**Hydrology and
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Modeling geophysical complexity: a case for geometric determinism

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Abstract. It has been customary in the last few decades to
employ stochastic models to represent complex data sets en-
countered in geophysics, particularly in hydrology. This arti-
cle reviews a deterministic geometric procedure to data mod-
eling, one that represents whole data sets as derived distribu-
tions of simple multifractal measures via fractal functions. It
is shown how such a procedure may lead to faithful holistic
representations of existing geophysical data sets that, while
complementing existing representations via stochastic meth-
ods, may also provide a compact language for geophysical
complexity. The implications of these ideas, both scientific
and philosophical, are stressed.

1 Introduction

The study of complex data sets is at the core of geophys-
ical research. This has resulted in the use of a variety of
techniques aimed at characterizing and modeling such sets.
With the development of ideas, a variety of qualifiers (e.g. au-
tocorrelation function, power spectrum, multifractal spec-
trum, probability distribution function, information function,
chaotic invariant properties, etc.) have been used for such
purposes, and the model's goodness has been defined in their
ability to preserve such quantities.

As illustrated in Fig. 1, many data sets (e.g. rainfall and
runoff time series in hydrology) exhibit high irregularity,
non-trivial intermittency, long-term persistence, power-law
power spectrum scaling and altogether "intrinsic random-
ness," which suggest, in a natural way, the usage of stochastic
models, such as multiplicative cascades with random mul-
tipliers, to represent such "$1/f^\beta$ noises" (e.g. Lovejoy and
Schertzer, 1990; Veneziano et al., 2000). The ability of the
stochastic models to reasonably represent important (statisti-
cal) characteristics of these data sets and the reasonably good

predictions reported on their underlying evolutions have fur-
ther strengthened our view on the usefulness and appropri-
ateness of such models.

This article argues that, given the irregularity present in
geophysical data sets, such stochastic modeling may be sup-
plemented with chaotic and other deterministic modeling to
provide a more complete representation of the records and
processes at hand. Besides the now classical use of phase-
space reconstruction techniques (e.g. Packard et al., 1980;
Takens, 1981) developed in nonlinear dynamics to qualify
the intrinsic dimensionality of sets (e.g. Rodriguez-Iturbe
et al., 1989; Tsonis et al., 1993; Sivakumar, 2000, 2004;
Sivakumar et al., 2001), this article makes a case for a ge-
ometric representation of geophysical records as derived dis-
tributions of deterministic multifractal measures via simple
deterministic fractal functions (Puente, 1992, 1994a), a novel
deterministic procedure that may be useful in archiving data
and in elucidating their dynamics.

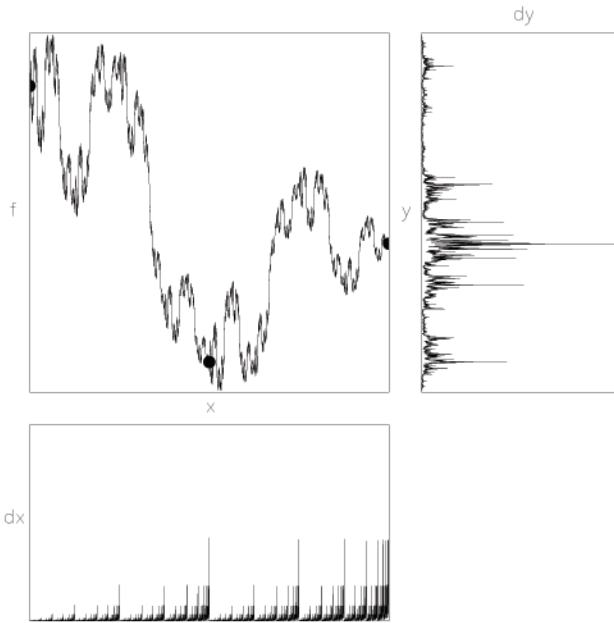## 2 The fractal-multifractal procedure and geophysical applications

This deterministic geometric procedure was inspired by the
simply quantized layering of energy dissipation in fully de-
veloped turbulence (Meneveau and Sreenivasan, 1987) and
by the automatic generation of the inherent multiplicative
cascade while constructing fractal interpolating functions
(Barnsley, 1988).

