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Standing Alfvén waves with m $\gg$ 1 in an axisymmetric magnetosphere excited by a stochastic source

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Abstract. In the framework of an axisymmetric magnetospheric model, we have constructed a theory for broad-band standing Alfvén waves with large azimuthal wave number $m \gg 1$ excited by a stochastic source. External currents in the ionosphere are taken as the oscillation source. The source with statistical properties of "white noise" is considered at length. It is shown that such a source drives oscillations which also have the "white noise" properties. The spectrum of such oscillations for each harmonic of standing Alfvén waves has two maxima: near the poloidal and toroidal eigenfrequencies of the magnetic shell of the observation. In the case of a small attenuation in the ionosphere the maximum near the toroidal frequency is dominated, and the oscillations are nearly toroidally polarized. With a large attenuation, a maximum is dominant near the poloidal frequency, and the oscillations are nearly poloidally polarized.

Key words. Ionosphere-magnetosphere interaction · Wave propagation · Magnetospheric physics · MHD waves and instabilities

1 Introduction

This paper is the continuation of Leonovich and Mazur (1993) where a comprehensive theoretical description of the spatial structure of standing Alfvén waves having a fixed frequency ω and a large wave number $m \gg 1$ is provided for a model of the axially symmetric magnetosphere. The major factor determining this structure is the Alfvén wave's transverse dispersion associated with the curvature of geomagnetic field lines as reported in Leonovich and Mazur (1990). For real geomagnetic field and plasma parameters, this dispersion exceeds greatly the well-known kinetic dispersion (Hasegawa, 1976; Hasegawa and Uberoi, 1982; Goertz, 1984), and it is precisely this dispersion that is responsible for the transverse (across the geomagnetic field) propagation of standing Alfvén waves.

We now outline the spatial structure of the monochromatic standing Alfvén wave as it follows from results reported in the cited work. A special role in this description is played by two particular magnetic shells: the poloidal and toroidal resonance surfaces, whose position is determined by the wave's frequency ω. On the former, the frequency ω coincides with the local frequency of poloidal eigen-oscillations, and on the latter, it coincides with the local frequency of toroidal Alfvén oscillations of the magnetosphere. The distance between the resonance surfaces is relatively short compared with the size of the surfaces themselves, but it is in the space between them that the mode described is localized. It is a standing mode in the longitudinal (along the geomagnetic field) direction. For fundamental harmonics ($N = 1, 2, 3, \ldots$), the longitudinal wavelength is of the order of the field line length. Transversally, the mode is an extremely small-scale one, and this is the case both azimuthally ($m \gg 1$) and normally to magnetic shells – many wavelengths find room between the resonance surfaces. And this warrants the applicability of the WKB approximation in a direction normal to the magnetic shells. The square of an appropriate component of a quasi-classical wave vector k_n^2 goes to zero on the poloidal surface and becomes infinite on the toroidal surface. In other words, the poloidal surface constitutes a usual turning point, while the toroidal surface represents a singular turning point. The space located outside the layer between the resonance surfaces is an opacity region for the waves under consideration, that is, $k_n^2 < 0$ holds in it. In a normal direction, the wave travels from poloidal to toroidal surface, and it travels azimuthally, so that its

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ray trajectories in transverse coordinates are spirals, and the sense of their twisting depends on the sign of the wave number m.

The wave is generated by an external source in the neighbourhood of the poloidal surface. This is not to say that the source is localized near this surface (recall that its position depends on the frequency ω); on the contrary, it is anticipated that it is a reasonably dispersed source. The matter is that the effectiveness of generation is highest near the poloidal surface and decreases dramatically with distance from it because of spatial refinement of the mode (i.e. a growth of k_n^2). The source can have a different nature. The cited paper considers external currents in the ionosphere produced by neutrals drifting in the E layer to be such a source.

On the toroidal surface the wave is entirely absorbed. Such an absorption is a well-known property of the singular turning-point and is associated with the absence of the wave reflected from this point (see Stix and Swanson, 1983). Physically, the absorption is ensured by some dissipation mechanism. In the cited paper the ohmic dissipation in the ionosphere is treated as such a mechanism.

In the process of propagation from the poloidal surface to the toroidal surface, the wave's polarization also changes from poloidal (the disturbed magnetic field and plasma oscillate normally to the magnetic shell, and the electric field oscillates azimuthally) to toroidal (the magnetic field and plasma oscillate azimuthally, and the electric field oscillates normally). We must emphasize that, in accordance with general polarization properties of Alfvén waves, such a behaviour of the polarization excellently correlates with a change of k_n from zero to infinity.

If the dissipation is neglected, then the wave amplitude with the distance from the poloidal surface initially decreases; after that it increases abruptly and becomes infinite on the toroidal surface. Taking the dissipation into account leads to an additional decrease in amplitude and regularizes the singularity on the toroidal surface. On the other hand, if there is a sufficiently strong instability of the waves in hand, then their amplitude can build up in the process of transverse propagation.

By investigating the monochromatic oscillations, it is possible to reveal their non-trivial spatial structure, but this, as such, is inadequate for a description of the real oscillations, which always have a relatively broad frequency spectrum (Takahashi and McPherron, 1982; 1984; Engebretson *et al.*, 1986; Mitchell *et al.*, 1990). This paper has attempted a switch-over to a description of broadband standing Alfvén waves with $m \gg 1$. To accomplish this, results reported in our previous paper provide the framework on which all our study is based.

Broadband oscillations may be arbitrarily categorized into two large classes: one includes stochastic oscillations excited by noisy sources, that is, sources whose amplitude is a random function of time. Such oscillations are typically exemplified by the certain class of Pc3 oscillations constituting the major portion of the daytime Pc3 driven by magnetosonic waves of extra-magnetospheric origin (Takahashi and McPherron, 1982; Wolfe *et al.*, 1990; Waters *et al.*, 1991; Potapov and Mazur, 1994). External currents in the ionosphere, suggested in our paper as a possible source for standing Alfvén waves with $m \gg 1$ are also more likely to be of a noisy origin. Stochastic oscillations are characterized by their correlational properties (Rytov, 1974). The simplest example is "white noise", with an almost totally lacking correlation on different frequencies. It seems reasonable to class most geomagnetic pulsations with stochastic osciallations.

The other class of broadband oscillations comprises non- stationary oscillations excited by a correlated non-random source. A most representative example is a source of the type of instantaneous impulse, the time dependence of which is described by the δ-function (Hasegawa *et al.*, 1983; Allan *et al.*, 1986). Oscillations of this class include all waves caused by transient processes in the magnetosphere. Typical examples are the SSC phenomenon or Pi2 pulsations (Baumjohann *et al.*, 1984; Yumoto *et al.*, 1990). Such oscillations are less common than stochastic oscillations, but they also play a vital part in the magnetosphere physics.

In accordance with this categorization, this work is also divided into two papers. The first (this) paper is devoted to stochastic oscullations, and the second addresses non-stationary correlated oscillations. Each of these papers makes wide use of formulas obtained in our previous paper. The next section gives a summary of these formulas. In Sect. 3 of this paper, stochastic properties of the oscillation source are defined. In Sect. 4 we determine spectral and polarization properties of stochastic standing Alfvén waves. Main results of this paper are formulated in the conclusion.

2 Results of the theory of monochromatic waves

In describing the axially symmetric magnetosphere, we shall be using a curvilinear orthogonal coordinate system x^1, x^2, x^3, related to the geomagnetic field in such a way that the surfaces $x^1 = const$ are geomagnetic shells and the lines $x^1 = const$, $x^2 = const$ are field lines (see Fig. 1). The square of a length element

$$ds^2 = g_1 dx^{1^2} + g_2 dx^{2^2} + g_3 dx^{3^2}$$

where $g_i = g_i(x^1, x^3)$ are diagonal elements of the metric tensor. Notation: $g = g_1 g_2 g_3$ is its determinant, and $p = (g_2/g_1)^{1/2}$ is the quantity that plays the key role in the theory presented. On each given field line, one can introduce, instead of the coordinate x^3, a physical length whose coordinates are related by $dl = \sqrt{g_3} dx^3$. Among all surfaces $x^3 = const$ is one particular equatorial surface which is a separatrix. On it we put $x^3 = 0$ and $l = 0$. Let x^3_+ and x^3_- represent the values of the coordinate x^3 at points where the field line intersects the ionospheres of the magneto-conjugate hemispheres. They are functions of the magnetic shell: $x^3_\pm = x^3_\pm(x^1)$. Let $x^3_+ > 0 > x^3_-$. The designations $l_\pm = l_\pm(x^1)$ have an analogous meaning.

A disturbance in the Alfvén wave can be described using a so-called transverse potential $\Phi(x^1, x^2, x^3, t)$, in

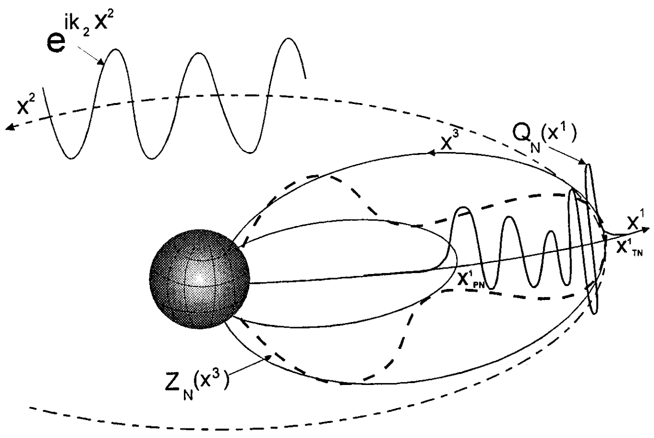

Fig. 1. Coordinate system (x^1, x^2, x^3) and schematic spatial structure

terms of which the disturbed electric and magnetic fields are expressed. It may be represented as a Fourier integral:

$$\Phi(x^1, x^2, x^3, t) = \int_{-\infty}^{\infty} \tilde{\Phi}(x^1, x^3, \omega)\, e^{ik_2x^2 - i\omega t}\, d\omega. \quad (1)$$

