained by

La Barbera and Rosso (1987, 1989) to the fact that formula (1) includes only bifurcation effects, ignoring individual stream fractality (Mandelbrot, 1982; Hjelmfelt, 1988; Helmlinger *et al.*, 1993); stream fractality contributes to the measure in a relevant way in the case of direct estimation through box-counting. The fractal dimension of natural channel networks, D_{cn}, is therefore equal to the dimension of a Hortonian branching structure, D_b (given by (1)), times the dimension of a single stream, D_s (which is approximately 1.1; Mandelbrot, 1977; Hjelmfelt, 1988):

$$D_{cn} = D_s D_b \tag{2}$$

Claps and Oliveto (1996) studied 23 river networks in Southern Italy and found the average values $D_{cn} = 1.7$, $D_b = 1.5$, $D_s = 1.1$ with very low variability. Commenting on their results, in addition to other comparable data taken from the literature, they hypothesised that natural networks tend to the same fractal dimension. They stated (p. 3132): "from these results it can be concluded that channel network structures are definitely non-plane-filling".

Beauvais and Montgomery (1997) argued that the Hortonian ratios are inadequate for the estimation of fractal dimension, because their use is possible by assuming statistical self-similarity of river networks, without any demonstration of this assumption; in addition, the application of such laws often gave values of fractal dimension even greater than 2 (Tarboton *et al.*, 1988; Helmlinger *et al.*, 1993). Eventually Beauvais and Montgomery (1997) concluded that channel networks cannot be considered statistically self-similar, namely fractals, "in spite of their seductive branching architecture" (p. 1066).

In this paper, a direct generalised box-counting measure technique is proposed and an analysis of the geometry of natural river networks is performed to confirm their multifractal behaviour, already shown in a previous study for a single case only (De Bartolo *et al.*, 1995).

Multifractal Formalism

If one considers N members of a population (N points of a set S) distributed over a geometric support and covered by cells of side size δ, the measure (or probability or mass) of the content in the i-th cell may be defined as follows:

$$\mu_i(\delta) = N_i(\delta)/N \tag{3}$$

where $N_i(\delta)$ is the number of points falling in the i-th cell at the resolution δ.

The family of parameters $q \in \Re$ (where $\Re$ is the set of real numbers) of fractal dimensions, based on the concept of generalised entropies (Rényi, 1955), is expressed as follows (Feder, 1988; Olsen, 1995):

$$D(q) = \frac{1}{q-1} \lim_{\delta \to 0} \frac{\ln Z_q(\delta)}{\ln \delta} \quad \text{for} \quad q \neq 1 \tag{4}$$

where:

$$Z_q(\delta) = \sum_{i=1}^{N_c(\delta)} [\mu_i(\delta)]^q \tag{5}$$

is the partition function (Halsey *et al.*, 1986), $N_c(\delta)$ being the number of covering cells at the resolution δ. As q varies, the function D(q) describes the spectrum of generalised fractal dimensions, following a multifractal formalism, previously proposed by Hentschel and Procaccia (1983), Grassberger and Procaccia (1983) and Grassberger (1983), parallel to the formalism of the $f(\alpha)$ spectrum. In particular, $D(q = 0)$ is the fractal dimension of the set S, over which the measure is performed; $D(q = 1)$ is the information entropy, that can be obtained from (4) as $q \to 1$:

$$D(1) = \lim_{\delta \to 0} \frac{\sum_{i=1}^{N_c(\delta)} \mu_i(\delta) \ln \mu_i(\delta)}{\ln \delta} \tag{6}$$

The following relationship holds between the spectrum of generalised fractal dimensions, D(q), and the sequence of mass exponents, $\tau(q)$:

$$\tau(q) = (1 - q) \cdot D(q) \tag{7}$$

The estimation of the D(q) function is possible using a generalised box-counting algorithm (Block *et al.*, 1990). The given set of N "vectors" (d-uples, i.e. coordinates of the points) lying in Euclidean d-dimensional space, $S \subset E^d$, is covered with a sequence of d-dimensional hyper-cubes (d-cubic boxes) having side size, δ, exponentially decreasing to a minimum value. The minimum value depends on the minimum distance between the points of the set S.