As illustrated in Fig. 2, the fractal-multifractal procedure
transforms a turbulence-related measure over x, say dx (i.e.,
a multifractal one found via a cascade and, hence, one with-
out a density), into the unique measure dy, over y, defined as
the derived distribution of dx via a fractal function f (Puente,
1992, 1994a). The ideas turned out to provide, in a way remi-
niscent of Plato's notions of "reality" as "shadows", a host of
interesting patterns over y, whose peculiar shapes suggested,
also in a natural manner, their usage as suitable geometric

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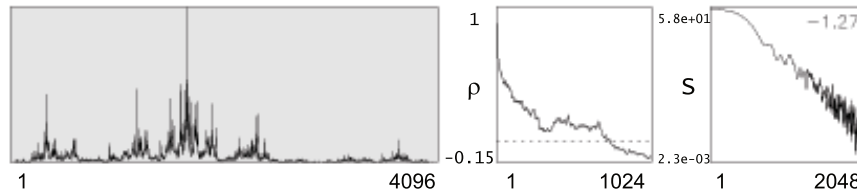
Fig. 1. A prototypical complex data set made of 2^{12} data points, followed by its autocorrelation function and its (log-log) power spectrum, with scaling exponent $\beta=1.27$.

Fig. 2. An example of the fractal-multifractal procedure. The measure dy is obtained adding the corresponding values of dx for horizontal crossings of the mapping f.

models for geophysical data sets, including rainfall (Puente, 2004).

It so happens that the time series in Fig. 1 corresponds to dy in Fig. 2. Specifically, such a complex set is found, at any resolution, as the unique invariant measure over y (at a projection angle of zero degrees) generated by the two simple affine mappings, from the plane to the plane,

$$w_1(x, y) = (\frac{1}{2}x, -0.51x + d_1 y) \tag{1}$$

$$w_2(x, y) = (\frac{1}{2}x + \frac{1}{2}, 0.03x + d_2 y - 0.35) \tag{2}$$

with the scaling parameters $d_1 = -0.8$ and $d_2 = -0.6$, when such mappings are iterated in an independent fashion according to a biased coin such that w_1 is used 30% of the time and w_2 the remaining 70%. As illustrated in Fig. 2, these notions also define a fractal function that passes by the three points

{(0,0), (1/2, –0.35), (1, –0.2)} whose graph has a fractal dimension of 1.485, and the invariant (and hence deterministic) measure dx, as identified in turbulence studies (Puente and Obregón, 1999). As a way of clarification, it should be emphasized that although it may appear to a casual reader that the obtained patterns may depend on the "coin" mentioned above, such is not the case, since the successive iterations are just a suitable Monte Carlo approach that always converges to the same deterministic pattern (Barnsley, 1988).

By varying the parameters of the geometric construction, that is, the points by which a given fractal function passes (which could contain more than three points), the scaling parameters that determine the fractal dimension of the transformation, the iteration frequencies that define a parent (multinomial) multifractal over x, and the projection angle (other than zero degrees in the y direction), one may indeed find a plethora of interesting patterns sharing the observed features of natural data sets (Puente, 2004). These resemble geophysical sets such as rainfall records and width functions of natural channels (usually modeled via stochastic cascades; e.g. Veneziano et al., 2000), among others (e.g. Puente and Obregón, 1996; Obregón et al., 2002; Puente and Sivakumar, 2003) and both chaotic and stochastic signals (with $1/f^\beta$ power spectra), as classified via phase-space reconstruction techniques (Puente et al., 2002).

3   Scientific and philosophical implications

As the advent of chaos theory has resulted in a debate of determinism vs. stochasticity as possible competing mechanisms to model natural data sets (e.g. Osborne and Provenzale, 1989; Koutsoyiannis and Pachakis, 1996; Sivakumar, 2000; Schertzer et al., 2002; Sivakumar et al., 2002a, b), it is pertinent to explain where the geometric ideas herein may fit within such a debate.

