Ann. Geophys., 29, 1179–1188, 2011
www.ann-geophys.net/29/1179/2011/
doi:10.5194/angeo-29-1179-2011

Annales Geophysicae

On the nature of particle energization via resonant wave-particle interaction in the inhomogeneous magnetospheric plasma

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Received: 1 January 2011 – Revised: 18 June 2011 – Accepted: 20 June 2011 – Published: 29 June 2011

Abstract. When a quasi-monochromatic wave propagating in an inhomogeneous magnetoplasma has sufficiently large amplitude, there exist phase-trapped resonant particles whose energy increases or decreases depending on the "sign" of inhomogeneity. The variation of energy density of such particles can greatly exceed the wave energy density which contradicts energy conservation under the prevalent assumption that the wave serves as the energy source or sink. We show that, in fact, the energy increase (or decrease) of phase-trapped particles is related to energy transfer from (to) phase untrapped particles, while the wave basically mediates the energization process. Virtual importance of this comprehension consists in setting proper quantitative constraints on attainable particle energy. The results have immediate applications to at least two fundamental problems in the magnetospheric physics, i.e. particle dynamics in the radiation belts and whistler-triggered emissions.

Keywords. Space plasma physics (Charged particle motion and acceleration; Nonlinear phenomena; Wave-particle interactions)

1 Introduction

Search for and understanding of mechanisms for particle energization is a key problem in physics of the Earth's radiation belts (e.g., Summers et al., 2007a,b; Trakhtengerts and Rycroft, 2008, early and recent references can be found in the latter citation). A good deal of suggested mechanisms is related to resonant wave-particle interactions (e.g., Albert, 2001, 2002; Shklyar and Kliem, 2006; Omura et al., 2007, and references therein). These mechanisms may be divided roughly into two groups. The mechanisms of the first group, which can be traced back to classical works by Andronov and Trakhtengerts (1964) and Kennel and Petschek (1966), deal with wave-particle interactions in the case of a wide wave spectrum, and are generally treated on the basis of quasi-linear theory as applied to magnetospheric conditions. Those mechanisms lead to particle diffusion in the phase space, and have characteristic times of the order of minutes or hours. The mechanisms of the second group involve resonant wave-particle interactions with a quasi-monochromatic waves. Quite a few references to the corresponding studies can be found in a review paper by Shklyar and Matsumoto (2009) and the above-mentioned works. In this case, the consideration is usually based on the analysis of particle motion in a given wave field, with a little concern about a back influence of resonant particles upon the wave. The characteristic time of the process is now determined by the duration of particle crossing of the wave packet, and, in the case of whistler-electron interactions, is of the order of seconds.

An important feature of wave-particle interaction with a quasi-monochromatic wave in a homogeneous or weakly inhomogeneous plasma is the existence of phase-trapped resonant particles. As we will show in the next sections, in the inhomogeneous case, the variation of energy density of the phase-trapped particles can be larger, or even much larger than the wave energy density. This raises the question of the corresponding energy source or sink. The answer to this question is given by the following consideration, which aims to reveal the nature of energization process.

In this paper, we consider energetic electron interaction with a whistler-mode wave, although the consideration is readily generalized to proton interaction with ion-cyclotron waves.

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Published by Copernicus Publications on behalf of the European Geosciences Union.

2 Particle equations of motion

Consider a whistler-mode wave propagating along a non-uniform ambient magnetic field $\boldsymbol{B}_0$. For the sake of definiteness, and keeping in mind applications of the following consideration to the outer radiation belt, we will assume the wave propagation along the field line $L \sim 4$. The wave electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ of a right-hand polarized wave propagating along the geomagnetic field may be written in the form:

$$E_x = -E\cos\xi;\ E_y = E\sin\xi;\ B_x = -B\sin\xi;\ B_y = -B\cos\xi; \tag{1}$$
$$\xi = \int^z k(z')dz' - \omega t\ ;\ \ B = \frac{kc}{\omega}E\ ,$$

where z is the coordinate along the ambient magnetic field, and it is assumed that the wave frequency ω and the wave number $k(z)$ are connected by a local dispersion relation for parallel propagating whistler-mode wave in a dense ($\omega_p^2 \gg \omega_c^2$) plasma:

$$\frac{k(z)^2c^2}{\omega^2} = \frac{\omega_p^2(z)}{\omega[\omega_c(z)-\omega]}\ , \tag{2}$$

where ω_p is electron plasma frequency, $\omega_c = eB_0/mc$ is the magnitude of electron gyrofrequency, $-e$ and m being the electron charge and mass, respectively, and c is the speed of light.

The variation of electron kinetic energy w caused by the interaction with the wave is determined by the equation

$$\frac{dw}{dt} = -e\boldsymbol{v}\cdot\boldsymbol{E}\ , \tag{3}$$

where $\boldsymbol{v}$ is electron velocity. Introducing the magnitude of electron perpendicular velocity $v_\perp$ and electron gyrophase φ according to relations

$$v_x = v_\perp\cos\varphi\ ;\ \ v_y = v_\perp\sin\varphi\ , \tag{4}$$

and taking into account Eqs. (1) and (4) we rewrite Eq. (3) in the form:

$$\frac{dw}{dt} = eEv_\perp\cos\zeta\ ;\ \ \zeta = \int^z k(z')dz' - \omega t + \varphi\ . \tag{5}$$

In the absence of the wave, the gyrophase φ varies according to equation $d\varphi/dt = \omega_c$. The energy variation is most significant for resonant particles whose parallel velocity $v_\parallel$ is close to the resonance value

$$v_R(z) = \frac{\omega - \omega_c(z)}{k(z)}\ , \tag{6}$$

because for such particles the total phase in the expression (5) is varying slowly, i.e.

$$\frac{d\zeta}{dt} \simeq k(v_\parallel - v_R) \tag{7}$$

is close to zero. It is important that the resonance velocity v_R varies along the field line due to variations of the quantities ω_c and k. Since for whistler-mode waves $\omega < \omega_c$, the quantity $v_R < 0$, so that the wave and resonant particles move in opposite directions. As has been demonstrated by many authors (e.g., Nunn, 1974; Karpman et al., 1975), and can easily be checked with the help of expressions (1), for resonant particles, the variation of transversal adiabatic invariant $\mu = mv_\perp^2/2\omega_c$ is connected with the variation of kinetic energy w by the following integral of motion:

$$w - \omega\mu \equiv C^2 = \text{const.}\ \ (C^2 > \frac{mv_R^2}{2})\ . \tag{8}$$