The vectors are firstly normalised by proper choice of the origin, in order to obtain non-negative coordinates, and then dividing the coordinates by the maximum of their values. This operation produces vectors confined to the unit d-cube, i.e. each coordinate assumes values between 0 and 1. It is important to remark that normalisation does not alter the measure, since transformations such as translations, rotations, similarities and affinities are geometrically invariant (Falconer, 1990). Notice that the edge of the d-cubic boxes, δ, becomes non-dimensional after normalisation, assuming values between 0 and 1.

For each value of δ, the number of points falling in the i-th cell, $N_i(\delta)$, is counted and the measure, $\mu_i(\delta)$, is computed. The values of $\ln Z_q(\delta)/(q-1)$ vs. $\ln \delta$ for $q \neq 1$ and $\sum_{i=1}^{N_c} \mu_i \ln \mu_i$ vs. $\ln \delta$ for $q = 1$ – see (4) and (6) – are then plotted and a 'fractal curve' is obtained for each q.

The assessment of D(q) values is performed through the least squares linear regression, in the range $\ln(\delta_{lower})$ to $\ln(\delta_{upper})$, where the fractal dimension is significantly determined as the slope of the fractal curve. In fact, it is well known that some problems arise in the interpretation of the results of the box-counting method. Helmlinger *et al.*

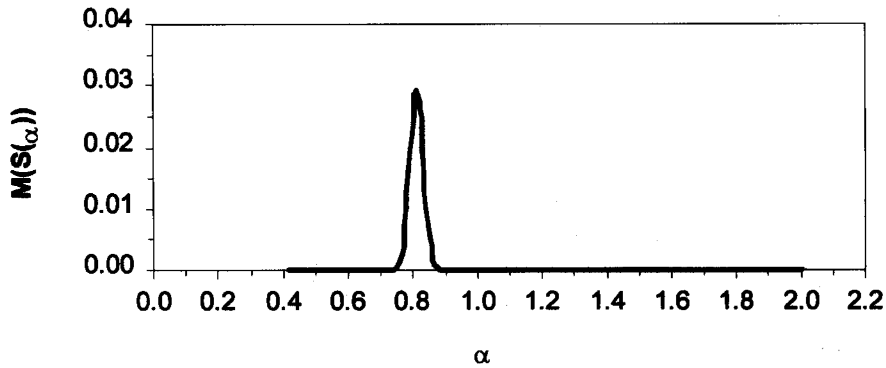

Fig. 1. *Example of measure M of sub-set S_α for a binomial multiplicative process. Most of the information is contained in the sub-set S_{α_s} for which the measure shows a peak.*

(1993) noticed that, at the smallest δ, each box contains a single point and, therefore, the resulting fractal dimension is zero. At the largest box sizes, only a few boxes include points and the fractal dimension approaches 2. The fractal dimension is meaningfully determined in the intermediate range of δ, whose limits can be called respectively lower and upper cut-off (δ_{lower} and δ_{upper}). The same concept is expressed by Meakin (1998), who calls them 'the inner and outer cut-off lengths' (p. 80). The generalised box-counting method can assess $1 < D(q) < 2$ only in the scale range $\delta_{lower} \leq \delta \leq \delta_{upper}$.

Once D(q) values are determined, the sequences of mass exponents can be obtained from (7), and the Lipschitz-Hölder exponents, α, and the multifractal spectra, $f(\alpha)$, are deduced by means of a Legendre transform:

$$\alpha(q) = -\frac{d\tau(q)}{dq} \qquad (8)$$
$$f(\alpha(q)) = \tau(q) + q\alpha(q)$$

Mandelbrot (1989) defined 'manifest', 'virtual' and 'latent' parts of the spectrum as the parts lying respectively in the I, II-III and IV quadrant of the coordinate plane. The virtual and latent parts are still under investigation. The manifest part can be split into left side ($q > 0$) and right side ($q < 0$); the left side characterises fractal dimensions of sub-sets, whose measure attributes greater weight to cells containing a larger quantity of information. Mandelbrot (1977, 1982) pointed out that there could be sub-sets with greater mass concentration, and called this process 'curdling'.