It ought to be stressed first that the fractal-multifractal procedure was not conceived as a means to endorse determinism and hence disprove randomness, but rather as a possible new framework for geophysical (hydrologic) complexity. The idea was developed as the first author attempted to employ extensions of fractal interpolating functions to represent mountain profiles from which to study the evolution of river networks (unsuccessfully) and when he later realized

that the natural measure generated via the iterations over x correspond to the spiky measures found in atmospheric turbulence and generated via deterministic multiplicative cascades. Such realization triggered the question: if turbulence along one line is just a permutation of a rather simple multiplicative cascade, could it be that rainfall and other turbulence-driven processes are also simple enough to be transformations (fractional integrations) of such turbulence? As seen in Fig. 2, and as illustrated in the diverse applications already mentioned and in others, the ideas turned out to be sensible indeed in one and higher dimensions (Puente 1994b; Puente et al., 2001a, b), even if a complete physical explanation for the procedure remains a challenge.

It should also be noted that the Platonic approach has not been developed to add to the aforementioned determinism vs. stochasticity debate, although the label of determinism may suggest otherwise. That the outcomes of the fractal-multifractal approach are deterministic is unquestionable; but they may also be interpreted, again in a natural way, as specific realizations of existing stochastic processes, as soundly developed elsewhere (e.g. Lovejoy and Schertzer, 1990; Veneziano et al., 2000). In this context, the geometric approach provides an alternative way of generating interesting sets, which may be used as simulations to drive a variety of water resource applications.

Clearly, however, there are noticeable differences between such stochastic approaches and the geometric methodology explained herein. While the latter, as applied to one data set, naturally tries to find an underlying multiplicative (multinomial) process which when fractally transformed provides a representation of a whole normalized set, stochastic approaches, by construction, concentrate on relevant statistical properties (e.g., power spectrum, multifractal spectrum, co-dimension function) to find a suitable cascade mechanism that may generate them as a realization. While the classical approaches can not control the specific realization and instead concentrate on relevant statistics of the sets, the geometric procedure, once an inverse problem is solved, aims at the overall geometry of the sets, one that if properly captured includes also its statistics. This makes the geometric procedure all the more ambitious.

Although producing similar outcomes, there are two potential features of the geometric procedure (absent by definition from the more classical approaches) that we believe deserve further study as they may result in relevant methodological breakthroughs.

First, a Platonic methodology (similar to the one herein but defined possibly via alternative transformations not using affine mappings) furnishes us with the real possibility of encoding holistically complete data sets, with rather substantial compression ratios. This can be readily noticed in Fig. 2, as the rather complex set therein is totally described by only 9 parameters. For, even if it is argued that the geometric approach uses "many more" parameters than the stochastic representations based on faithful fittings of co-dimension functions, the latter, in its "characterization" of the record's intermittency structure, can not uniquely define a given realization, which the geometric approach can in principle do, including records that may be termed chaotic or stochastic (Puente et al., 2002), as mentioned earlier.

Second, a geometric approach may open new vistas to study the dynamics of geophysical processes. The evolution of records may perhaps be discerned in the compressed parameter space of subsequent sets, as has been found in a contamination problem when the geometric ideas are extended to produce complex patterns over two dimensions (Puente et al., 2001a, b). Notwithstanding the difficulties involved in thinking in a surrogate space as done in other disciplines (e.g. strings in theoretical physics), the notions may perhaps provide a dual representation for understanding dynamics, one that may prove useful as it has the potential of capturing the ever important details (i.e., timing and magnitude) present in the data sets at hand.

Whether simplicity can be found at the root of complexity remains a tremendous challenge in science (see, however, Wainright and Mulligan, 2004; for a similar issue in environmental modeling). Undoubtedly, geophysical complexity is very hard to quantify and as such there are, no doubt, opportunities for further improvement. In regards to the geometric ideas herein, more research is needed in trying to solve a complex inverse optimization problem and in finding ways by which physical knowledge, as defined via conservation principles and differential equations, may be coupled with the geometric ideas. It is our hope that such issues will have a successful ending.

4 Conclusions

A geometric procedure aimed at modeling complex geophysical data sets as derived distributions of multifractal measures via fractal functions has been reviewed. It has been argued, both scientifically and philosophically, that such an approach may provide a suitable language for geophysical complexity, one that may be useful to simulate distinct phenomena of diverse types and one that one day may lead to a new vision to study geophysical dynamics. We envision that this fractal-based approach and other procedures aiming to capture mathematical morphology (e.g. Maragos, 1989) would result in improved understandings of complex natural patterns, thresholds and their dynamics.

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