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ON CERTAIN ASTRONOMICAL CONDITIONS FAVORABLE TO GLACIATION.

By G. F. Becker.

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ART. XIII.—*On Certain Astronomical Conditions favorable to Glaciation*; by GEO. F. BECKER.

THE influence of local terrestrial conditions on glaciation is manifest to observers, and few geologists will entertain the idea that cosmical conditions alone can have determined the glacial epoch. Yet variations in the elements of the earth's orbit have certainly influenced climate, and they must have influenced it more favorably to glaciation at some periods than at others. The nature and extent of this influence have been much discussed; but it seems to me that further light can be thrown upon the subject by considering in detail the distribution of solar energy with reference to latitude, and the rate at which this energy is received during the two great seasons separated by the equinoxes.

The elements of the earth's orbit undergo slow variations, and three of these variations affect climate. The time at which the earth is in perihelion affects the length of the two great seasons. If perihelion coincides with either equinox, the seasons are of equal length. If perihelion coincides with either solstice, the seasons differ in length as much as they can for a given eccentricity of the orbit. The whole time which intervenes between the occurrence of seasons of equal length and that of seasons of the most diverse length is five or six thousand years, being the time required for the precession of the equinoxes to amount to a right angle. The eccentricity of the earth's orbit determines the possible difference between the seasons, and it slightly affects the mean distance of the earth from the sun, so that at a period of high eccentricity the world receives a little more heat than it does at a period of zero eccentricity.* It is mechanically possible for the eccentricity to become zero,† and it is never a large quantity. The latitude of the tropical circles also fluctuates within somewhat narrow limits, and of course the polar circles fluctuate correspondingly. There appear to be no other changes in the orbit which can affect the accumulation of ice, and all of these have been considered before now.

Dr. James Croll, as is well known, attributed the glacial epoch to the more or less indirect action of the difference in the length of the seasons, some 35 days, which occurs when the eccentricity is high.‡ Sir Robert Ball dwells upon the difference in the amount of heat received in the two seasons by an entire hemisphere, and he regards the low rate at which the winter hemisphere receives sunshine when the winter has its greatest length as an explanation of the ice age. This astronomer computed that the proportion of heat received during warm season (irrespective of its length), is expressed by $\frac{1}{2} + \sin\varepsilon/\pi$, where ε is the latitude of the tropical circles, and at present this fraction is expressed numerically by 0·627. Thus at the period of greatest eccentricity three eighths of the entire heat of the year may be spread over a winter some two hundred days in length.§

Sir Robert quotes a passage from Sir John Herschell from which it appears that this famous astronomer, at least momentarily, assumed that each hemisphere would receive the same amount of heat in each of the two great seasons, so that the difference of climate would depend solely on the length of the season.‖ If Herschell was under this impression, the mistake was a temporary one; for a page or two before the passage in which the erroneous statement is found he says:—"Now the temperature of any part of the earth depends mainly on its exposure to the sun's rays. . Whenever then the sun remains more than twelve hours above the horizon of any place, and less beneath, the general temperature of that place will be above the average; when the reverse below. As the earth, then, moves from A to B, the days growing longer, and the nights shorter in the northern hemisphere, the temperature of every part of that hemisphere increases and we pass from spring to summer."

This is of course the usual explanation of the seasons to be found in all astronomies and physical geographies. If it were true that a hemisphere received the same amount of heat in

* See the note appended to this paper on the calculation of sunshine per unit area.

† Cf. J. N. Stockwell, Smiths. Cont. to Knowledge, vol. xviii, 1873.

‡ Climate and Time.

§ Cause of the Ice Age, 1892.

‖ Outlines of Astronomy, § 368 (c) in 9th ed. 1867.

each great season, the winter would always be the long season, and therefore winter would occur in both hemispheres at the same time! It appears from the paragraph following that in which the mistake occurs, that Herschell was thinking particularly of tropical climates. Now at the equator it is literally true that the heat received in summer and in winter is the same, as will be shown in the second part of this paper, and the application which Herschell makes of his equal division is to explain the alleged great intensity of summer heat in the tropical regions of Australia as compared with those of northern Africa.

While at or close to the equator the heat received in each season is the same, at the poles all of the heat is received in summer. Hence while it is true that with the present obliquity only about three-eighths of the entire heat received by a hemisphere is received in winter, this fact helps but little towards an explanation of climate. The distribution of heat between the seasons varies with the latitude, and to form any just idea of the effect of eccentricity on climatic conditions one must know the heat received per unit area in any latitude.