In correspondence to $q = 1$, a particular point of the spectrum is identified, with equal abscissa and ordinate: $\alpha_S = f(\alpha_S) \equiv S$, S being the information entropy (Mandelbrot, 1982). In other words, if one considers that a multifractal set is a union of sub-sets, each characterised by a value of the Lipschitz-Hölder exponent, α, and a different fractal dimension, the sub-set S_{α_s} has a fractal dimension equal to the entropy of the partition of the measure, and most of the information concentrates in it (e.g. Fig. 1, for a binomial multiplicative process).

Application to Natural River Networks

The above theory may be applied to characterise the geometry of river networks. This paper shows results from the analysis of five fluvial networks, extracted from the Calabrian basins of the Ancinale, Crati, Petrace, Savuto and Trionto rivers.

GEOLOGICAL AND CLIMATIC DESCRIPTION OF CATCHMENTS

The Ancinale river basin is located on the north-eastern slope of the Serre mountains, and drains Paleozoic granitoid (granites, granodiorite and tonalite) and metamorphic rocks. Miocene to Quaternary sedimentary rocks comprise a minor part of the basin and only occur on the low-course. The lithology of the Crati river basin includes Paleozoic metamorphic (phyllite, gneiss and minor schist) and plutonic (mainly granodiorite and granite, minor tonalite), early Mesozoic ophiolitiferous (metagabbros and metabasalts, serpentinite) and sedimentary (limestone, marble, argilloschist and quartzite), and Tertiary to Quaternary sedimentary rocks. The Petrace river basin is located on the western slope of the Aspromonte mountains. This fluvial system drains predominantly high-grade metamorphic rocks (granulite to gneiss) and minor plutonic rocks on its upper course, with Quaternary sedimentary rocks on its lower course. The Savuto river basin has, on the upper course, Paleozoic gneissic rocks thrust over Paleozoic schist and phyllite (mid-course); the lower course has phyllite, and Mesozoic to Quaternary sedimentary rocks. The Trionto river basin has Paleozoic granites and phyllite-schist on its upper reach and, on the middle and lower course, Mesozoic to Quaternary sedimentary (sandstone, marl, limestone, claystone, gypsum, conglomerate) rocks.

The five basins experience a typical Mediterranean climate, characterised by dry summer periods and rainy autumns. The annual average rainfall height is about

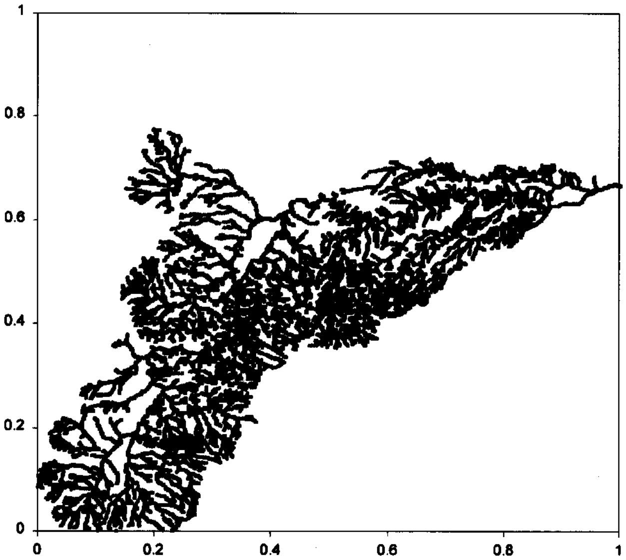

Fig. 2. *River Ancinale normalised net-points extracted from the representation of the natural basin on 1:25000 topographic sheet.*

800 mm for the Ancinale and Trionto catchments, 900 mm for the Crati, 1200 mm for the Petrace and the Savuto.

APPLICATION OF THE GENERALISED BOX-COUNTING ALGORITHM AND RESULTS

Each network was represented by a set of points, obtained by digitising 1:25000 topographic sheets (e.g. Fig. 2). These points will be referred to in the rest of the paper as 'net-points'. A set of net-points is a set of vectors in the Euclidean plane, $S \subset E^2$, over which it is the intention to perform the measurement.