Since the variation of w and μ is determined by the variation of the total phase ζ, its behaviour constitutes the essence of the problem of resonance wave-particle interaction. Equation for the phase ζ has the form (e.g., Nunn, 1974; Karpman et al., 1975):

$$\frac{d^2\zeta}{dt^2} = \frac{1}{\tau^2}\cos\zeta - \alpha\ , \tag{9}$$

where nonlinear time τ and inhomogeneity parameter α are determined by the expressions:

$$\frac{1}{\tau^2} = hkv_{\perp R}\omega_c\ ;\ \ h \equiv \frac{B}{B_0}\ ; \tag{10}$$
$$\alpha = \frac{k}{2}\left(\frac{dv_R^2}{dz} + \frac{v_{\perp R}^2}{\omega_c}\frac{d\omega_c}{dz}\right),$$

with

$$v_{\perp R}^2 = \frac{\omega_c}{\omega_c - \omega}\left(\frac{2}{m}C^2 - v_R^2\right)\ . \tag{11}$$

It is easy to see that for $v_\parallel = v_R$, the quantity $v_{\perp R}$ coincides with particle perpendicular velocity $v_\perp$, which explains its physical meaning. Equation (9) that can also be written as a set of two first order equations is not a closed one since, in general, the quantities τ and α are slowly varying functions of the coordinate z. However, since this dependence caused by plasma inhomogeneity is slow, and since we are interested only in resonant particles, we may write missing equation as:

$$\frac{dz}{dt} = v_R(z)\ . \tag{12}$$

Solution to this equation which depends on an arbitrary constant z_i may be written in the form

$$t = \int_{z_i}^{z}\frac{dz'}{v_R(z')}\ . \tag{13}$$

For a given value of the parameter z_i, relation (13) defines the coordinate z as a one-to-one function of time: $z = z(t\ ;\ z_i)$. This function should now be substituted into Eq. (9) making it dependent on two parameters C^2 and z_i (see Eqs. 10, 11).

These parameters will further be assumed, although not written explicitly. We should mention that since t and z are biunique functions, either may be used as an independent variable in particle equations of motion.

Equation (9) describes particle motion in effective potential

$$P = \alpha\zeta - \frac{1}{\tau^2}\sin\zeta\,. \tag{14}$$

To avoid misunderstanding we should underline that P represents a valid potential related to the variables $(\zeta,\dot{\zeta})$, and conservation of the corresponding effective energy ϵ (see Eq. 29 below), which takes place in the approximation of constant τ and α, by no means imply conservation of particle kinetic energy in laboratory frame of reference. Nevertheless, consideration of particle dynamics in variables $(\zeta,\dot{\zeta})$ is most convenient and permits to find variations of particle kinetic energy and transversal adiabatic invariant in the most efficient way. Following the basic idea first formulated by O'Neil (1965), we will perform this consideration in the approximation of a given field, which is valid under condition

$$\gamma\tau \ll 1\,, \tag{15}$$

where γ is the wave growth (damping) rate. In this approximation, particle dynamics is considered in the given wave field, while the back influence of particles upon the wave is estimated from energy conservation. As we will see below, although the approximation of a given field requires slow variation of the wave amplitude, finite values of γ are essential for visible effect of particle energization.

For $\alpha\tau^2 < 1$, the potential P (Eq. 14) has potential wells and, hence, there are phase-trapped particles (hereinafter "trapped particles"). Since for such particles the coordinate ζ varies in a limited interval, the quantity $d\zeta/dt$ is zero on average; thus, particle parallel velocity oscillates around resonance value $v_{\rm R}$ (see Eq. 7). As in an inhomogeneous plasma the quantity $v_{\rm R}$ varies in space monotonically, the same is true for the average value of trapped particle parallel velocity while it moves along the geomagnetic field line inside the wave packet:

$$\overline{v}_{\parallel} = v_{\rm R}(z)\,. \tag{16}$$

Two Eqs. (8) and (16) permit to determine kinetic energy of a trapped particle as a function of coordinate and the integral of motion $C^2 = \text{const}$, namely:

$$w = C^2 + \frac{\omega}{\omega_{\rm c}(z)-\omega}\left[C^2 - \frac{mv_{\rm R}^2(z)}{2}\right]. \tag{17}$$

The rate of energy variation for a trapped particle that follows from Eq. (17) with the account of $dz/dt = v_{\rm R}$ and definition (10) is equal to

$$\frac{dw_{\rm T}}{dt} = \frac{m\omega}{k^2}\alpha\,, \tag{18}$$

thus, the energy variation of trapped particles has the sign of α. As we will see below, for reasonable behaviour of cold plasma density along the geomagnetic field line, the quantity $v_{\rm R}^2$ increases with increasing ω_c and vice versa. Equations (17), (18) then shows that kinetic energy of a trapped particle increases when it moves from a pole toward the equator ($\alpha > 0$) and decreases when it moves from the equator to another pole. (We remind the reader that in the case of parallel propagation considered here, a whistler-mode wave and resonant particles always move in opposite directions.) The absolute value of the corresponding energy variation is determined by the variations of $\omega_{\rm c}$ and $v_{\rm R}^2$ in the region where the particle remains trapped by the wave. As has been mentioned above, phase trapping is possible only for $\alpha\tau^2 < 1$. However, for $\alpha\tau^2 = 1$, the phase volume of trapped particles is equal to zero; it gradually increases with decreasing of $\alpha\tau^2$. We will define trapping region by the inequality:

$$\alpha\tau^2 < 1/3\,. \tag{19}$$

As the estimations show (see below), for $\alpha\tau^2 = 1/3$, the phase volume of trapped particles is equal to one half of its maximum value which is achieved at $\alpha = 0$. Clearly, the trapping region is centered on the equator where $\alpha = 0$. For reasonable wave amplitudes, quadratic approximation for electron cyclotron frequency (Trakhtengerts and Rycroft, 2008, e.g.,):

$$\omega_{\rm c} = \omega_{\rm c\,eq}\left[1 + \frac{9}{2}\left(\frac{z}{LR_{\rm E}}\right)^2\right] \tag{20}$$

may be used throughout the trapping region. Here L is McIlwain's parameter, $R_{\rm E}$ is the Earth's radius, and subscript "eq" denotes the equatorial value. Relation (20) shows that the inhomogeneity scale lengths of the problem is equal to $LR_{\rm E}$.