Considering the axial symmetry of the magnetosphere, we restrict ourselves to a separate harmonic in the azimuthal coordinate x^2. If the azimuthal angle φ is used as this coordinate, then $k_2 = m$ is the azimuthal wave number. Here we examine the waves with $m \gg 1$. A theory developed by Leonovich and Mazur (1993) refers to monochromatic oscillations, that is, to separate harmonics $\tilde{\Phi}(x^1, x^3, \omega)$. Using also in what follows the symbol tilde to designate the Fourier harmonics, for disturbed fields we have

$$\tilde{E}_1 = -\nabla_1\tilde{\Phi}, \tilde{E}_2 = -\nabla_2\tilde{\Phi}, \tilde{E}_3 = 0, \quad (2a)$$

$$\tilde{B}_1 = -i\frac{c}{\omega}\frac{1}{p}\nabla_2\frac{\partial\tilde{\Phi}}{\partial l}, \;\; \tilde{B}_2 = i\frac{c}{\omega}p\nabla_1\frac{\partial\tilde{\Phi}}{\partial l}, \quad (2b)$$

$$\tilde{B}_3 = i\frac{c}{\omega}\frac{g_3}{\sqrt{g}}\beta, \quad (2b)$$

where the function β is defined by the equation

$$\Delta_\perp\beta = 2\frac{\partial \ln p}{\partial l}\nabla_1\nabla_2\frac{\partial\tilde{\Phi}}{\partial l}, \quad (3)$$

and $\Delta_\perp = g_1^{-1}\nabla_1^2 + g_2^{-1}\nabla_2^2$ is a transverse (two-dimensional) Laplacian. According to results reported in the cited paper, the potential Fourier harmonic may be represented as the product of three functions

$$\tilde{\Phi}(x^1, l, \omega) = \tilde{\Phi}_N(x^1, \omega)\tilde{Q}_N(x^1, \omega)\tilde{Z}_N(x^1, l, \omega). \quad (4)$$

The function $\tilde{\Phi}_N(x^1, \omega)$ fulfils the role of the amplitude. It is determined by the source (external currents in the ionosphere) and depends relatively slightly on x^1. The function $\tilde{Q}_N(x^1, \omega)$ defines the transverse structure of the wave, and – by virtue of its small- scale character in this direction – it depends strongly on x^1. The function $\tilde{Z}_N(x^1, l, \omega)$ describes the longitudinal structure of a standing wave. It has also a relatively weak dependence on x^1. By substituting the expression for the potential $\tilde{\Phi}$ in such a form into the equation for standing Alfvén waves with $m \gg 1$, it is possible to use in solving it perturbation theory based on the method of different scales. This leads to the separation of the solution of the input partial differential equation into two ordinary differential equations for the function $\tilde{Z}_N$ which describe the structure of the oscillation field along field lines, and for $\tilde{Q}_N$ which describe their structure across the magnetic shell. We now turn to a detailed description of these functions.

As has already been pointed out in the introduction, the transverse small-scale character of the waves in hand makes it possible to apply the WKB approximation in the coordinate x^1. Within the limits of this approximation, the function $\tilde{Z}_N(x^1, l, \omega)$ is the solution of the eigenvalue problem:

$$\frac{\partial}{\partial l}q\frac{\partial\tilde{Z}}{\partial l} + q\frac{\omega^2}{A^2}\tilde{Z} = 0, \qquad \tilde{Z}|_{l_\pm} = 0. \quad (5)$$

Here $A = A(x^1, l) \equiv B_0/\sqrt{4\pi\rho}$ is the local Alfvén velocity, $q = pk_1^2 + p^{-1}k_2^2$, and k_1 is a covariant component of the quasi-classical wave vector. The latter plays in Eq. (5) the role of the eigenvalue. The solutions of the problem (5), i.e. eigenvalues

$$k_1 = \tilde{k}_{1N}(x^1, \omega) \quad (6)$$

and the eigenfunctions

$$\tilde{Z} = \tilde{Z}_N(x^1, l, \omega), \quad (7)$$

depend on the variables x^1 and ω as the parameters. In Eqs. (6) and (7) the index N is the harmonic number, that is, the number of half-waves along the field line $(N = 1, 2, \ldots)$.

If Eq. (6) is solved for the variable ω, then the resulting function

$$\omega = \omega_N(x^1, k_1)$$

can be thought of as a local dispersion equation for the waves under consideration.

The function $\tilde{Z}_N$ are normalized by the condition

$$\left\langle\frac{q_N}{A}\tilde{Z}_N^2\right\rangle = \frac{q_N^{(0)}}{A_0}.$$

Here $q_N = p\tilde{k}_{1N}^2 + p^{-1}k_2^2$, the index zero designates equatorial values of corresponding quantities (values at $l = 0$), and the brackets stand for the mean value along the field line. For an arbitrary function $F(l)$, it is defined as

$$\langle F(l)\rangle = \frac{1}{t_A}\oint F(l)\frac{dl}{A},$$

where the integral sign on a closed contour means integration along a field line "there and back", and

$$t_A = \oint\frac{dl}{A}$$

is the run time with Alfvén speed "there and back".

At given frequency ω the poloidal and toroidal resonant surfaces are defined by the equations

$$\tilde{k}_{1N}(x^1, \omega) = 0, \qquad \tilde{k}_{1N}(x^1, \omega) = \infty. \tag{8}$$

We shall designate their solutions accordingly

$$x^1 = x^1_{PN}(\omega), \qquad x^1 = x^1_{TN}(\omega). \tag{9}$$

The ratio in Eq. (8) may be considered as the equations on ω for given x^1. They determine the poloidal and toroidal frequencies

$$\omega = \Omega^1_{PN}(x^1), \qquad \omega = \Omega^1_{TN}(x^1). \tag{10}$$

Obviously the functions (9) and (10) are mutually reverse.

We shall enter designations

$$P_N(x^1, l) = \tilde{Z}_N(x^1, l, \Omega^1_{PN}(x^1)),$$

$$T_N(x^1, l) = \tilde{Z}_N(x^1, l, \Omega^1_{TN}(x^1)).$$

It is natural to name these functions as poloidal and toroidal eigenfunctions. It is easy to see that they, and also appropriate frequencies Ω_{PN} and Ω_{TN}, are the solutions of the following problems for eigenvalues

$$\frac{\partial}{\partial l}\frac{1}{p}\frac{\partial P_N}{\partial l} + \frac{1}{p}\frac{\omega^2}{A^2}P_N = 0, \qquad P_N|_{l_\pm} = 0,$$

$$\left\langle \frac{1}{pA}P_N^2 \right\rangle = \frac{1}{p_0 A_0},$$

$$\frac{\partial}{\partial l}p\frac{\partial T_N}{\partial l} + p\frac{\omega^2}{A^2}T_N = 0, \qquad T_N|_{l_\pm} = 0,$$

$$\left\langle \frac{p}{A}T_N^2 \right\rangle = \frac{p_0}{A_0},$$

where the frequency ω is treated as the eigenvalue. Hence it is possible to offer a definition for P_N, T_N and Ω_{PN}, Ω_{TN} that is not related to the WKB approximation in the coordinate x^1. This factor is important because the WKB approximation is violated near the poloidal and the toroidal surface (or, equivalently, near the frequencies $\omega = \Omega_{PN}$ and $\omega = \Omega_{TN}$).

The splitting of the frequencies $\Delta\Omega_N = \Omega_{TN} - \Omega_{PN}$ for realistic models of the magnetosphere is positive and small compared with the frequencies themselves, that is, the parameter

$$\alpha_N \equiv \Delta\Omega_N / \Omega_{TN} \ll 1. \tag{11}$$

In a narrow (of the order of $\Delta\Omega_N$) range of variation of the values of functions (10), a linear expansion can be used for them and may be represented as

$$\Omega_{PN}(x^1) = \omega\left(1 - \frac{x^1 - x^1_{PN}}{2l_N}\right),$$
$$\Omega_{TN}(x^1) = \omega\left(1 - \frac{x^1 - x^1_{TN}}{2l_N}\right). \tag{12}$$

These functions are taken to be decreasing, as is the case in most of the magnetosphere. By virtue of the inequality (11) the scale of decreasing l_N can be considered the same for both functions, and the validity ranges of the expansions (12) can be thought of as overlapping. By subtracting one from the other, we get $\Delta x^1_N \equiv x^1_{TN} - x^1_{PN} = 2\alpha_N l_N$. Calculations show that a maximum splitting between the poloidal and toroidal resonance surfaces occurs for the fundamental harmonic of eigen- oscillations of standing Alfvén waves (Leonovich and Mazur, 1993). When projected onto the ionosphere, it is $\Delta x^1_1 \approx 800\,\text{km}$. The splitting for higher harmonics is significantly smaller $\Delta x^1_2 \approx 60\,\text{km}$, $\Delta x^1_3 \approx 50\,\text{km}$, $\Delta x^1_4 \approx 30\,\text{km}$, $\Delta x^1_5 \approx 10\,\text{km}$.

The dependence $\tilde{k}^2_{1N}(x^1, \omega)$ on x^1 is plotted diagrammatically in Fig.2. Near the poloidal and toroidal surfaces analytical expressions can be obtained. When $|x^1 - x^1_{PN}| \ll \Delta x^1_N$ (or equivalently $|\omega - \Omega_{PN}| \ll \Delta\Omega_N$)

$$\tilde{k}_{1N} = \frac{1}{\lambda_{PN}}\left(\frac{x^1 - x^1_{PN}}{\lambda_{PN}}\right)^{1/2}$$
$$\equiv \frac{1}{\lambda_{PN}}\left(\frac{\omega - \Omega_{PN}}{\omega_{PN}}\right)^{1/2}, \tag{13}$$

and when $|x^1 - x^1_{TN}| \ll \Delta x^1_N$ (or $|\omega - \Omega_{TN}| \ll \Delta\Omega_N$)

$$\tilde{k}_{1N} = \frac{1}{\lambda_{TN}}\left(\frac{\lambda_{TN}}{x^1_{TN} - x^1}\right)^{1/2}$$
$$\equiv \frac{1}{\lambda_{TN}}\left(\frac{\omega_{TN}}{\Omega_{TN} - \omega}\right)^{1/2}, \tag{14a}$$

Here it is designated

$$\lambda_{PN} = \left(h_{PN}\frac{l_N}{k_2^2}\right)^{1/3}, \quad \lambda_{TN} = \frac{1}{h_{TN}k_2^2 l_N}, \tag{14b}$$

$$\omega_{PN} = \frac{\lambda_{PN}}{2l_N}\Omega_{PN}, \quad \omega_{TN} = \frac{\lambda_{TN}}{2l_N}\Omega_{TN}, \tag{14c}$$