Table I. Characteristics of river networks (A, drainage basin area; *N*, number of net-points).

Set, S	A (km^2)	N	N/A ($1/km^2$)
Ancinale	173.84	19380	111
Crati	2447.79	268286	110
Petrace	406.62	30039	74
Savuto	411.54	16476	40
Trionto	288.49	61167	212

Table I shows the following characteristics of the analysed river networks: the drainage basin area, A; the number of net-points, N; the specific number of net-points, N/A, which is an index of how accurately the digitised point sets approximate river networks.

In order to verify the multifractal behaviour of river networks, the spectrum of generalised fractal dimensions was computed, by using the previously discussed generalised box-counting algorithm. As an example, Fig. 3 shows the scaling of the partition function, defined by (5), with the box size for the Ancinale river network.

Many authors underlined the fact that some experience is needed to select the scale range between the lower and upper cut-off lengths (see Helmlinger *et al.*, 1993, and Meakin, 1998). In the present application, it was determined as the range of box sizes where the coefficient, R^2, of the least squares linear regression is maximum.

The generalised fractal dimensions for values of the moment order q = 0 to 20 are shown in Fig. 4; the natural

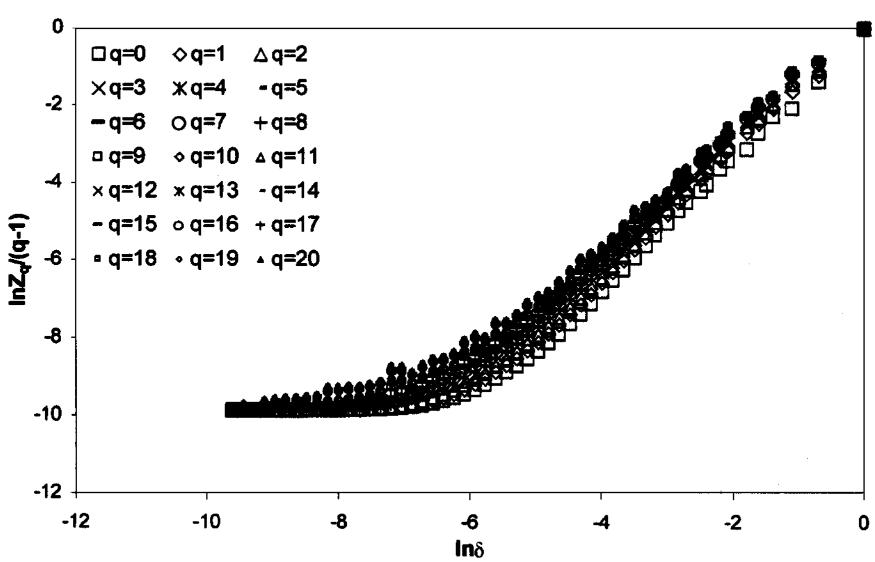

Fig. 3. *Example of scaling of the partition function,* Z_q, *with the box size (Ancinale river network).*