Assuming the relation between electron plasma and cyclotron frequencies in the form

$$\omega_{\rm p} \propto \omega_{\rm c}^{\eta}\,,\ \ (0 < \eta \lesssim 1/2)\,,$$

which includes both gyrotropic distribution of cold plasma density along a field line ($\eta = 1/2$) and constant density ($\eta = 0$) (e.g., Trakhtengerts and Rycroft, 2008), and using the definitions (10), (6), together with the dispersion relation (2) and the expansion (20), we rewrite the trapping conditions (19) in the form:

$$|z| < z_m \equiv \frac{2}{27}\frac{hv_{\perp\rm R}\omega_{\rm c}(LR_{\rm E})^2}{(v_{\perp\rm R}^2 + bv_{\rm R}^2)}\bigg|_{\rm eq}\,, \tag{21}$$

where equatorial values are taken for all quantities that depend on coordinate z, and

$$b = \frac{3\omega_{\rm c\,eq}}{\omega_{\rm c\,eq}-\omega} - \eta\,. \tag{22}$$

The consideration above is by no means a novelty (see, e.g., Karpman et al., 1975; Karpman and Shklyar, 1977; Trakhtengerts and Rycroft, 2008, and references therein), but it is necessary for deriving results that we aim at. To run a few steps forward we will mention that recently, in the investigations of particle acceleration in the field of a whistler wave in the magnetosphere, the main focus was on trapped particles, because the variation of their energy is much larger than for untrapped ones. Due to this fact, and since the analysis was usually made for a given wave field, it was somehow assumed that the energy for particle acceleration is provided by the wave. The fallacy of this conception, which does not take into account that the number of untrapped particles greatly exceeds the number of trapped ones, can be understood from the following consideration. If the energy balance would be between the wave and trapped particles, then the sign of the growth rate would be opposite to the sign of inhomogeneity (see Eq. 18), while, in fact, it depends mainly on resonant particle distribution, as well as on inhomogeneity (e.g., Karpman et al., 1975, see Sect. 4 for details).

3 Energy constraints on particle energization

Let us estimate the energy variation of a trapped particle while it moves inside the wave packet over the trapping region from z_m to the equator. From Eq. (17), (8) we get

$$\Delta w_{\mathrm{T}} = -\frac{\omega}{\omega_{\mathrm{c\,eq}}-\omega}\left(\mu\Delta\omega_{\mathrm{c}}+\frac{m}{2}\Delta v_{\mathrm{R}}^2\right) = -\frac{m}{2}\frac{\omega}{\omega_{\mathrm{c\,eq}}-\omega}\left(v_{\perp\mathrm{R\,eq}}^2+bv_{\mathrm{R\,eq}}^2\right)\frac{\Delta\omega_{\mathrm{c}}}{\omega_{\mathrm{c\,eq}}}\,. \tag{23}$$

As follows from Eq. (23), energy increase of trapped particles is accompanied by decrease of magnitude of their parallel velocity. This process is often called "acceleration", although the term "energization" that we use seems to be more apt. Nevertheless, referring to earlier works, we retain the original term "acceleration". Using Eqs. (20), (21) we obtain from (23):

$$\Delta w_{\mathrm{T}} = \frac{m}{81}\,\frac{\omega}{\omega_{\mathrm{c}}-\omega}\,\frac{h^2 v_{\perp\mathrm{R}}^2\omega_{\mathrm{c}}^2(LR_{\mathrm{E}})^2}{(v_{\perp\mathrm{R}}^2+bv_{\mathrm{R}}^2)}\Bigg|_{\mathrm{eq}}\,. \tag{24}$$

To get the energy density variation of trapped particles, ΔW_{T}, we should multiply Eq. (24) by the trapped particle density n_{T} that may be estimated as:

$$n_{\mathrm{T}} \sim n_{\mathcal{E}}\frac{\Delta v_{\|\mathrm{T}}}{v_{\mathcal{E}}}\,, \tag{25}$$

where $n_{\mathcal{E}}$ is the density of energetic electrons, $v_{\mathcal{E}}$ is their thermal velocity, $\Delta v_{\|\mathrm{T}}$ is an effective width, on the axis of parallel velocities, occupied by trapped particles, and it is assumed that the resonance velocity is of the order of thermal velocity of energetic electrons. Using

$$\Delta v_{\|\mathrm{T}} = \frac{1}{2}\left(\Delta v_{\|\mathrm{T}}\right)_{\max} = \frac{4}{\pi k\tau}\,, \tag{26}$$

(see Eq. 36 and the comments after it) with τ determined in Eq. (10), we get

$$\Delta W_{\mathrm{T}} \equiv n_{\mathrm{T}}\Delta w_{\mathrm{T}} = \frac{4n_{\mathcal{E}}}{\pi k\tau v_{\mathcal{E}}}\Delta w_{\mathrm{T}}\,, \tag{27}$$

where Δw_{T} is given by Eq. (24). Estimation (27) is valid if the wave amplitude B changes slowly in the trapping region that may assume the energy relation $U > \Delta W_{\mathrm{T}}$, where

$$U = \frac{B^2}{8\pi}\frac{\omega_{\mathrm{c}}}{\omega_{\mathrm{c}}-\omega} \tag{28}$$

is the wave energy density. However, in fact, the opposite strong inequality holds that can be proved by numerical estimations. Using the values of parameters typical of $L=4$ (cf. Gurnett et al., 2001; Katoh and Omura, 2004):

$\omega_{\mathrm{c\,eq}} = 8.5\times10^4\,\mathrm{rad\,s^{-1}}$

$\omega_{\mathrm{p\,eq}} = 7.9\times10^5\,\mathrm{rad\,s^{-1}}$; $n_{\mathrm{c}} = 195\,\mathrm{cm^{-3}}$;

$n_{\mathcal{E}} = 0.2\,\mathrm{cm^{-3}}$; $v_{\mathcal{E}} = 2.7\times10^9\,\mathrm{cm\,s^{-1}}$,

wave frequency $\omega = 3.14\times10^4\,\mathrm{rad\,s^{-1}}$ ($f = 5\,\mathrm{kHz}$), and wave amplitude $B = 3\times10^{-7}$ gauss (30 pT), we find:

$U = 5.7\times10^{-15}\,\mathrm{erg\,cm^{-3}}$; $\Delta W_{\mathrm{T}} = 2.3\times10^{-13}\,\mathrm{erg\,cm^{-3}}$,

while the total energy density of trapped particles $W_{\mathrm{T}} \sim 3\times10^{-12}\,\mathrm{erg\,cm^{-3}}$. We should mention that increasing the wave amplitude does not "improve" the situation, because $U \propto B^2$, while the quantity ΔW_{T}, as it follows from Eqs. (27), (10), and (24) is proportional to $B^{5/2}$.