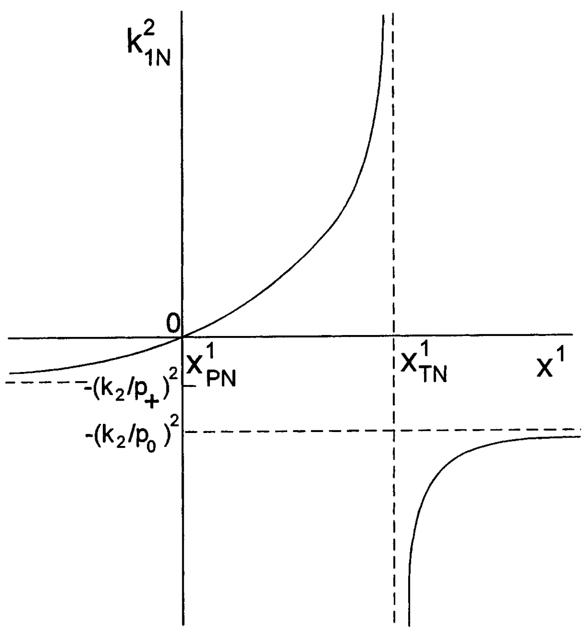

Fig. 2. Dependence of the component $\tilde{k}^2_N(x^1, \omega)$ of a quasi-classical wave vector of the standing Alfvén wave on the radial coordinate x^1. The point x^1_N, where $\tilde{k}^2_N(x^1_{PN}, \omega) = 0$, defines the poloridal resonance surface, and x^1_{TN}, where $\tilde{k}^2_{1N}(x^1_{TN}, \omega) = \infty$, defines the toroidal resonance surface

and h_{PN} and h_{TN} are constants defined by the relationships

$$h_{PN} = \frac{p_0 A_0}{\Omega_{PN}^2} \left\langle A \frac{\partial^2 p}{\partial l^2} P_N^2 \right\rangle,$$

$$h_{TN} = -\frac{A_0}{p_0 \Omega_{TN}^2} \left\langle a \frac{\partial^2 p^{-1}}{\partial l^2} T_N^2 \right\rangle.$$

In realistic models of the magnetosphere, these constants are positive. If the coordinates x^1 and x^2 have the same dimensions such that the quantity p is dimensionless, then h_{PN} and h_{TN} are dimensionless and, by order of magnitude, are α_N. The parameters λ_{PN} and λ_{TN} characterize the transverse dispersion of the wave near the poloidal and toroidal surfaces. If x^1 and x^2 have the dimensions of length, then λ_{PN} and λ_{TN} also have the dimensions of length and by the order of magnitude

$$\lambda_{PN} \sim \frac{\alpha_N^{1/3} a}{m^{2/3}}, \quad \lambda_{TN} = \frac{a}{\alpha_N m^2},$$

where a is a typical scale of transverse inhomogeneity of the magnetosphere (it is assumed that $l_N \sim a$).

A quasi-classical phase is defined by the relationship

$$\tilde{\Psi}_N(x^1, \omega) = \int\limits_{x_{PN}^1(\omega)}^{x^1} \tilde{k}_{1N}(x^{1'}, \omega) \mathrm{d}x^{1'}. \tag{15a}$$

Near the poloidal surface, from Eq. (13) we have

$$\tilde{\Psi}_N(x^1, \omega) = \frac{2}{3} \left(\frac{x^1 - x_{PN}^1}{\lambda_{PN}} \right)^{2/3} \equiv \frac{2}{3} \left(\frac{\omega - \Omega_{PN}}{\omega_{PN}} \right)^{2/3}$$

and near the toroidal surface, from Eq. (14a) we get

$$\tilde{\Psi}_N(x^1, \omega) = \tilde{\Psi}_N(\omega) - 2\left(\frac{x_{TN}^1 - x^1}{\lambda_{TN}} \right)^{1/2} \equiv \tilde{\Psi}_N(\omega) - 2\left(\frac{\Omega_{TN} - \omega}{\omega_{TN}} \right)^{1/2}$$

where

$$\tilde{\Psi}_N(\omega) = \int\limits_{x_{PN}^1(\omega)}^{x_{TN}^1(\omega)} \tilde{k}_{1N}(x^1, \omega) \mathrm{d}x^1. \tag{15b}$$

is a total run-on of the quasi-classical phase between the resonance surfaces.

The countervariant component of the group velocity in the coordinate x^1 is defined in the usual fashion:

$$v_N^1(x^1, \omega) = \left[\frac{\partial \omega_N(x^1, k_1)}{\partial k_1} \right]_{k_1 = \tilde{k}_{1N}(x^1, \omega)} \equiv \left[\frac{\partial \tilde{k}_{1N}(x^1, \omega)}{\partial \omega} \right]^{-1}. \tag{16}$$

According to Eqs. (13) and (14a), near the poloidal and toroidal surfaces, respectively, we have

$$\begin{aligned} v_N^1(x^1, \omega) &= v_{PN}^1 \left(\frac{\omega - \Omega_{PN}}{\omega_{PN}} \right)^{1/2} \equiv v_{PN}^1 \left(\frac{x^1 - x_{PN}^1}{\lambda_{PN}} \right)^{1/2}, \\ v_N^1(x^1, \omega) &= v_{TN}^1 \left(\frac{\Omega_{TN} - \omega}{\omega_{TN}} \right)^{3/2} \equiv v_{TN}^1 \left(\frac{x_{TN}^1 - x^1}{\lambda_{TN}} \right)^{3/2}, \end{aligned} \tag{17a}$$

where

$$v_{PN}^1 = \frac{\lambda_{PN}^2 \Omega_{PN}}{l_N}, \qquad v_{TN}^1 = \frac{\lambda_{TN}^2 \Omega_{TN}}{l_N}. \tag{17b}$$

The group velocity component $v_N^2(x^1, \omega)$ can be determined from the relationship $\tilde{k}_{1N} v_N^1 + k_2 v_N^2 = 0$. More specifically, near the resonance surface we have

$$v_N^2 = -\frac{v_{PN}^1}{k_2 \lambda_{PN}} \frac{x^1 - x_{PN}^1}{\lambda_{PN}}, \quad v_N^2 = -\frac{v_{TN}^1}{k_2 \lambda_{TN}} \frac{x_{TN}^1 - x^1}{\lambda_{TN}}.$$

The transit time of the wave between the poloidal surface to the magnetic shell x^1 is

$$\tau_N(x^1, \omega) = \int\limits_{x_{PN}^1(\omega)}^{x^1} \frac{\mathrm{d}x^{1'}}{v_N^1(x^{1'}, \omega)}. \tag{18a}$$

It is a straightforward matter to establish from the definitions (15a), (16) and (18a) that

$$\tau_N(x^1, \omega) = \frac{\partial \tilde{\Psi}_N(x^1, \omega)}{\partial \omega}. \tag{18b}$$

Near resonance surfaces

$$\tau_N(x^1, \omega) = \frac{1}{\omega_{PN}} \left(\frac{\omega - \Omega_{PN}}{\omega_{PN}} \right)^{1/2} \equiv \frac{1}{\omega_{PN}} \left(\frac{x^1 - x_{PN}^1}{\lambda_{PN}} \right)^{1/2}, \tag{18c}$$

$$\tau_N(x^1, \omega) = \tilde{\tau}_N(\omega) + \frac{1}{\omega_{TN}} \left(\frac{\omega_{TN}}{\Omega_{TN} - \omega} \right)^{1/2} \equiv \tilde{\tau}_N(\omega) + \frac{1}{\omega_{TN}} \left(\frac{\lambda_{TN}}{x_{TN}^1 - x^1} \right)^{1/2} \tag{18d}$$

Here it is designated

$$\tilde{\tau}_N(\omega) = \frac{\mathrm{d}\tilde{\Psi}_N(\omega)}{\mathrm{d}\omega}.$$

Noteworthy are the useful relationships

$$\frac{\partial \tilde{k}_{1N}(x^1, \omega)}{\partial \omega} = \frac{\partial \tau_N(x^1, \omega)}{\partial x^1}$$

$$= \frac{1}{v_N^1(x^1,\omega)} = \frac{\partial^2 \tilde{\Psi}_N(x^1,\omega)}{\partial x^1 \partial \omega}.$$

By the order of magnitude

$$\begin{aligned} &\tilde{\Psi}_N \sim \alpha_N m, && \tau_N \sim \frac{\tilde{\Psi}_N}{\Delta\Omega_N} \sim \frac{m}{\omega}, \\ &\tilde{k}_{1N} \sim \frac{\tilde{\Psi}_N}{\Delta x_N^1} \sim \frac{m}{a}, && v_N^1 \sim \frac{\Delta x_N^1 \Delta\Omega_N}{\tilde{\Psi}_N} \sim \alpha_N \frac{\omega a}{m}. \end{aligned} \tag{19}$$

The transparency region of the mode in hand, that is, the region where $\tilde{k}_{1N}^2(x^1,\omega) > 0$, is in the plane (x^1,ω) a narrow band between the plots of the functions $\omega = \Omega_{PN}(x^1)$ and $\omega = \Omega_{TN}(x^1)$. Level lines of the function $\Psi(x^1,\omega)$ lie inside this band and are therefore approximately parallel to the plot of the function $\Omega_{PN}(x^1)$ (or, equivalently, to the plot of $\Omega_{TN}(x^1)$). This gives an approximate equality

$$\frac{\partial\tilde{\Psi}_N(x^1,\omega)}{\partial x^1} + \frac{\partial\tilde{\Psi}_N(x^1,\omega)}{\mathrm{d}\omega}\frac{\mathrm{d}\Omega_{PN}(x^1)}{\mathrm{d}x^1} = 0,$$

or

$$\tau_N(x^1,\omega) = \frac{1}{\Omega'_{PN}(x^1)}\ \tilde{k}_{1N}(x^1,\omega). \tag{20a}$$

This relationship will be used in the following.