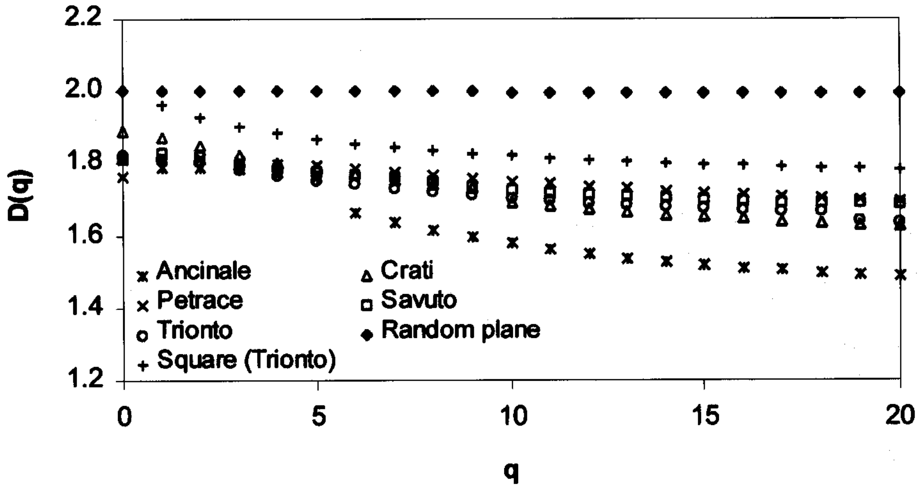

Fig. 4. *Spectra of generalised fractal dimensions.*

distribution of net-points has an evident multifractal behaviour, since D(q) varies as a function of q. Figures 4 to 6 include results from the analysis of the other two sets (a random plane and a square portion of the Trionto river network), which are discussed in the rest of the paper.

Figures 5 and 6 show respectively the sequences of mass exponents, $\tau(q)$, obtained from (7), and the multifractal spectra, $f(\alpha)$, deduced from (8). In Fig. 6 the experimental points describe the left sides of the manifest part of the spectra, interpolated with third order polynomial curves (solid lines).

The following results are shown in Table II: the regression coefficient, R^2, for the assessment of D(0); the bounds of the scaling behaviour, δ_{upper} and δ_{lower}, for q = 0 and for each normalised set; the support fractal dimension, $D(0) = f(\alpha_0)$; the information entropy, $D(1) = f(\alpha_S) = \alpha_S \equiv S$; the maximum and minimum values of the Lipschitz-Hölder exponents, α_{min} and α_0, in the given range of moment order, q = 0 to 20.

Notice that all the values of $R^2(q = 0)$ are very close to 1. The values of $R^2(q \neq 0)$, not shown in Table II, are also approximately 1.

Results show that the values of the Lipschitz-Hölder exponent lie in the range 1.43 to 1.86, the support fractal dimensions range from 1.76 to 1.89 while the information entropy assumes values 1.79 to 1.87. The Petrace and Savuto river networks, having a specific number of net-points, N/A, less than the other analysed networks, present the anomaly $D(1) > D(0)$, in spite of the regular fashion of the multifractal spectrum. This fact indicates that, when the cardinality of the set S is not sufficiently high, one cannot assess correctly the