For convenience of further estimations, we give characteristic values of some essential parameters that are secondary to those given above, namely: the index of refraction $N \equiv kc/\omega \simeq 19$, wave electric field $E \simeq 1.6\times10^{-8}$ statvolt $\mathrm{cm^{-1}}$ ($0.5\,\mathrm{mV\,m^{-1}}$), trapping length $z_m \simeq 2000\,\mathrm{km}$, trapping time, i.e. the time during which a particle remains trapped by the wave: $z_m/|v_{\mathrm{R}}| \simeq 0.075\,c$, and nonlinear time $\tau \simeq 0.002\,c$. Definition (19) then gives the maximum value of inhomogeneity parameter in the trapping region $\alpha_m \simeq 9.5\times10^4\,c^{-2}$.

Concerning the energy estimations presented above, the question arises as to where the energy increase of trapped particles comes from. The answer to this question consists in the following. Along with the trapped particles, there are untrapped resonant particles whose contribution to wave-particle interaction is equally important as that of trapped ones. While trapped particles remain in resonance with the wave for a long time and undergo significant energy variation, phase volume of untrapped particles is continuously renewing. This can easily be seen from the sketch of effective potential (14) shown in Fig. 1. The rate of this renewing is proportional to the magnitude of the inhomogeneity parameter α. Energy variation of an untrapped particle during the time of resonant interaction with the wave is much smaller than for trapped particles. In return, the total number of untrapped resonant particles is much larger, while the rates of energy variation and the phase volumes of trapped and untrapped particles interacting with the wave at a given instant of time are of the same order.

A peculiarity of resonance interaction in an inhomogeneous plasma is that, on the average, the energy variation has different sign for trapped and untrapped particles. A hint to understand this feature consists in that the phase volumes of trapped and untrapped particles are not symmetrical with respect to the phase ζ, while the energy variation is proportional to $\cos\zeta/\alpha\tau^2$ (see Eq. 5). For illustration, the quantity $\cos\zeta/\alpha\tau^2$ is plotted in Fig. 1 by dotted line. Moreover, it appears that the average energy variation for untrapped particles is proportional to the phase volume of trapped particles, while the rate of untrapped particle phase volume renewing is proportional to $|\alpha|$, i.e. to the rate of energy variation of trapped particles (see Eq. 18). Eventually, the net energy flux to the wave and, thus, the wave growth rate is determined, along with inhomogeneity parameter, by the difference in distribution functions of trapped and untrapped particles (see Sect. 4). As this difference is much smaller than the distribution function itself, the wave energy variation appears to be much smaller than energy variations of each group (trapped and untrapped) of particles. This very feature permits to treat wave-particle interaction in the approximations of a given field (O'Neil, 1965), i.e. to consider particle dynamics in the field of a given wave, and then find the variation of wave energy density from energy conservation, using the method of successive approximations.

To prove the statements expressed above, let us calculate average energy variation of an untrapped particle in the time of its resonant interaction with the wave. Since this variation is most significant close to reflection point $d\zeta/dt=0$ (see Fig. 1) where resonance conditions (7) are fulfilled exactly, we may replace the quantities τ and α by their values at this point. Equation (9) then has the integral of motion:

$$\epsilon=\frac{1}{2}\left(\frac{d\zeta}{dt}\right)^2+\alpha\zeta-\frac{1}{\tau^2}\sin\zeta\,. \qquad (29)$$

We now integrate Eq. (5) over the time of resonant interaction centered on the reflection time, making use of $dt=d\zeta/\sqrt{2(\epsilon-\alpha\zeta+\sin\zeta/\tau^2)}$ (see Eq. 29) that gives:

$$\Delta w_{\text{UT}}=\frac{\sqrt{2}m\omega}{k^2\tau^2}\Phi(\epsilon)\,,$$
$$\Phi(\epsilon)=\int_{-\alpha\infty}^{\zeta_r(\epsilon)}\frac{\text{sign}\alpha\,\cos\zeta\,d\zeta}{\sqrt{\epsilon-\alpha\zeta+\sin\zeta/\tau^2}}\,, \qquad (30)$$

where the reflection point $\zeta_r(\epsilon)$ satisfies $\epsilon-\alpha\zeta_r+\sin\zeta_r/\tau^2=0$. In deriving Eq. (30), we have used the relations (1), (10) and have taken into account a fast convergence of the integral in non-resonant region. The function $\Phi(\epsilon)$ defined in Eq. (30) is the essential factor that determines the quantity Δw_{UT}. It is easy to see that $\Phi(\epsilon)$ is

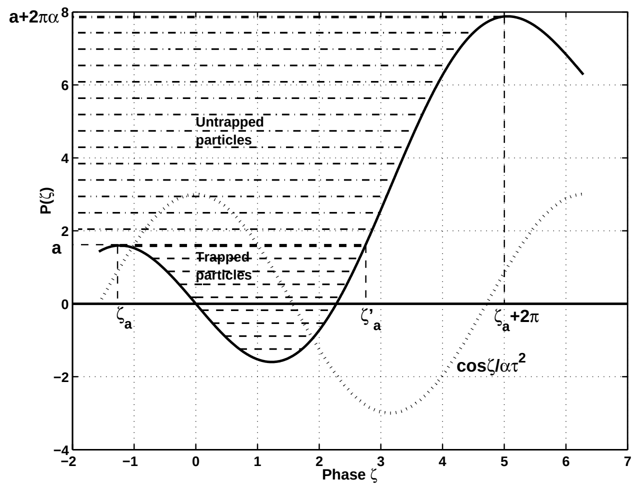

Fig. 1. Effective potential (14) for $\alpha>0$, with trapped and untrapped particles. It is clearly seen that $\text{sign}(\cos\zeta)=-\,\text{sign}\,\alpha$ for the majority of reflection coordinates of untrapped resonant particles, which ensures that the energy variations for trapped and untrapped particles have opposite signs.