The standing wave attenuation due to the ohmic dissipation in the ionosphere is characterized by a local decrement of damping $\tilde{\gamma}_N(x^1,\omega)$ [an appropriate expression may be found in Leonovich and Mazur (1993)]. The attenuation is taken to be small: $\tilde{\gamma}_N/\omega \ll 1$. At a fixed value of ω and with a variation of x^1 from x_{PN}^1 to x_{TN}^1, as well as at a fixed value of x^1 and with a variation of ω from Ω_{PN} to Ω_{TN}, the local decrement $\tilde{\gamma}_N(x^1,\omega)$ changes substantially (a relative change is of order unity), but it has no singularities in this case. Let use denote

$$\begin{aligned} &\gamma_{PN}(x^1) = \tilde{\gamma}_N\ [x^1, \Omega_{PN}(x^1)], \\ &\gamma_{TN}(x^1) = \tilde{\gamma}_N\ [x^1, \Omega_{TN}(x^1)], \\ &\tilde{\gamma}_{PN}(\omega) = \tilde{\gamma}_N\ [x_{PN}^1(\omega), \omega], \\ &\tilde{\gamma}_{TN}(\omega) = \tilde{\gamma}_N\ [x_{TN}^1(\omega), \omega]. \end{aligned} \tag{20b}$$

The attenuation coefficient of the wave as it propagates from the poloidal surface to the shell x^1 is defined by the relationship

$$\tilde{\Gamma}_N(x^1,\omega) = \int\limits_{x_{PN}^1(\omega)}^{x^1} \frac{\tilde{\gamma}_N(x^{1'},\omega)\mathrm{d}x^{1'}}{v_N^1(x^{1'},\omega)} \tag{20c}$$

Near resonance surfaces (resonance frequencies) simple approximate expressions hold. When $x^1 - x_{PN}^1 \ll \Delta x_N^1(\omega - \Omega_{PN} \ll \Delta\Omega_N)$ we have

$$\begin{aligned} &\tilde{\Gamma}_N(x^1,\omega) = \tilde{\gamma}_{PN}(\omega)\tau_N(x^1,\omega) \\ &\equiv \frac{\tilde{\gamma}_{PN}}{\omega_{PN}}\left(\frac{x^1 - x_{PN}^1}{\lambda_{PN}}\right)^{1/2} = \frac{\gamma_{PN}}{\omega_{PN}}\left(\frac{\omega - \Omega_{PN}}{\omega_{PN}}\right)^{1/2}, \end{aligned} \tag{21}$$

and when $x_{TN}^1 - x^1 \ll \Delta x_N^1(\Omega_{TN} - \omega \ll \Delta\Omega_N)$

$$\begin{aligned} \tilde{\Gamma}_N(x^1,\omega) &= \tilde{\Gamma}_N(\omega) + \tilde{\gamma}_{TN}(\omega)\tau_N(x^1,\omega) \\ &\equiv \tilde{\Gamma}_N(\omega) + \frac{\tilde{\gamma}_{TN}}{\omega_{TN}}\left(\frac{\lambda_{TN}}{x_{TN}^1 - x^1}\right)^{1/2} \\ &= \tilde{\Gamma}_N(\omega) + \frac{\gamma_{TN}}{\omega_{TN}}\left(\frac{\omega_{TN}}{\Omega_{TN-\omega}}\right)^{1/2}. \end{aligned} \tag{22a}$$

Here it is designated

$$\tilde{\Gamma}_N(\omega) = \int\limits_{x_{PN}^1}^{x_{TN}^1} \frac{\tilde{\gamma}_N(x^1,\omega) - \tilde{\gamma}_{TN}(\omega)}{v_N^1(x^1,\omega)}\mathrm{d}x^1, \tag{22b}$$

$$\tilde{\Gamma}_N(\omega) = \tilde{\Gamma}_N(\omega) + \tilde{\gamma}_{TN}(\omega)\tilde{\tau}_N(\omega). \tag{22c}$$

The foregoing formulas permit the expression to be developed for the transverse function $\tilde{Q}_N(x^1,\omega)$ in the WKB approximation:

$$\begin{aligned} \tilde{Q}_N(x^1,\omega) &= \left(\frac{v_{PN}^1}{v_N^1}\frac{p_0^{-1}k_2^2}{p_0\tilde{k}_{1N}^2 + p_0^{-1}k_2^2}\right)^{1/2} \\ &\times \exp\left[i\tilde{\Psi}_N(x^1,\omega) - \tilde{\Gamma}_N(x^1,\omega) + i\frac{\pi}{4}\right]. \end{aligned} \tag{23}$$

This expression holds good when $x^1 - x_{PN}^1 \gg \lambda_{PN}$, $(\gamma_N/\omega)l_N$; $x_{TN}^1 - x^1 \gg \lambda_{TN}$, $(\gamma_N/\omega)l_N$, that is, not too close to the resonance surfaces.

Near these surfaces the expressions for $\tilde{Q}_N$ can be obtained in terms of perturbation theory based on the smallness of deviation of appropriate solutions from the poloidal and toroidal modes, respectively.

When $|x^1 - x_{PN}^1| \ll \Delta x_N^1(|\omega - \Omega_{PN}| \ll \Delta\Omega_N)$, we have

$$\begin{aligned} &\tilde{Q}_N(x^1,\omega) = G\left(\frac{x^1 - x_{PN}^1}{\lambda_{PN}} + i\varepsilon_{PN}\right) \\ &\equiv G\left(\frac{\omega - \Omega_{PN}}{\omega_{PN}} + i\varepsilon_{PN}\right). \end{aligned} \tag{24a}$$

Here $G(z)$ is a function having an integral representation

$$G(z) = \frac{i}{\sqrt{\pi}}\int\limits_0^\infty \exp\left(isz - i\frac{s^3}{3}\right)\ \mathrm{d}s. \tag{24b}$$

It is the solution of the differential equation

$$G'' + zG = -\frac{1}{\sqrt{\pi}}$$

and has the following asymptotics when $|z| \gg 1$:

$$G(z) = \begin{cases} -\frac{1}{\sqrt{\pi}z}, & \frac{\pi}{3} < \arg z < \frac{2\pi}{3}, \\ \frac{1}{z^{1/4}}\exp\left(\frac{2}{3}iz^{3/2} + i\frac{\pi}{4}\right), & 0 < \arg\ z < \frac{\pi}{3}. \end{cases} \tag{25}$$

The parameter $\varepsilon_{PN} = 2(l_N/\lambda_{PN})(\gamma_{PN}/\omega)$ characterizes the relative role of the dispersion and attenuation near the poloidal surface. When $\varepsilon_{PN} \ll 1$ the dispersion is dominant, and when $\varepsilon_{PN} \gg 1$ the attenuation predominates.

When $|x^1 - x_{TN}^1| \ll \Delta x_N^1(|\omega - \Omega_{TN}| \ll \Delta\Omega_N)$

$$\tilde{Q}_N(x^1,\omega) = \frac{k_2\lambda_{PN}}{p_0}\exp\left[i\tilde{\Psi}_N(\omega) - \tilde{\Gamma}_N(\omega) + i\frac{\pi}{2}\right] \times g\left(\frac{x^1 - x^1_{TN}}{\lambda_{TN}} + i\varepsilon_{TN}\right)$$
$$= \frac{k_2\lambda_{PN}}{p_0}\exp\left[i\tilde{\Psi}_N(\omega) - \tilde{\Gamma}_N(\omega) + i\frac{\pi}{2}\right] \times g\left(\frac{\omega - \Omega_{TN}}{\omega_{TN}} + i\varepsilon_{TN}\right). \quad (26a)$$

The function $g(z)$, having an integral representation

$$g(z) = \frac{1}{\sqrt{\pi}}\int_0^\infty \exp\left(isz + \frac{i}{s}\right)\frac{ds}{s}, \quad (26b)$$

satisfies the equation

$$(zg')' - g = 0$$

and has, when $|z| \gg 1$, the asymptotics

$$g(z) = z^{-1/4}\exp\left(-2z^{1/2}\right), \; -\pi < \arg z < \pi. \quad (27)$$

In Eq. (26a) it is designated $\varepsilon_{TN} = 2(l_N/\lambda_{TN})(\gamma_{TN}/\omega)$. In Eqs. (24a) and (26a), power function branches are chosen, which are positive when $\mathrm{Im}z = 0$, $\mathrm{Re}z > 0$. The function $G(z)$ is regular in any finite part of the plane z, and $g(z)$ has a branch point when $z = 0$. Near this singularity

$$g(z) = -\frac{1}{\sqrt{\pi}}\ln z.$$

It is evident from Eqs. (24a) and (26a) that of special interest are the asymptotics of the functions $G(\eta + i\varepsilon)$ and $g(\eta + i\varepsilon)$ at fixed values of ε and at large absolute values of real η, When $\varepsilon \lesssim 1$ we have

$$G(\eta + i\varepsilon) = \begin{cases} \pi^{-1/2}(-\eta)^{-1}, & (-\eta) \gg 1, \\ \eta^{-1/4}\exp\left(\frac{2}{3}i\eta^{3/2} - \varepsilon\eta^{1/2} + i\pi/4\right), & \eta \gg 1, \end{cases}$$

$$g(\eta + i\varepsilon) = \begin{cases} (-\eta)^{-1/4} & \exp[-2i(-\eta)^{1/2} - \varepsilon(-\eta)^{-1/2} - i\pi/4], \\ & (-\eta) \gg 1, \\ \eta^{-1/4} & \exp(-2\eta^{-1/2} - i\varepsilon\eta^{-1/2}), \\ & \eta \gg 1. \end{cases} \quad (28a)$$

The validity ranges of Eqs. (23) and (24a), on the one hand, and of Eqs. (23) and (26a), on the other, overlap, and the respective expressions coincide in the ranges of overlapping. One can see from the set of the above formulas that $\tilde{Q}_N(x^1,\omega)$ is a wave that is generated in the neighbourhood of the poloidal surface, subsequently travels toward the toroidal surface and is totally absorbed in the neighbourhood of this surface. The factor $\tilde{\Phi}(x^1,\omega)$ in Eq. (4) is expressed in terms of external field-aligned currents in the ionosphere which are the source for the waves under discussion. Appropriate formulas are given in the next section.