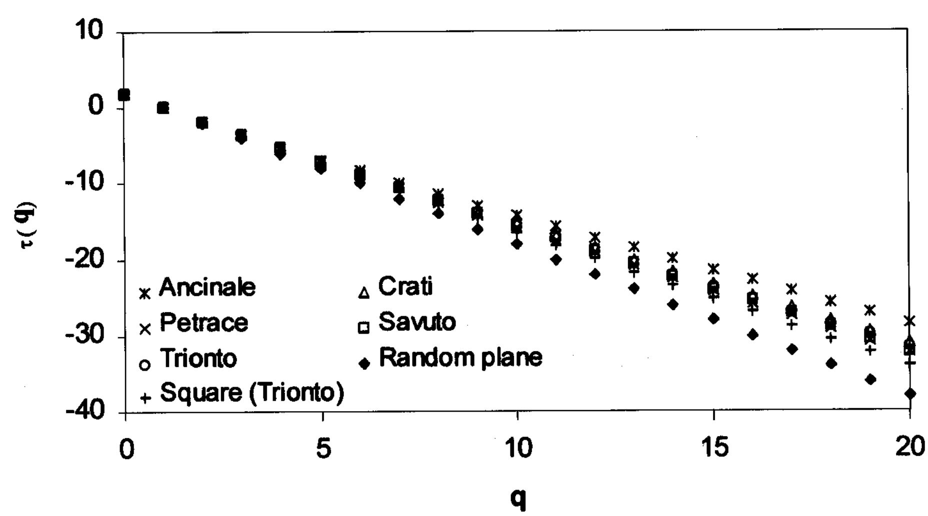

Fig. 5. *Sequences of mass exponents.*

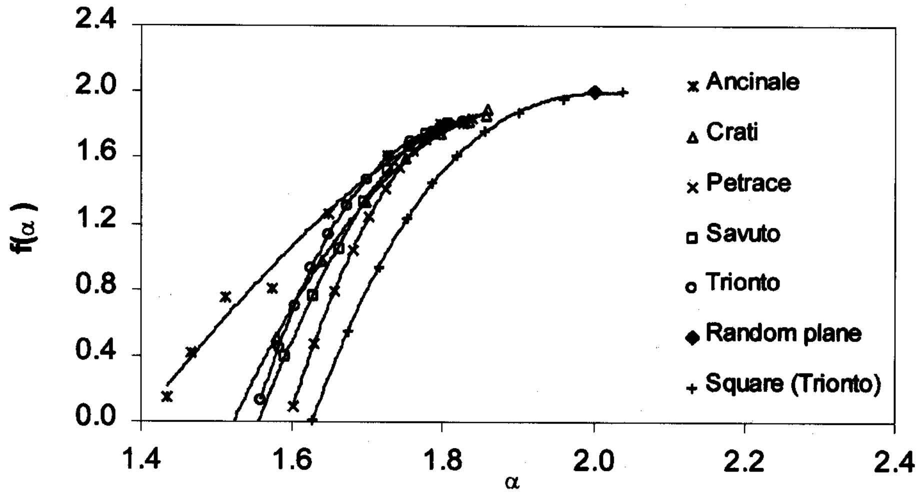

Fig. 6. *Multifractal spectra (left sides). Symbols indicate experimental points; solid lines are third order polynomial interpolating curves.*

support dimension, but only the information entropy, which is an index of the information content in the available data.

Discussion

The scale range [δ_{lower}, δ_{upper}] over which the generalised box-counting method yields $1 < D(q) < 2$ is narrow. This fact was already noticed by Helmlinger *et al.* (1993) and Beauvais and Montgomery (1997) in the application of the standard (not generalised) box-counting method to channel networks extracted from the digital elevation models. In that case, a dependence of results on the threshold value of the contributing area used to delineate network sources was evidenced by the authors. In the present case, the box-counting method was applied to sets of digitised points. The width of the scaling region, therefore, is probably dependent on the scale of the topographic sheets used to extract the net-points.

It is interesting to note that the values of $D(q = 0)$ for the five natural fluvial networks are in contrast with the findings of some authors who obtained $D(0) = 2$ and defined the river networks as "space filling" (Tarboton *et al.*, 1988; Ijjasz-Vasquez *et al.*, 1992). The present analysis seems to be in accordance with the outcome of Claps and Oliveto (1996), reported in the Introduction, and could support their opinion that natural networks are non-plane-filling. Nevertheless, Ijjasz-Vasquez *et al.* (1992) pointed out that the application of the box-counting algorithm may produce biased results due to the border effects of boxes overlaying the edges of the catchments and including a "white space" that belongs to adjacent catchments and may contain other net-points. In order to investigate possible border effects, another test was performed for a square portion of the Trionto river network (Fig. 7).

Results are shown in Figs. 4 to 6, which confirm the multifractal behaviour, but with support dimension $D(q = 0) = f(\alpha_0) = 2$. This outcome could induce supporters of the space filling hypothesis to think that border effects are responsible for the values $f(\alpha_0) < 2$ obtained in the analysis of the whole river networks. However, other problems are probably hidden in the application of the box-counting method, as noticed also by Claps and Oliveto (1996), but in the opposite conviction: "the box counting method could be responsible for often producing D = 2" (p. 3128). Really, the problem is that not all the properties of the classical Hausdorff dimension hold for the box-counting dimension (Falconer, 1990); in particular, the 'countable stability' – which states that if $S_{1 \leq i < \infty}$ is a countable

Table II. Results.