a periodic function of ϵ with the period $2\pi|\alpha|$ and, thus, has the Fourier expansion of the form

$$\Phi(\epsilon)=b_0+\sum_{n=1}^{\infty}b_n\cos\frac{n\epsilon}{|\alpha|}+c_n\sin\frac{n\epsilon}{|\alpha|}\,, \qquad (31)$$

where

$$b_0=\frac{1}{2\pi|\alpha|}\int_a^{a+2\pi|\alpha|}d\epsilon\int_{-\alpha\infty}^{\zeta_r(\epsilon)}\frac{\text{sign}\alpha\,\cos\zeta\,d\zeta}{\sqrt{\epsilon-\alpha\zeta+\sin\zeta/\tau^2}}\,. \qquad (32)$$

Changing the order of integrals and performing integration with respect to ϵ we obtain

$$b_0=-\frac{\tau^2}{\pi}\text{sign}\alpha\int_{\zeta_a}^{\zeta_a'}\sqrt{a-\alpha\zeta+\sin\zeta/\tau^2}\,d\zeta\,, \qquad (33)$$

where the meaning of the quantities a,ζ_a, and ζ_a' is clear from Fig. 1. In particular, for $\alpha>0$ and for the period of potential $P(\zeta)$ (Eq. 14) that includes $\zeta=0$, which is shown in Fig. 1, the corresponding quantities are equal to:

$$\zeta_a=-\text{acos}(\alpha\tau^2)\,;\;a=-\alpha\cdot\text{acos}(\alpha\tau^2)+\sqrt{\tau^{-4}-\alpha^2}\,, \qquad (34)$$

and ζ_a' is the second point in this interval of periodicity where the potential is equal to a: $P(\zeta_a)=P(\zeta_a')=a$.

The coefficient b_0 (Eq. 33) is uniquely related to the phase volume of trapped particles Ω_{T}, or, equivalently, to average width, over parallel velocities, of trapped particle region defined as:

$$\Delta v_{\parallel\text{T}}\equiv\frac{\Omega_{\text{T}}}{2\pi k}\,;$$
$$\Omega_{\text{T}}\equiv\int_{\Omega_{\text{T}}}\int d\zeta\,d\dot{\zeta}=2^{3/2}\int_{\zeta_a}^{\zeta_a'}\sqrt{a-\alpha\zeta+\sin\zeta/\tau^2}\,d\zeta\,, \qquad (35)$$

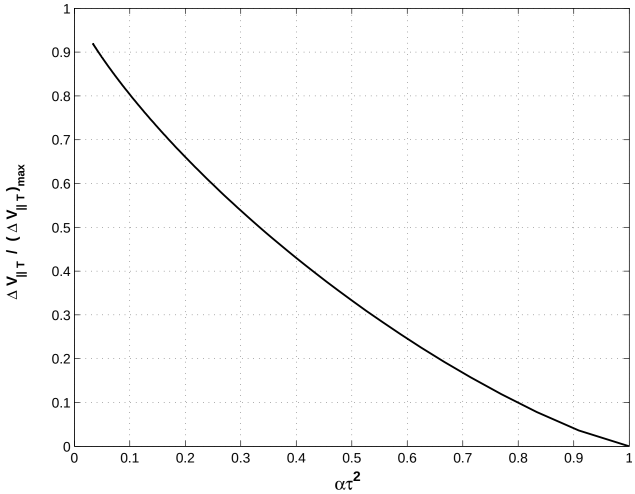

Fig. 2. Effective width (over parallel velocities) of trapped particle region, normalized to its maximum value $8/\pi k\tau$, as the function of dimensionless parameter $\alpha\tau^2$.

and, thus,

$$\sqrt{2}b_0 = -\mathrm{sign}\alpha \cdot \tau^2 \Omega_{\mathrm{T}}/2\pi \equiv -\mathrm{sign}\alpha \cdot \tau^2 k \Delta v_{\parallel \mathrm{T}} .$$

From (35) it follows that the maximum value of $\Delta v_{\parallel \mathrm{T}}$ is achieved at $\alpha\tau^2 \to 0$ and is equal to

$$\left(\Delta v_{\parallel \mathrm{T}}\right)_{\max} = 8/\pi k \tau . \tag{36}$$

The graph of $\Delta v_{\parallel \mathrm{T}}$ for arbitrary $\alpha\tau^2 < 1$ is shown in Fig. 2. It shows that at $\alpha\tau^2 = 1/3$, $\Delta v_{\parallel \mathrm{T}} \simeq (1/2)\left(\Delta v_{\parallel \mathrm{T}}\right)_{\max}$. These relations were used in Eq. (26). Using Eqs. (30)–(35) we can rewrite the expression for Δw_{UT} in the form:

$$\Delta w_{\mathrm{UT}} = -\mathrm{sign}\alpha \frac{m\omega}{k} \Delta v_{\parallel \mathrm{T}} \tag{37}$$

that proves in fact the statements made above.

Concerning the wave growth rate γ, it has earlier been calculated from the resonant particle current (Karpman et al., 1975), although the consequences of the expression for γ have not been discussed in detail. The above consideration permits to derive the corresponding expression in a simple and visual way. We give this derivation since the results we arrive at are essential for further analysis.

4 Whistler wave growth rate in an inhomogeneous plasma in the presence of trapped particles

The averaged energy variations of trapped and untrapped particles calculated above permit an estimation of the wave growth rate that sets in far enough from the wave packet front and depends on averaged distribution functions of trapped and untrapped particles. The wave growth rate γ is determined by the well known relation which expresses energy conservation in the system "wave- resonant particles":

$$\frac{dU}{dt} \equiv 2\gamma U = - < \boldsymbol{j}_{\mathrm{R}} \boldsymbol{E} > , \tag{38}$$

where the wave energy density U is determined by Eq. (28), $\boldsymbol{j}_{\mathrm{R}}$ is resonant particle current:

$$\boldsymbol{j}_{\mathrm{R}} = -e \int \boldsymbol{v} f_{\mathrm{R}} d\boldsymbol{v} , \tag{39}$$

and $< ... >$ stands for averaging over the wave period. The integral in Eq. (39) is extended over the resonance region. Using Eq. (39) and taking into account that $-e\boldsymbol{v}\boldsymbol{E}$ is the rate of particle kinetic energy variation we obtain from Eq. (38)

$$\gamma = -\frac{1}{2U} \left\langle \int \frac{dw_{\mathrm{R}}}{dt} f_{\mathrm{R}} d\boldsymbol{v} \right\rangle . \tag{40}$$