In closing this section, we develop the expressions for physical components of a disturbed magnetic field $\tilde{B}_i(x^1,l,\omega) = \tilde{B}_i(x^1,l,\omega)/\sqrt{g_i}$ which can be obtained from Eqs. (4), (23), (24a) and (26a) using the relationships of Eqs. (2b) and (3). The following designations are used:

$$\tilde{B}_N(x^1,\omega) = \frac{ck_2}{A_0\sqrt{g_2^{(0)}}}\tilde{\Phi}_N(x^1,\omega), \quad (28b)$$

$$\tilde{Y}_N(x^1,l,\omega) = \frac{A_0}{\omega}\frac{\partial\tilde{Z}_N(x^1,l,\omega)}{\partial l}, \quad (28c)$$

$$\tilde{Y}_N^{(1)} = \sqrt{\frac{g_2^{(0)}}{g_2}}\tilde{Y}_N, \quad \tilde{Y}_N^{(2)} = \sqrt{\frac{g_1^{(0)}}{g_1}}\tilde{Y}_N,$$
$$\tilde{Y}_N^{(3)} = \frac{\sqrt{g_2^{(0)}}}{k_2}\frac{\partial \ln p}{\partial l}\frac{q_N^{(0)}}{q_N}\tilde{Y}_N. \quad (28d)$$

When $\omega = \Omega_{PN}(x^1)$, the respective functions are

$$Y_{PN}(x^1,l) = \tilde{Z}_N(x^1,l,\Omega_{PN}(x^1)) = \frac{A_0}{\omega}\frac{\partial P_N(x^1,l)}{\partial l},$$

$$Y_{PN}^{(1)} = \sqrt{\frac{g_2^{(0)}}{g_2}}Y_{PN}, \quad Y_{PN}^{(2)} = \sqrt{\frac{g_1^{(0)}}{g_1}}Y_{PN},$$

$$Y_{PN}^{(3)} = \frac{\sqrt{g_2^{(0)}}}{k_2}\frac{\partial \ln p}{\partial l}\frac{p}{p_0}Y_{PN}.$$

Similarly, when $\omega = \Omega_{TN}(x^1)$

$$Y_{TN}(x^1,l) = \tilde{Z}_N(x^1,l,\Omega_{TN}(x^1)) = \frac{A_0}{\omega}\frac{\partial T_N(x^1,l)}{\partial l},$$

$$Y_{TN}^{(1)} = \sqrt{\frac{g_2^{(0)}}{g_2}}Y_{TN}, \quad Y_{TN}^{(2)} = \sqrt{\frac{g_1^{(0)}}{g_1}}Y_{TN},$$

$$Y_{TN}^{(3)} = \frac{\sqrt{g_2^{(0)}}}{k_2}\frac{\partial \ln p}{\partial l}\frac{p_0}{p}Y_{TN}.$$

Furthermore, we designate

$$\tilde{Q}_N^{(1)} = \tilde{Q}_N, \quad \tilde{Q}_N^{(2)} = i\frac{p_0}{k_2}\nabla_1\tilde{Q}_N \quad (29)$$

and introduce the function $\tilde{Q}_N^{(3)}$ which satisfies the equation

$$\Delta_\perp^{(0)}\tilde{Q}_N^{(3)} = \frac{2k_2}{\sqrt{g_1^{(0)}g_2^{(0)}}}\nabla_1\tilde{Q}_N.$$

In the validity range of the WKB approximation

$$\tilde{Q}_N^{(2)} = -p_0\frac{\tilde{k}_{1N}}{k_2}\tilde{Q}_N, \quad \tilde{Q}_N^{(3)} = -\frac{2i\tilde{k}_{1N}k_2}{q_N^{(0)}}\tilde{Q}_N. \quad (30)$$

Near the poloidal surface

$$\tilde{Q}_N^{(3)} = -\frac{2p_0}{k_2}\nabla_1\tilde{Q}_N,$$

and near the toroidal surface

$$\tilde{Q}_N^{(3)}(x^1, \omega) = \frac{2k_2}{p_0} \int_{-\infty}^{x^1} Q_N(x^1, \omega)\, \mathrm{d}x^{1'}.$$

Using these designations the physical components of a disturbed magnetic field may be represented as

$$\hat{\tilde{B}}_i(x^1, l, \omega) = \tilde{B}_N(x^1, \omega)\tilde{Q}_N^{(i)}(x^1, \omega)\tilde{Y}_N^{(i)}(x^1, l, \omega). \quad (31)$$

Note that all functions $\tilde{Q}_N^{(i)}$ are dimensionless and of order unity, and the functions $\tilde{Y}_N^{(i)}$ are also dimensionless, with $\tilde{Y}_N^{(1)}$ and $\tilde{Y}_N^{(2)}$ being of order unity, and

$\tilde{Y}_N^{(3)} \sim (1/m)\tilde{Y}_N^{(1,2)}$.

3 Statistical properties of the oscillation source

Under the hypothesis suggested by Leonovich and Mazur (1993), the source for the oscillations considered is external current in the ionosphere. A total current density in the ionosphere $\mathbf{j}$ may be represented as

$\mathbf{j} = \hat{\sigma}\mathbf{E} + \mathbf{j}^{(ext)}$,

where $\hat{\sigma}$ is the conductivity tensor, and $\mathbf{j}^{(ext)}$ is the density of external current of a non- electromagnetic origin. At the ionospheric E-layer level, electrons are magnetized, while ions are not. For this reason, the movement of neutral (for example, in internal gravity or acoustic-gravity waves) entrains in a different manner these plasma components and hence produces an electric current. This current is not associated directly with the macroscopic large- scale electric field in the plasma; therefore, it is usually referred to as the external current. Let $\tilde{j}_\parallel^{(\pm)}$ represent the Fourier-component of the longitudinal component $\mathbf{j}^{(ext)}$ in conjugate ionospheres (plus for the northern ionosphere, and minus for the southern ionosphere). We now derive the functions $\tilde{J}^{(\pm)}(x^1, \omega)$ by defining them by equations

$$\Delta_\perp^{(\pm)}\tilde{J}^{(\pm)} = \tilde{j}_\parallel^{(\pm)}. \quad (32)$$

Here

$$\Delta_\perp^{(\pm)} = \frac{1}{g_1^{(\pm)}}\nabla_1^2 - \frac{k_2^2}{g_2^{(\pm)}}$$

is the transverse Laplacian at the ionospheric level. The standing wave amplitude appearing in (31) is expressed in terms of $\tilde{J}^{(\pm)}$ as

$$\tilde{B}_N = \frac{2\pi k_2}{\sqrt{g_1^{(0)}}c\omega A_0 t_A}\frac{l_N}{\lambda_{PN}} \times \left[-\frac{\sum_P^{(+)}}{p_+}\tilde{J}^{(+)}\tilde{Y}_N\Big|_{l_+} + \frac{\sum_P^{(-)}}{p_-}\tilde{J}^{(-)}\tilde{Y}_N\Big|_{l_-}\right]. \quad (33)$$

Here $\sum_P^{(\pm)}$ are integral Pedersen conductivities of conjugate ionospheres.

Within the framework of the accepted hypothesis for the oscillation source, it would appear reasonable to consider the functions $\tilde{j}^{(\pm)}(x^1, \omega)$ to be random functions of frequency ω. Their statistical properties are characterized by the specification of correlations. It will be assumed that external currents of the conjugate hemispheres are not correlate with each other:

$$\left\langle \tilde{j}_\parallel^{(+)*}(x^1, \omega)\tilde{j}_\parallel^{(-)}(x^{1'}, \omega')\right\rangle = 0. \quad (34)$$

From this point on the brackets designate averaging over a statistical ensemble. For autocorrelators, we take the simplest model, the "white noise" model. In other words, we put

$$\left\langle \tilde{j}_\parallel^{(\pm)*}(x^1, \omega)\tilde{j}_\parallel^{(\pm)}(x^{1'}, \omega')\right\rangle = f^{(\pm)}(x^1, x^{1'}, \omega)\delta(\omega - \omega'), \quad (35)$$

where $f^{(\pm)}(x^1, x^{1'}, \omega)$ are correlation functions.

By virtue of Eq. (32), the functions $\tilde{J}^{(\pm)}(x^1, \omega)$ are also random functions ω and have statistical "white noise" properties. Indeed, let $G^{(\pm)}(x^1, y^1, \omega)$ be Green's function of Eq. (28a)

$$\tilde{J}^{(\pm)}(x^1, \omega) = \int_{-\infty}^{\infty} G^{(\pm)}(x^1, y^1, \omega)\tilde{j}_\parallel^{(\pm)}(y^1, \omega)\, \mathrm{d}y^1. \quad (36)$$

Integration over y^1 is formally extended to the interval $(-\infty, \infty)$, but it is implied that in actual fact it is performed over the region of localization of external current $\tilde{j}_\parallel$. From Eqs. (34), (35) and (36) it follows that

$$\langle \tilde{J}^{(+)*}(x^1, \omega)\tilde{J}^{(-)}(x^{1'}, \omega')\rangle = 0, \quad (37)$$

$$\langle \tilde{J}^{(\pm)*}(x^1, \omega)\tilde{J}^{(\pm)}(x^{1'}, \omega')\rangle = F^{(\pm)}(x^1, x^{1'}, \omega)\delta(\omega - \omega'), \quad (38)$$

where

$$F^{(\pm)}(x^1, x^{1'}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}y^1 \int_{-\infty}^{\infty} \mathrm{d}y^{1'} \tilde{G}^{(\pm)}(x^1, y^1, \omega) \times G^{(\pm)}(x^{1'}, y^{1'}, \omega) f^{(\pm)}(y^1, y^{1'}, \omega).$$

Using Eqs. (33), (37) and (38) it is easy to see that the amplitude function $\tilde{B}_N(x^1, \omega)$ also has the properties of "white noise". In the subsequent discussion we need to use the correlator

$$\langle \tilde{B}_N^*(x^1, \omega)\tilde{B}_N(x^1, \omega')\rangle = \frac{1}{|\omega|}B_N^2(x^1, \omega)\delta(\omega - \omega'). \quad (39)$$

Here it is designated

$$B_N^2(x^1, \omega) = \frac{1}{g_1^{(0)}|\omega|}\left(\frac{8\pi k_2}{cA_0 t_A}\right)^2\left(\frac{l_N}{\lambda_{PN}}\right)^2 \times \left[\frac{\sum_P^{(+)^2}}{p_+^2}\left(\tilde{Y}_N\big|_{l_+}\right)^2 F^{(+)}(x^1, x^1, \omega) + \frac{\sum_P^{(-)^2}}{p_-^2}\left(\tilde{Y}_N\big|_{l_-}\right)^2 F^{(-)}(x^1, x^1, \omega)\right].$$

Note that the function $B_N(x^1, \omega)$ has the dimensions of a magnetic field.