Set, S	$R^2_{q=0}$	$\delta_{lower,\ q=0}$	$\delta_{upper,\ q=0}$	$D(0) = f(\alpha_0)$	$D(1) = f(\alpha_S)$	α_{min}	α_0
Ancinale	0.99959	0.036	0.067	1.809	1.789	1.434	1.838
Crati	0.99996	0.007	0.023	1.890	1.873	1.579	1.859
Petrace	0.99999	0.167	0.500	1.764	1.814	1.601	1.798
Savuto	0.99993	0.013	0.083	1.809	1.830	1.590	1.806
Trionto	0.99999	0.011	0.023	1.824	1.813	1.559	1.826

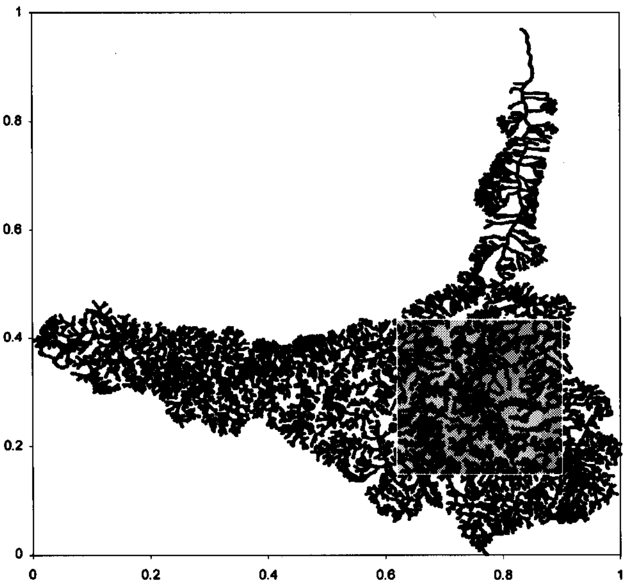

Fig. 7. *Square portion (N = 13404) of the Trionto river network.*

sequence of sets then $\dim(\bigcup_i S_i) = sup_i(\dim S_i)$, where $sup_i(\dim S_i)$ is the smallest upper bound of $\dim S_i$ – does not hold for the box-counting dimension computed on countable sets. This fact can explain how it is possible to find different values of D(0) if a whole network (union of sub-sets) or a square inside it (a single sub-set) is analysed.

Another issue arises about the density of sets. It is known that if S is a dense sub-set of an open region of $\Re^n$, then its box-counting dimension is equal to n (Falconer, 1990), so that the problem of achieving D(0) = 2 depends on the density of the set under investigation. If one considers that often river networks are assimilated to tree-graphs, and that trees with a finite number of nodes are not dense (West, 1996), it is evident that more research is needed to investigate these open mathematical aspects. The multifractal behaviour of the natural networks was confirmed by the analysis of the square portion of the Trionto river network, as stated above. To verify whether the generalised box-counting algorithm adopted is able to recognise different geometric structures, a random set of 50000 vectors, filling a plane area (random plane) was generated. The spectrum of generalised fractal dimensions proved to be a very good approximation of a horizontal line, expressed by D(q) = 2 (Fig. 4). This constant value indicates that the random plane does not have a multifractal behaviour, as is obvious for such a Euclidean structure. The sequence of mass exponents is depicted in Fig. 5. The random plane spectrum, f(α), coincides in a unique point with the abscissa and ordinate equal to 2 (Fig. 6).

Conclusions

The multifractal analysis of five digitised Calabrian river networks was performed by means of an efficient generalised box-counting algorithm. The spectrum of generalised fractal dimensions, D(q), and the sequence of mass exponents, τ(q), were obtained. The multifractal spectrum, f(α), was deduced with a Legendre transform. Results showed that the digitised points extracted from 1:25000 topographic sheets constitute a multifractal natural distribution on the shape of each river network, with estimated values of the fractal support dimension between 1.8 and 1.9.

Border effects were investigated in a square portion of the Trionto river network. Results confirm the multifractal behaviour, but show a plane filling support dimension.

Some open mathematical problems are pointed out. Firstly, the countable stability of the classical Hausdorff dimension does not hold for the box-counting dimension computed on countable sets. This fact can explain the different values of D(0) estimated for a whole network and a square portion of it. Secondly, the problem of achieving D(0) = 2 depends on the density of the set under consideration, and more knowledge about density of river networks is needed.

Nevertheless, the ability of the generalised box-counting algorithm adopted to recognise different geometric structures was tested on a random set of 50000 vectors filling a plane area. Results showed that the random plane is obviously a Euclidean structure, with D(q) = 2 for each q.

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