As was mentioned above, we are interested in asymptotic value of the growth rate related to energization processes and determined by averaged particle distribution functions. Since the resonance region of untrapped particles is continuously renewing, their averaged distribution function is close to unperturbed one F_0:

$$\bar{f}_{\mathrm{UT}} = F_0(w, \mu)_{w = m v_{\mathrm{R}}^2/2 + \mu\omega_{\mathrm{c}}} , \tag{41}$$

and it is assumed that F_0 depends on particle invariants of motion in the absence of wave field, i.e. kinetic energy w and transversal adiabatic invariant μ. For trapped particles, the averaged distribution function is equal, by the Liouville's theorem, to unperturbed distribution function corresponding to initial values w_0 and μ_0 at which a particle has been trapped by the wave:

$$\bar{f}_{\mathrm{T}} = F_0(w_0, \mu_0) . \tag{42}$$

According to Eqs. (18), (8)

$$w - w_0 = \omega(\mu - \mu_0) = \frac{m\omega}{k^2} \int_{z_0}^{z} \alpha(z') \frac{dz'}{v_{\mathrm{R}}(z')} , \tag{43}$$

where the integral is taken over the trapping region, with the account of $dt = dz/v_{\mathrm{R}}(z) > 0$ which holds for trapped particles, and z_0 is the coordinate at which a particle has been trapped by the wave. If the sign of α does not change along the integration region, i.e. if the wave packet is situated from one side of the equator, which will further be assumed, then the integral on the right-hand side of Eq. (43) has always the same sign as $\alpha(z)$. We should stress that the quantity z_0, as well as α, τ and $\Delta v_{\parallel \mathrm{T}}$ depends on μ as a parameter, which we omit for the sake of shortness.

We now transform the variables of integration in Eq. (40) to $\zeta, \dot{\zeta}, \mu$, divide the integral into contributions from trapped and untrapped particles, and make use of Eqs. (42) and (41). As the result we obtain:

$$\gamma \equiv \gamma_{\mathrm{T}} + \gamma_{\mathrm{UT}} = -\frac{\omega_c}{2mkU} \left[\int d\mu \bar{f}_{\mathrm{T}} \int_{\Omega_{\mathrm{T}}} \int \frac{dw_{\mathrm{T}}}{dt} d\dot{\zeta} d\zeta + \int d\mu \bar{f}_{\mathrm{UT}} \int_{\Omega_{\mathrm{UT}}} \int \frac{dw_{\mathrm{UT}}}{dt} d\dot{\zeta} d\zeta \right] , \tag{44}$$

where $\Omega_{\rm T}$ and $\Omega_{\rm UT}$ are elementary (corresponding to $\Delta\zeta = 2\pi$) phase volumes on the $(\zeta,\dot{\zeta})$-plane of trapped and untrapped particles, respectively. Here and further, the integral with respect to μ is taken over the domain of μ where $\Omega_{\rm T}$ and $\Delta v_{\parallel \rm T}$ are greater than zero.

The quantity $dw_{\rm T}/dt$ is determined by Eq. (18), so that the contribution of trapped particles to the wave growth rate is equal to:

$$\gamma_{\rm T} = -\frac{\pi\omega\omega_c}{k^2 U}\int d\mu\, \bar{f}_{\rm T}\,\alpha(z)\Delta v_{\parallel \rm T}\,, \tag{45}$$

with $v_{\parallel \rm T}$ being determined in Eq. (35).

To find the contribution of untrapped particles we notice that far from resonance, the quantity $\dot{\zeta}$ varies according to $d\dot{\zeta}/dt \simeq -\alpha$. Due to phase volume conservation, the phase volume (per interval $\Delta\zeta = 2\pi$) of particles which enter the resonance region on the $(\zeta,\dot{\zeta})$-plane during a time interval Δt is equal to $2\pi|\alpha|\Delta t$, each particle experiencing an average energy variation determined by Eq. (37), thus

$$\int_{\Omega_{\rm UT}}\int \frac{dw_{\rm UT}}{dt}d\dot{\zeta}d\zeta = -\frac{2\pi m\omega}{k}\alpha(z)\Delta v_{\parallel \rm T}\,, \tag{46}$$

giving

$$\gamma_{\rm UT} = \frac{\pi\omega\omega_{\rm c}}{k^2 U}\int d\mu\, \bar{f}_{\rm UT}\,\alpha(z)\Delta v_{\parallel \rm T}\,. \tag{47}$$

Combining Eqs. (45) and (47) we finally obtain:

$$\gamma = \frac{\pi\omega\omega_{\rm c}}{k^2 U}\int d\mu(\bar{f}_{\rm UT} - \bar{f}_{\rm T})\alpha(z)\Delta v_{\parallel \rm T}\,. \tag{48}$$

Formula (48) gives the required asymptotic expression for growth rate of whistler-mode wave propagating along a non-uniform magnetic field in an inhomogeneous plasma, in the presence of phase trapped particles. Equivalent expression has earlier been obtained in a different way by Karpman et al. (1975). For numerical estimations, it is useful to bear in mind that the integral $\int d\mu\, \bar{f}_{\rm T}\Delta v_{\parallel \rm T}$ is proportional to total density of trapped particles, namely:

$$\int d\mu\, \bar{f}_{\rm T}\Delta v_{\parallel \rm T} = \frac{m}{2\pi\omega_c}n_{\rm T}\,.$$

5 Analysis of the expression for γ – formation of "beams" and "holes"

For coordinates z inside the wave packet, such that the quantity $(w - w_0)$ determined by Eq. (43) is small as compared to thermal energy of resonant particles, the difference $\bar{f}_{\rm UT} - \bar{f}_{\rm T}$ in Eq. (48) may be expanded to the first order in $(w_0 - w)$ and $(\mu_0 - \mu)$, which gives, with the account of (43):

$$\bar{f}_{\rm UT} - \bar{f}_{\rm T} = F_0'(\mu)\cdot\frac{m}{k^2}\int_{z_0}^{z}\alpha(z')\frac{dz'}{v_{\rm R}(z')}\,;$$
$$F_0'(\mu) \equiv \left(\frac{\partial F_0}{\partial\mu} + \omega\frac{\partial F_0}{\partial w}\right)_{w = mv_{\rm R}^2/2 + \mu\omega_{\rm c}}. \tag{49}$$