4 Spectral and polarization properties of Alfvén noise

The stochastic origin of the source leads to a statistical character of standing Alfvén waves generated by it. In an effort to describe their statistical properties, we shall examine correlators of the components of a disturbed magnetic field. From Eqs. (31) and (39) we get

$$\left\langle \hat{\tilde{B}}_i^*(x^1, l, \omega)\hat{\tilde{B}}_j(x^1, l, \omega')\right\rangle = \mathscr{P}_{ij}(x^1, l, \omega)\delta(\omega - \omega'), \quad (40)$$

where

$$\mathscr{P}_{ij}(x^1, l, \omega) = \frac{1}{|\omega|}B_N^2(x^1, \omega)\tilde{Q}_N^{(i)*}(x^1, \omega) \times \tilde{Q}_N^{(j)}(x^1, \omega)\tilde{Y}_N^{(i)}(x^1, l, \omega)\tilde{Y}_N^{(j)}(x^1, l, \omega). \quad (41)$$

Thus, standing Alfvén waves also represent "white noise", and the functions $\mathscr{P}_{ij}(x^1, l, \omega)$ have a spectral density meaning at a given point of space (x^1, l). In what follows we shall restrict our study to the functions $\mathscr{P}_{11}$, $\mathscr{P}_{12}$, $\mathscr{P}_{22}$ and $\mathscr{P}_{33}$. The first three functions and the last function, respectively, describe transverse and longitudinal components of a disturbance.

The Alfvén wave amplitude at the point (x^1, l) is defined by the correlators

$$P_{ij}(x^1, l) = \langle \hat{B}_i(x^1, l, t)\hat{B}_j(x^1, l, t)\rangle.$$

From Eqs. (1) and (40) it is easy to obtain

$$P_{ij}(x^1, l) = 2\int_0^\infty \mathscr{P}_{ij}(x^1, l, \omega)\mathrm{d}\omega. \quad (42)$$

By virtue of the stationarity of our accepted model for a random source, these correlators are independent of time t.

The properties of spectral densities as functions of frequency ω are determined predominantly by the functions $\tilde{Q}_N^{(i)*}(x^1, w)$ and $\tilde{Q}_N^{(j)}(x^1, w)$ involved in Eq. (41). Their presence leads to the fact that the functions $\mathscr{P}_{ij}$ have sharp peaks near the frequencies Ω_{PN} and Ω_{TN} and are concentrated mainly in the interval $(\Omega_{PN}, \Omega_{TN})$ and decrease rapidly with the distance from it. A much smoother dependence of the functions $\tilde{Y}_N^{(i)}$ on ω does not affect qualitatively the behaviour of $\mathscr{P}_{ij}$. This applies for the function B_N^2 to still a greater extent. It will be assumed that the spectrum width of the source is much larger than $\Delta\Omega_N$; therefore, the function $B_N^2(x^1, \omega)$ can be considered constant in the interval $(\Omega_{PN}, \Omega_{TN})$.

Of considerable interest is the question: At what frequencies is the Alfvén noise spectrum concentrated (either near Ω_{PN}, or near Ω_{TN}, or in the entire interval between them)? In other words: what is the frequency region where the integral (42) is mainly taken? As will be shown, the answer to this question depends on the amount of attenuation of standing waves in the ionosphere. There are two simple limiting cases of a large and small attenuation. In the first case monochromatic waves constituting Alfvén noise and travelling from the poloidal surface to the toroidal surface attenuate near the poloidal surface on a length much shorter than the distance between the resonance surfaces. According to Eq. (19), a typical transit time between these surfaces $\tau \sim m/\omega$. A strong attenuation implies that $\gamma_N\tau_N \sim m(\gamma_N/\omega) \gg 1$. In this case the energy of Alfvén oscillations is concentrated near Ω_{PN}, and the oscillations have a poloidal character, i.e. the radial component of a disturbed magnetic field is much larger compared with the azimuthal component. With a small attenuation. $m(\gamma_N/\omega) \ll 1$, monochromatic waves reach the neighbourhoods of the toroidal surface without any appreciable attenuation and accumulate there because of a fast decrease in group velocity v_N^1 as one approaches the toroidal surface [see Eq. (17a)]. As a result, the noise spectrum is concentrated near the frequency Ω_{TN}, and the noise polarization has a toroidal character.

We now calculate the correlators (42) in these limiting cases. With a large attenuation when the oscillations are concentrated (on a given magnetic shell) near the poloidal frequency, Eq. (24a) will suffice for $\tilde{Q}_N(x^1, \omega)$. We have

$$\mathscr{P}_{11}(x^1, l, \omega) = \frac{1}{\Omega_{PN}}B_N^2(x^1, \Omega_{PN}) \times \left|G\left(\frac{\omega - \Omega_{PN}}{\omega_{PN}} + i\varepsilon_{PN}\right)\right|^2 Y_{PN}^{(1)^2}(x^1, l).$$

In this case, for the slowly varying functions ω, their values are taken at the point $\omega = \Omega_{PN}$. Using results of Appendix 1 we have

$$P_{11}(x^1, l) = \frac{2\omega_{PN}}{\Omega_{PN}}B_N^2(x^1, \Omega_{PN}) \times Y_{PN}^{(1)^2}(x^1, l)\int_{-\infty}^{\infty}|G(\eta + i\varepsilon_{PN})|^2\,\mathrm{d}\eta = \frac{\mu_N}{\nu_N}B_N^2(x^1, \Omega_{PN})Y_{PN}^{(1)^2}(x^1, l). \quad (43)$$

Here it is designated

$$\mu_N = \frac{k_2\lambda_{PN}^2}{p_0 l_N}, \quad \nu_N = \frac{2k_2 l_N}{p_0}\frac{\gamma_{PN}}{\Omega_{PN}}.$$

By the order of magnitude

$$\mu_N \sim \frac{\alpha_N^{2/3}}{m^{1/3}}, \quad \nu_N = m\frac{\gamma}{\omega}.$$

In the present case $\nu_N \gg 1$.

Remarkably, precisely the same result can be obtained by using for $\tilde{Q}_N(x^1, \omega)$ the WKB approximation of Eq. (23) despite the fact that it is inapplicable in the small vicinity of the point $\omega = \Omega_{PN}$. In this approximation

$$\mathscr{P}_{11}(x^1, l, \omega) = \frac{1}{\omega} B_N^2(x^1, \omega) \frac{v_{PN}^1}{v_N^1(x^1, \omega)} \times \frac{k_2^2/p_0^2}{\tilde{k}_{1N}^2(x^1, \omega) + k_2^2/p_0^2} \mathrm{e}^{-2\Gamma_N(x^1, \omega)} \tilde{Y}_N^{(1)^2}(x^1, l, \omega).$$

Thus, passing in Eq. (42) on to the variable of integration $k_1 = \tilde{k}_{1N}(x^1, \omega)$ and using Eq. (16) we have

$$\mathscr{P}_{11}(x^1, l, \omega) = 2\mu_N B_N^2(x^1, \Omega_{PN}) \int_0^\infty \mathrm{d}k_1 \frac{k_2/p_0}{k_1^2 + k_2^2/p_0^2} \times \mathrm{e}^{-2\Gamma_N[x^1, \omega_N(x^1, k_1)]} \tilde{Y}_N^{(1)^2}(x^1, l, \omega_N(x^1, k_1)). \quad (44)$$

From Eqs. (20a), (21) and (22a) we have

$$\tilde{\Gamma}_N[x^1, \omega_N(x^1, k_1)] = \begin{cases} \frac{\gamma_{PN}}{|\Omega'_{PN}|} k_1, & k_1 \ll k_2/p_0, \\ \bar{\Gamma}_N + \frac{\gamma_{PN}}{|\Omega'_{PN}|} k_1, & k_1 \gg k_2/p_0. \end{cases} \quad (45)$$

Based on this expression it is easy to see that when $v_N \gg 1$, because of a fast decrease in the exponent, the integral (44) is taken when

$$k_1 \lesssim \frac{|\Omega'_{PN}|}{\gamma_{PN}} \sim \frac{1}{v_N} \frac{k_2}{p_0} \ll \frac{k_2}{p_0}.$$

In terms of the variable ω, according to Eq. (13), this means that the spectrum width

$$\Delta\omega \sim \omega - \Omega_{PN} \lesssim \frac{\Delta\Omega_N}{v_N^2}.$$

The integral (44) in this case is readily evaluated, and we again arrive at Eq. (43).

Similarly, when calculating the other correlators, the WKB approximation and the use of Eq. (23) gives identical results. We have

$$P_{12} = -\frac{\mu_N}{2v_N^2} B_N^2(x^1, \Omega_{PN}) Y_{PN}^{(1)}(x^1, l) Y_{PN}^{(2)}(x^1, l),$$

$$P_{22} = \frac{\mu_N}{2v_N^3} B_N^2(x^1, \Omega_{PN}) Y_{PN}^{(2)^2}(x^1, l),$$

$$P_{33} = \frac{\mu_N}{2v_N^3} B_N^2(x^1, \Omega_{PN}) Y_{PN}^{(3)2}(x^1, l).$$

Using results reported in Appendix 2 we find the semi-axes of the ellipse of transverse polarization and its slope to the coordinate line x^1:

$$\langle \hat{B}_{\perp\max}^2 \rangle = \frac{\mu_N}{v_N} B_N^2(x^1, \Omega_{PN}) Y_{PN}^{(1)2}(x^1, l),$$

$$\langle \hat{B}_{\perp\min}^2 \rangle = \frac{\mu_N}{4v_N^3} B_N^2(x^1, \Omega_{PN}) Y_{PN}^{(2)^2}(x^1, l),$$

$$\varphi_0 = -\frac{1}{2v_N} \frac{Y_{PN}^{(2)}(x^1, l)}{Y_{PN}^{(1)}(x^1, l)}.$$

Thus, when $v_N \gg 1$. Alfvén noise has a political character. The noise spectrum is concentrated near the frequency $\omega = \Omega_{PN}$, and the typical spectrum width $\Delta\omega \sim \Delta\Omega_N/v_N^2 \ll \Delta\Omega_N$. The ellipse of polarization is greatly elongated.

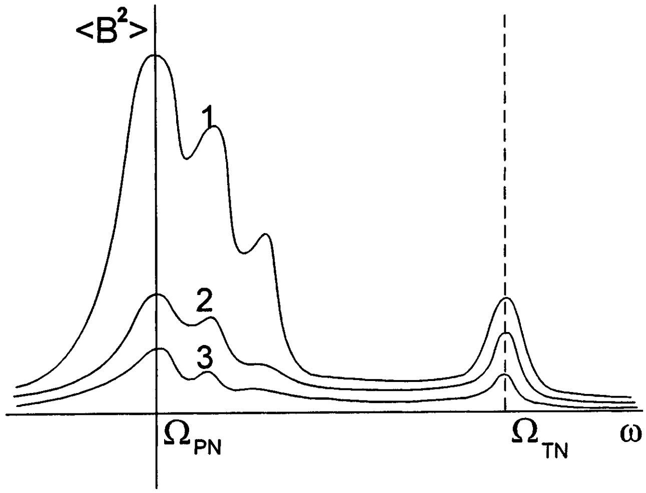

Fig. 3. Schematic diagrams of spectral density for components of a disturbed magnetic field in the case of a large attenuation ($v_N \gg 1$)

$$\langle \hat{B}_{\perp\min}^2 \rangle / \langle \hat{B}_{\perp\max}^2 \rangle \sim \frac{1}{v_N^2} \ll 1,$$

and is almost radially aligned. Note also that

$$\langle \hat{B}_{\parallel}^2 \rangle / \langle \hat{B}_{\perp\max}^2 \rangle \sim \frac{1}{m^2 v_N^2} \ll 1.$$

Spectral densities for different components of a disturbed magnetic field when $v_N \gg 1$ are plotted diagrammatically in Fig. 3.

We turn to the case $v_N \ll 1$. From the relationships (45) it is evident that the exponent appearing in the integrand in Eq. (44) in this case decreases very slowly, and the typical scale of its decrease

$$k_1 \sim \frac{|\Omega'_{PN}|}{\gamma_{PN}} \sim \frac{1}{v_N} \frac{k_2}{p_0} \gg \frac{k_2}{p_0}.$$

Hence this exponent may be omitted because the integral remains convergent:

$$P_{11}(x^1, l, \omega) = 2\mu_N B_N^2(x^1, \Omega_{PN}) \times \int_0^\infty \mathrm{d}k_1 \frac{k_2/p_0}{k_1^2 + k_2^2/p_0^2} \tilde{Y}_N^{(1)^2}(x^1, l, \omega_N(x^1, k_1)). \quad (46)$$

This integral is taken when $k_1 \sim k_2/p_0$, that is, in terms of the variable ω, when

$$\omega - \Omega_{PN} \sim \Delta\Omega_N.$$

It is impossible to take the integral (46) in a closed form because of the presence of the factor $\tilde{Y}_N^{(1)^2}$, but it is easy to obtain an order-of-magnitude estimate

$$P_{11} \sim \mu_N B_N^2. \quad (47)$$

If the WKB approximation is also used for the correlator $\mathscr{P}_{22}$, then instead of Eq. (44) we obtain

$$P_{22} = \frac{2p_0\mu_N}{k_2} B_N^2(x^1, \Omega_{TN}) \int_0^\infty \mathrm{d}k_1 \frac{k_1^2}{k_1^2 + k_2^2/p_0^2} \times \mathrm{e}^{-2\Gamma_N[x^1,\omega_N(x^1,k_1)]} \tilde{Y}_N^{(2)^2}(x^1, l, \omega_N(x^1, k_1)). \quad (48)$$

Here the exponent now cannot be omitted in the integrand – otherwise the integral becomes divergent. Using Eq. (45) it is easy to find that in this case the characteristic region of integration over k_1

$$k_1 \sim \frac{|\Omega'_{PN}|}{\gamma_{TN}} \sim \frac{1}{v_N}\frac{k_2}{p_0} \gg \frac{k_2}{p_0}.$$

In terms of the variable ω this implies that the spectrum is concentrated near $\omega = \Omega_{TN}$, and its characteristic width, as follows from Eq. (14a)

$$\Delta\omega \sim \Omega_{TN} - \omega \sim v_N^2 \Delta\Omega_N \ll \Delta\Omega_N.$$

Based on the foregoing, the integral (48) is readily evaluated:

$$P_{22} = \frac{\mu_N}{v_N}\frac{\gamma_{PN}}{\gamma_{TN}} B_N^2(x^1, \Omega_{TN}) Y_{TN}^{(2)^2}(x^1, l). \quad (49)$$

Here it is taken into consideration that $\bar{\Gamma}_N \sim v_N \ll 1$.

Since in the last case the spectrum is concentrated in the small vicinity of the toroidal frequency Ω_{TN}, the question arises as to whether the use of the WKB approximation is correct, because this approximation is inapplicable when $|\omega - \Omega_{TN}| \lesssim \omega_{TN}$. For that reason, we calculate P_{22} using for Q_N Eq. (26a). From Eq. (29) we get

$$P_{22} = \frac{2}{\Omega_{TN}} B_N^2(x^1, \Omega_{TN}) Y_{TN}^{(2)^2}(x^1, l) \times \int_{-\infty}^{\infty} \left| g'\left(\frac{\omega - \Omega_{TN}}{\omega_{TN}} + i\varepsilon_{TN}\right)\right|^2 \mathrm{d}\omega.$$

Using the values of the integral from Appendix 1 we again arrive at Eq. (49).

Reasoning along similar lines, we find that the spectral density $\mathscr{P}_{12}(x^1, l, \omega)$ is also concentrated near the frequency $\omega = \Omega_{TN}(x^1)$ on the interval $\Delta\omega = v_N^2 \Delta\Omega_N$. The corresponding correlator is found to be

$$P_{12} = -2\mu_N \left(\ln\frac{1}{v_N}\right) \times B_N^2(x^1, \Omega_{TN}) Y_{TN}^{(1)}(x^1, l) Y_{TN}^{(2)}(x^1, l), \quad (50)$$

and calculations in the WKB approximation and using Eq. (26a) yield the same result.

The relationships given by Eqs. (47), (49) and (50) can be used to determine the semi-axis of the ellipse of transverse polarization and its slope to the axis x^1:

$$\langle \hat{B}_\perp^2 \max\rangle = P_{22} = \frac{\mu_N}{v_N}\frac{\gamma_{PN}}{\gamma_{TN}} B_N^2(x^1, \Omega_{TN}) Y_{TN}^{(2)^2}(x^1, l),$$

$$\langle \hat{B}_\perp^2 \min\rangle = P_{11} \sim \mu_N B_N^2(x^1, \Omega_{TN}),$$

$$\varphi_0 = \frac{\pi}{2} + 2v_N \frac{Y_{TN}^{(1)}(x^1, l)}{Y_{TN}^{(2)}(x^1, l)} \ln\frac{1}{v_N}.$$

As far as the correlator P_{33} is concerned, in the WKB approximation, in view of Eq. (30), we have

$$P_{33} = 8\mu_N B_N^2(x^1, \Omega_{TN}) \int_0^\infty \mathrm{d}k_1 \frac{k_1^2 (k_2/p_0)^3}{(k_1^2 + k_2^2/p_0^2)^3} \times \mathrm{e}^{-2\Gamma_N[x^1,\omega_N(x^1,k_1)]} \tilde{Y}^{(3)^2}{}_N(x^1, l, \omega_N(x^1, k_1)).$$

Here as in the case of calculating P_{11}, the exponential factor may be omitted. The spectral density is distributed throughout the interval $(\Omega_{PN}, \Omega_{TN})$. The integral cannot be taken in a closed form, but it is easy to obtain an order-of-magnitude estimate:

$$P_{33} \equiv \left\langle \hat{B}_\parallel^2 \right\rangle \sim \mu_N B_N^2 \tilde{Y}_N^{(3)^2} \sim \frac{\mu_N}{m^2} B_N^2.$$

Thus, when $v_N \ll 1$ the oscillations have a toroidal character. The major semi-axis of the ellipse of polarization is directed almost azimuthally. The oscillations along this axis have a narrow spectrum concentrated near the frequency $\omega = \Omega_{TN}(x^1)$ with a typical width $\Delta\omega \sim v_N^2 \Delta\Omega_N$. The oscillations along the minor semi-axis of the ellipse of transverse polarization as well as along the geomagnetic field have a broader spectrum lying within the range $(\Omega_{PN}, \Omega_{TN})$, and their amplitudes are relatively small:

$$\frac{\langle \hat{B}_{\perp\,\min}^2\rangle}{\langle \hat{B}_{\perp\,\max}^2\rangle} \sim v_N, \quad \frac{\langle \hat{B}_\parallel^2\rangle}{\langle \hat{B}_{\perp\,\max}^2\rangle} \sim \frac{v_N}{m^2}.$$

Schematic diagrams of spectral densities for different components of a disturbed magnetic field when $v_N \ll 1$ are shown in Fig. 4.

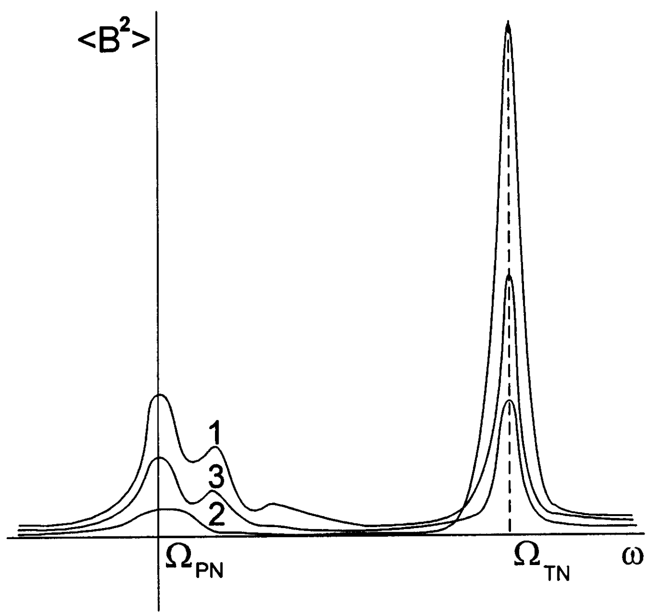

Fig. 4. Schematic diagrams of spectral density for components of a disturbed magnetic field in the case of a small attenuation $(v_N \ll 1)$

5 Discussion

There are currently several theoretical publications devoted to the study of MHD oscillations driven in the magnetosphere by stochastic sources. Specifically, the investigation has been concerned with magnetospheric oscillations with small azimuthal wave numbers $m \sim 1$. As is known (Kivelson and Southwood, 1985) two types of MHD eigen-oscillations exist in the magnetosphere in this case. One represents magnetosonic eigen-oscillations of the magnetospheric cavity (global modes), and the other includes standing Alfvén waves. In an inhomogeneous plasma of the magnetosphere, these kinds of MHD oscillations can interact with each to other in those regions where their eigen-frequencies coincide.

Stochastic magnetosonic broadband oscillations can form in the magnetosphere a continuously distributed (across magnetic shells) oscillation field of standing Alfvén waves characteristic for daytime Pc3 (Leonovich and Mazur, 1989). A numerical simulation of MHD oscillations of the magnetosphere excited by stochastic oscillations of its boundary was reported by Wright and Richard (1995). They showed that a broad spectrum of the oscillations excited in the magnetosphere, one can distinguish, by amplitude, oscillations with frequencies close to those of the global modes. Besides, on a particular magnetic shell Alfvén oscillations are distinguished, the frequencies of which coincide with those of the global modes.