Substituting Eq. (49) into Eq. (48) we get:

$$\gamma = \frac{\pi m\omega\omega_{\rm C}}{k^4 U}\int d\mu F_0'(\mu)\alpha(z)\Delta v_{\parallel \rm T}\int_{z_0}^{z}\alpha(z')\frac{dz'}{v_{\rm R}(z')}\,. \tag{50}$$

Expressions (48), (50) show that in an inhomogeneous plasma, and in the presence of trapped particles, the wave growth rate is nonlocal, i.e., the growth rate at a given point z depends on the position of wave packet along the geomagnetic field line, especially with respect to the equator. In particular, if the whole wave packet is situated at one side from the equator, then the sign of γ does not depend on the sign of inhomogeneity parameter α, but is determined by the sign of the combined derivative $F_0'(\mu)$ (Eq. 49), similar to linear growth rate (Sagdeev and Shafranov , 1961):

$$\gamma_{\rm L} = \omega\frac{\pi^2 e^2 B^2\omega_{\rm c}^2}{m^2 k^3 c^2 U}\int_0^\infty d\mu F_0'(\mu)\mu\,. \tag{51}$$

Since the wave energy density U is proportional to B^2, the linear growth rate does not depend on wave amplitude, of course. In contrast to this, in the case under discussion the growth rate is essentially nonlinear, the corresponding expressions being valid only under conditions $\gamma\tau \ll 1$ and $\alpha\tau^2 < 1$ (cf. Eqs. 15 and 19) that prevents transition to the linear case.

The above consideration that divides the wave growth rate into contributions from trapped and untrapped particles gives the conditions under which the wave energy variation is much smaller than the rate of energy variation of trapped as well as untrapped particles. Clearly, these conditions have the form:

$$|\bar{f}_{\rm UT} - \bar{f}_{\rm T}| \sim \left|F_0'(\mu)\cdot\frac{m}{k^2}\int_{z_0}^{z}\alpha(z')\frac{dz'}{v_{\rm R}(z')}\right| \ll |\bar{f}_{\rm UT}|\,, |\bar{f}_{\rm T}|\,, \tag{52}$$

i.e., the difference between distribution functions of trapped and untrapped particles should be small as compared to the distribution functions themselves. The fact that the wave energy variation determined by growth (or damping) rate γ is indeed much smaller than the rate of energy variation of each group of particles, which is necessary for the inequality $U \ll \Delta W_{\rm T}$ demonstrated above to be fulfilled, indicates that conditions (52) are fulfilled in real situation, at least with the parameters used. It means that, under conditions (52), energy exchange between trapped and untrapped particles is much more significant than between wave and particles. Thus, if the energy of trapped particles increases, the source of energy is not the wave, as was habitually assumed, but the untrapped particles, while the wave only mediates the energy transfer.

It is easy to see that in the process of energy exchange between trapped and untrapped particles the energy is always transferred from more energetic to less energetic particles, because, on the average, the energy variations of trapped and untrapped particles are differently directed, while at exact resonance their energies are equal. These arguments apply to particles with the same value of the integral of motion C^2 (Eq. 8). Consistent with this is the above-mentioned fact that the sign of growth rate is not determined by the sign of inhomogeneity, but depends on the features of distribution function.

An inevitable result of particle dynamics described above consists in formation of "beams" or "holes" on particle distribution function. (In analysing this effect, we assume that the unperturbed distribution function depends on C^2 and w which permits to express its variation through the variation of particle kinetic energy w.) Indeed, according to Liouville's theorem, distribution function is conserved along phase trajectories, thus the distribution function of trapped particles is typical of the region of the phase space where the particles become trapped by the wave. At the same time, the distribution function of untrapped particles is, on average, close to the unperturbed one since the phase space of untrapped particles is continuously renewing. As the result, there appear sharp gradients of the distribution function in the resonant region of the phase space. This effect depends on both the inhomogeneity and the shape of unperturbed distribution. If, for example, the wave packet moves toward a pole and is located at one side from the equator, then trapped particle carry over their distribution function from lower energy region of the phase space, where they have been trapped by the wave at the packet front, to higher energy region. If the unperturbed distribution function decreases with increasing energy, then at the trailing edge of the wave packet a "beam" in the resonance region of the distribution function will be formed. If, on the contrary, the unperturbed distribution function increases with increasing energy, which corresponds to an unstable distribution, then at the trailing edge of the wave packet a "hole" in the resonance region of the phase space will be formed. For a wave packet moving toward the equator the situation will be opposite to that described above, namely, a "hole" and a "beam" will be formed in the cases of stable and unstable distributions, respectively. An important consequence of the inequality (52) consists in that no strongly pronounced beams or holes on the distribution function, which assume significant difference between $\bar{f}_{\mathrm{UT}}$ and $\bar{f}_{\mathrm{T}}$, may be formed without essential variation of the wave energy.

As has already been emphasised, our consideration is not valid for large growth rates. Since the difference $\bar{f}_{\mathrm{UT}} - \bar{f}_{\mathrm{T}}$ is only one factor in the expression for γ, the inequality (52) is not sufficient condition for the employed approximation of a given field (15) to be valid. To find the corresponding conditions, we rewrite the expression (50) for γ with the help of Eq. (51) that gives:

$$\gamma \sim \gamma_{\mathrm{L}} \left\langle \alpha(z)\tau^3 \int_{z_0}^{z} \alpha(z') \frac{dz'}{v_{\mathrm{R}}(z')} \right\rangle , \quad (53)$$

where < ... > stands for averaged (with respect to μ) value of the corresponding expression calculated with the weighting function $F_0'(\mu)\mu$. Using the estimation for γ (Eq. 53) and relations (49) we obtain from Eq. (15):

$$\tau \int_{z_0}^{z} \alpha(z') \frac{dz'}{v_{\mathrm{R}}(z')} \ll \frac{1}{(\gamma_{\mathrm{L}}\tau)(\alpha\tau^2)} , \quad (54)$$

or, equivalently,

$$|\bar{f}_{\mathrm{UT}} - \bar{f}_{\mathrm{T}}| \ll F_0' \cdot \frac{m}{k^2\tau} \cdot \frac{1}{(\gamma_{\mathrm{L}}\tau)(\alpha\tau^2)} . \quad (55)$$