It should be noted that the oscillations with $m \gg 1$ do not break down in the magnetosphere into two different modes (Alfvén and magnetosonic), as is the case in oscillations with $m \sim 1$. In a cold inhomogeneous plasma of the magnetosphere, oscillations with $m \gg 1$ exist in the form of a single mode with the polarization which is customary categorized as the Alfvén wave ($B_3 \ll B_1, B_2$). Unlike oscillations with $m \sim 1$, monochromatic oscillations with $m \gg 1$ cannot propagate across magnetic shells to distances comparable with the typical scales of the magnetosphere (l_N). They are localized on significantly smaller scales determined either by the curvature of geomagnetic field lines (Δx^1_N) or by the dissipation of the waves in the ionosphere ($\varepsilon_{TN} \lambda_{TN}$). Inside the magnetosphere it is therefore impossible to observe oscillations with $m \gg 1$ which can be driven by disturbances on its boundary. The source for such oscillations must be located immediately on magnetic shells where these oscillations are excited. In this paper we consider external currents in the ionosphere to be such a source.

As our calculations show, such a stochastic source in the ionosphere will produce the field of stochastic oscillations of standing Alfvén waves in the magnetosphere. The polarization of these oscillations essentially depends on the magnitude of their dissipation in the ionosphere. If the dissipation is small ($m(\gamma/\omega) \ll 1$), two peaks close located near each to other will be observed in the oscillation spectrum on a given magnetic shell near the frequency of each of the eigen-harmonics of the standing waves. One lies at the frequency coincident with that of toroidal eigen-oscillations of this harmonic (Ω_{TN}), and the other is located at the frequency of poloidal eigen-oscillations (Ω_{PN}). In the case of a large damping ($m(\gamma/\omega) \gg 1$), the frequency spectrum of the excited oscillations will show only one peak at the frequency of poloidal eigen-oscillations (Ω_{PN}).

6 Conclusion

Main results of this work may be formulated as follows:

1. We have presented a summary of basic notions and formulas describing the spatial structure of a standing Alfvén wave with $m \gg 1$ excited by a monochromatic source. This summary defines concretely the picture of generation, propagation and dissipation of the monochromatic wave described qualitatively in the introduction.

2. We have formulated mathematically the hypothesis of the stochastic source of standing Alfvén waves. It has been shown that if external currents in the ionosphere generating Alfvén waves have statistical properties of "white noise", then a random function playing in the equations the role of the oscillation source also has the "white noise" properties. The source spectral density was expressed in terms of spectral densities of external currents in magneto-conjugate ionospheres.

3. It has been shown that stochastic standing Alfvén waves excited by such a source have statistical characteristics of "white noise". We have determined the spectral densities and root-mean-square values of three components of a disturbed magnetic field. Two limiting cases: a large and small attenuation in the ionosphere, were considered at length.

4. In the case of a large attenuation, $m(\gamma/\omega) \gg 1$, each of monochromatic waves constituting "white noise", after the excitation on its poloidal surface, attenuates rapidly and traverses only a small part of the distance Δx^1_N between the resonance surfaces. As a result, only those monochromatic waves reach this magnetic shell, whose poloidal surfaces are at a distance much shorter than Δx^1_N from this shell, therefore, they still conserve their poloidal character. But the entire oscillation as a whole then has a poloidal character – the spectrum is concentrated near the frequencies $\omega = \Omega_{PN}(x^1)$ on an interval of a width $\Delta\omega \sim \Delta\Omega_N / v_N^2$, and the ellipse of transverse polarization of a disturbed magnetic field is strongly elongated and is almost radially aligned.

5. In the case of a small attenuation $m(\gamma/\omega) \ll 1$, the monochromatic wave traverses almost without any attenuation the entire distance between the resonance surfaces, and its energy is stored near the toroidal surface because the wave's group velocity tends rapidly to zero. Therefore, this magnetic shell presents those monochromatic waves for which the toroidal surfaces lie near this shell. As a consequence, the entire stochastic oscillation has a toroidal character. The oscillation energy is concentrated near the frequency $\omega = \Omega_{TN}(x^1)$ on an interval of width $\Delta\omega \sim \Delta\Omega_N v_N^2$, and the ellipse of

polarization of the disturbed magnetic field is strongly elongated and aligned almost azimuthally. At the same time, the oscillations along the minor semi-axis of the ellipse of transverse polarization (that is, in the radial direction) and the longitudinal oscillations have a broader spectrum lying within the interval $(\Omega_{PN}, \Omega_{TN})$. The reason has to do with the fact that the corresponding components in the monochromatic wave become zero as it approaches the toroidal surface, which turns out to be a stronger effect compared with the group velocity's becoming zero.

Appendix 1

To evaluate the integral

$$I_1 = \int_{-\infty}^{\infty} |G(\eta + i\varepsilon)|^2 \, d\eta$$

we use the integral representation of Eq. (47). We have

$$\begin{aligned} \mathrm{I}_1 &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\eta \int_0^{\infty} ds \int_0^{\infty} ds' \\ &\quad \times \exp\left[i(s-s')\eta - \varepsilon(s+s') - i\frac{s^3}{3} + i\frac{s'^3}{3}\right] \\ &= \frac{1}{\pi}\int_0^{\infty} ds \int_0^{\infty} ds' \exp\left[-i\frac{s^3}{3} + i\frac{s'^3}{3} - \varepsilon(s+s')\right] \\ &\quad \times \int_{-\infty}^{\infty} d\eta \, \exp[i(s-s')\eta] \\ &= 2\int_0^{\infty} ds \int_0^{\infty} ds' \exp\left[-i\frac{s^3}{3} + i\frac{s'^3}{3} - \varepsilon(s+s')\right] \\ &\quad \times \delta(s-s') \\ &= 2\int_0^{\infty} ds e^{-2\varepsilon s} = \frac{1}{\varepsilon}. \end{aligned}$$

Similarly, to calculate

$$\mathrm{I}_2 = \int_{-\infty}^{\infty} |g'(\eta + i\varepsilon)|^2 \, d\eta$$

we use the representation (5). This yields

$$g'(z) = \frac{i}{\sqrt{\pi}} \int_0^{\infty} \exp\left(isz + \frac{i}{s}\right) ds.$$

Whence

$$\begin{aligned} \mathrm{I}_2 &= 2\int_0^{\infty} ds \int_0^{\infty} ds' \exp\left[\frac{i}{s} - \frac{i}{s'} - \varepsilon(s+s')\right] \\ &\quad \times \delta(s-s') = 2\int_0^{\infty} ds e^{-2\varepsilon s} = \frac{1}{\varepsilon}. \end{aligned}$$

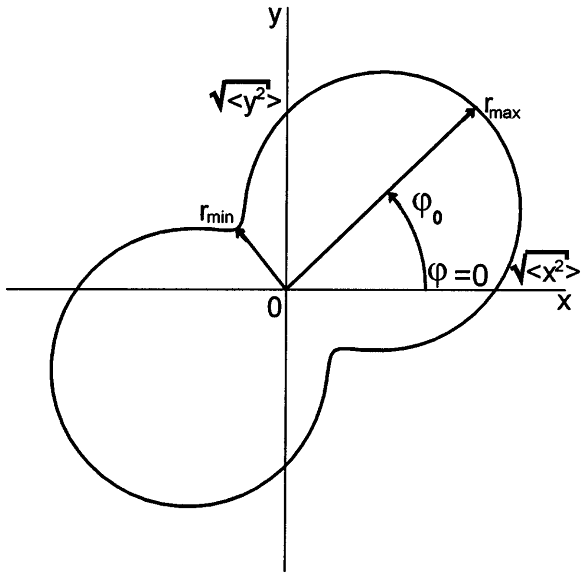

Fig. 5. Mean square of the amplitude of a two- dimensional random vector $\mathbf{r} = (x, y)$ with specified quadratic correlators $< x^2 >, < y^2 >, < xy >$ as a function of polar angle φ

Appendix 2

Let $\mathbf{r} = (x, y)$ be a two-dimensional vector whose components $x = x(t), y = y(t)$ are random functions of time. Suppose that there exist correlators $< x^2 >$, $< y^2 > < xy >$. Consider the projection of the vector $\mathbf{r}$ onto a straight line tilted at the angle φ to the axis x:

$$r_\varphi = x \cos\varphi + y \sin\varphi.$$

The mean value of its square may be represented as

$$\left\langle r_\varphi^2 \right\rangle = r_{\min}^2 + (r_{\max}^2 - r_{\min}^2)\cos^2(\varphi - \varphi_0), \tag{A1}$$

where

$$r_{\min}^2 = \frac{1}{2}\left[\langle x^2\rangle + \langle y^2\rangle . -\sqrt{(\langle x^2\rangle - \langle y^2\rangle)^2 + 4\langle xy\rangle^2}\right],$$

$$r_{\max}^2 = \frac{1}{2}\left[\langle x^2\rangle + \langle y^2\rangle , . +\sqrt{(\langle x^2\rangle - \langle y^2\rangle)^2 + 4\langle xy\rangle^2}\right],$$

$$\tan 2\varphi_0 = \frac{2\langle xy\rangle}{\langle x^2\rangle - \langle y^2\rangle}$$

The figure described by Eq. (A1) in polar coordinates (φ – polar angle, and $r = \sqrt{< r_\varphi^2 >}$ – radius) is not an ellipse. Its typical form is shown in Fig. 5. This figure has a maximum diameter $r_{\max}$ in the direction $\varphi = \varphi_0$ and a minimum diameter $r_{\min}$ in the perpendicular direction. At the same time, if the extreme point of the vector $\mathbf{r}$ is harmonically rotated along an ellipse with the semi-axes a and b tilted at the angle α to the axis x, that is

$$x = a \cos \alpha \cos \omega t - b \sin \alpha \sin \omega t,$$

$$y = a \sin \alpha \cos \omega t - b \cos \alpha \sin \omega t,$$

then, by calculating the time-averaged $< x^2 >$, $< y^2 >, < xy >$, and substituting into the above formulas, we obtain $\varphi_0 = \alpha, r_{\max} = a, r_{\min} = b$. Also in the general case, this gives grounds to call the quantities $r_{\min}$ and $r_{\max}$ the semi-axes of the ellipse of polarization and to call φ_0 its slope.

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