In Eqs. (54), (55), which have a character of estimations, average values of quantities that depend on μ are assumed. An essential quantity which enters into these inequalities is the linear growth rate γ_{L} determined by the expression (51). To find an explicit expression for γ_{L}, we will use a distribution of energetic electrons with a loss-cone of the Dory-Guest-Harris type (Dory et al., 1965):

$$f_0(w,\mu) = \left(\frac{m}{2\pi}\right)^{3/2} \frac{n_{\mathcal{E}} \omega_c^j \mu^j}{\Gamma(j+1) w_{\mathcal{E}}^{j+3/2}} e^{-w/w_{\mathcal{E}}} , \quad (56)$$

where $n_{\mathcal{E}}$ is the energetic electron density, $w_{\mathcal{E}}$ is a characteristic thermal energy of energetic electrons equal to $mv_{\mathcal{E}}^2/2$, j is a positive quantity determining the loss cone, and Γ is the Gamma-function. Evaluation of the linear growth rate (Eq. 51) with the help of Eq. (56) gives:

$$\gamma_{\mathrm{L}} = \frac{4\pi^{3/2} n_{\mathcal{E}} e^2 \omega\omega_c}{mk^3c^2 v_{\mathcal{E}}} e^{-v_{\mathrm{R}}^2/v_{\mathcal{E}}^2} \left[j - \frac{\omega}{\omega_c}(j+1) \right] \left(1 - \frac{\omega}{\omega_c} \right) . \quad (57)$$

Numerical estimation of γ_{L} with the help of parameters listed in Sect. 3 and $j = 1.1$ gives $\gamma_{\mathrm{L}} \simeq 11.6\, c^{-1}$, which permits to estimate the value of nonlinear growth rate (Eq. 53) as $\gamma \simeq 10$, so that $\gamma\tau \simeq 0.02$ which justifies the approximation of a given field (15) used in the present study. We remind the reader that inequalities (54) and (55) are different forms of the condition (15).

An obvious consequence of the energization process described above is the appearance of highly energetic electrons in the Earth's radiation belts. Another one is related to excitation of whistler-triggered emissions (Helliwell, 1969; Omura et al., 1991; Nunn, 2003). These emissions may be generated due to described above specific features of the distribution function at the trailing edge of the main wave packet (see, e.g., Istomin et al., 1976; Omura et al., 1991; Nunn, 2003; Trakhtengerts and Rycroft, 2008, and references therein).

Before concluding this section, one remark is in order. Imagine that the unperturbed distribution F_0 is "flat" in the sense that the combined derivative $F_0'(\mu)$ (Eq. 49) is zero in the whole resonance region. All features described above, namely, energy transfer between trapped and untrapped particles mediated by the wave, whose energy does not change at all in this case, remain in effect, but do not produce observable consequences as both wave and particle distribution function remain unchanged. Thus, an essential characteristic of energization process is the variation in distribution function which is produced. As we have seen above, in the frame of present consideration, which uses the approximation of a given field, this variation is limited by the condition (55) that follows from Eq. (15).

6 Concluding remarks

A systematic investigation of resonant wave-particle interaction in an inhomogeneous plasma has been undertaken in 1970s, and the idea of electron acceleration caused by interaction with quasimonochromatic whistler-mode waves in the magnetosphere can be traced back to the corresponding works from this period. We will refer to a review paper by Matsumoto (1979) where the references to the most important studies on this subject fulfilled by that time can be found. Recently, the interest to this issue has been recommenced in connection with the problem of spacecraft safe functioning in the Earth's radiation belts. The idea of electron acceleration by whistler-mode waves has been developed and enriched by including relativistic effects into consideration (e.g., Demekhov et al., 2006; Omura et al., 2007), and by considering electron acceleration by whistler-mode waves of varying frequency (e.g., Demekhov et al., 2006; Trakhtengerts and Rycroft, 2008). Most works devoted to particle acceleration, at least analytical ones, used the approximation of a given wave field and treated electron acceleration as an acceleration by a wave, which may assume that electron energy is derived from the wave energy.

The notion originating from the present study may be summarized as follows. The interacting system under consideration consists of the wave with energy density U that includes oscillation energy of "cold" (non-resonant) particles, resonant trapped particles with kinetic energy density W_{T}, and resonant untrapped particles with kinetic energy density W_{UT}. The energy conservation requires that the variations of these quantities satisfy the equation:

$$\Delta U + \Delta W_{\mathrm{T}} + \Delta W_{\mathrm{UT}} = 0; \qquad (58)$$

There are only two possibilities to satisfy this equation: either all three quantities are of the same order, or one is much smaller than other two, which are close in magnitude, but have different signs. The question is, which of these two possibilities is in point of fact. This is not an idle question, and the answer to it is not obvious. Since energy variation for an untrapped particle is much smaller than for a trapped particle, one may think that in the Eq. (58), $\Delta W_{\mathrm{UT}} \ll \Delta W_{\mathrm{T}}$ and, thus, $\Delta U \simeq -\Delta W_{\mathrm{T}}$, so that the energy balance is basically between wave and trapped particles. We have shown that this assumption is incorrect since it does not take into account that, in an inhomogeneous plasma, the number of untrapped resonant particles which interact with the wave on a time scale greater than τ is much larger than the number of trapped particles. It appears that, in fact, $\Delta U \ll \Delta W_{\mathrm{T}}$, W_{UT}, while $\Delta W_{\mathrm{UT}} \simeq -\Delta W_{\mathrm{T}}$. Thus, in resonant wave-particle interaction in an inhomogeneous plasma, the energy increase (decrease) of trapped particles, for the most part, comes from (goes to) untrapped particles, while the wave mainly mediates the energy transfer. It implies, among other things, that in the case of a lack of untrapped particles, the accelerating (decelerating) trapped particles will strongly damp (enhance) the wave, because, as has been shown above, the wave energy density is usually much smaller than the energy density increase (decrease) of trapped particles. On the contrary, in the case of an abundance of untrapped particles (or, which is the same, a lack of trapped particles that is equivalent to a "hole" in the phase space), the situation will be opposite to that described above, namely: accelerating trapped particles will enhance the wave and vice versa. The outcome of this work is important not only for understanding the nature of energization process, but also for setting proper quantitative constraints on attainable electron energy increase (under appropriate conditions of course) that can be orders of magnitude larger than it would be expected if the energy source were the wave energy. The results obtained have immediate applications to particle dynamics in the radiation belts and to generation of whistler triggered emissions.

Acknowledgements. This work was supported by grant N10-02-93115 of RFBR-CNRS.

Topical Editor I. A. Daglis thanks J. Bortnik and two other anonymous referees for their help in evaluating this paper.

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