The method of finding the amount of sunshine per unit area between equinoxes is very simple in principle. The great circle bounding the illuminated half of the earth is called the circle of illumination, and is represented by the right hand circle in figure 1. Any parallel of latitude projected onto the

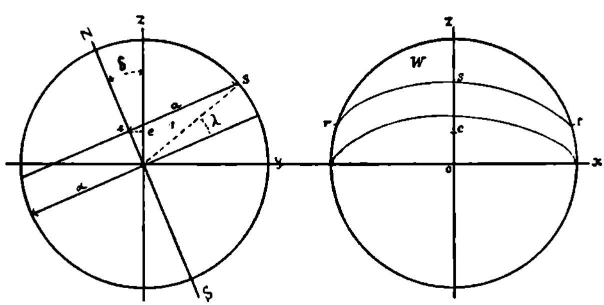

FIGURE 1.—Circle of Illumination.

circle of illumination becomes an ellipse, e. g., the ellipse *r*, *s*, *t*. The heat received between this parallel and the pole while the sun remains in the same position will be proportional to the crescentic area marked W. If one supposes a second parallel very close to the first, say at the unit distance from it, the narrow interval between their projections will be proportional to the heat received upou a zone of the earth's surface. These conditions apply only to an instant of time because the declination of the sun varies from zero at the equinoxes to 23° 27′ at the solstices. The next step is to find the average area on the circle of illumination representing the projection of the narrow zone from equinox to equinox. When this is accomplished, one has only to multiply the result by the time interval and divide by the length of the parallel of latitude to obtain the area on the circle of illumination representing the solar radiation received per unit area of the earth's surface between the autumnal and the vernal equinoxes. The unit in which radiation is measured may be arbitrarily chosen and as arbitrarily changed, provided that it is employed for all latitudes. I have chosen one which is convenient. It and the formulas for computation are explained in a note appended to this paper.

The heat received per unit area between equinoxes is independent of the length of the season being proportional to the change of the earth's longitude in its orbit.* The average rate at which heat is received during one of the great seasons is therefore merely the total heat per unit area divided by the length of the season.

The radiant energy per unit area depends to a slight extent upon the eccentricity of the orbit. If u_0 is the energy per unit area for zero eccentricity, and u the energy for eccentricity e, then

$$u = \frac{u_0}{\sqrt{1-e^2}}.$$

Though the difference is small, it is perfectly easy to take it into consideration, and this I have done.

At the present time the warm season in the northern hemisphere is approximately 186 days 10 hours long† or 1·0208 times half the year. The eccentricity of the orbit is 0·01677, so that if P is the present mean rate at which sunshine is received per unit area in summer in the northern hemisphere and L the length of the summer‡

$$P = \frac{u_0}{L\sqrt{1-e^2}} = 0·97978\ u_0.$$

So too if p is the present winter rate I find

$$p = 1·02138\ u_0.$$

* The heat received on the area when the earth is in a given position is directly as the time and inversely as the square of the distance, r, from the sun. In an instant therefore it is proportional to dt/r^2. By Kepler's first law this is equal to $d\vartheta/h$ where ϑ is longitude and h a constant.

† See Nautical Almanac for 1895.

‡ The coefficients are stated to five figures not because they are of themselves of interest to this degree of accuracy, but because in checking the tabulated values the numbers really used should be known.

The exact value of the greatest possible eccentricity of the earth's orbit is somewhat uncertain. It is dependent upon the masses of the planets, and these are not determined with final accuracy. It is not far from 0·07 and I shall assume that it is 0·0745, the value taken by Sir Robert Ball. The greatest possible difference between the seasons occurs when the long season is $(1 + 4e/\pi)$ times half the year; and if X and x are the mean rates of receipt of sunshine during the long and short seasons respectively for greatest eccentricity and greatest difference between seasons in the hemisphere where the winter is long, I find

$$X = 1{\cdot}10788\, u_0$$
$$x = 0{\cdot}91590\, u_0.$$

The values are tabulated below with two other sets of values to be explained presently. The diagrams, figures 2 and 3, show the rates graphically, but before commenting upon the differences between the curves it is desirable to consider how the present distribution of heat-rates is related to climate as known by observation.

No one doubts that temperature is dependent in some manner upon solar radiation, but the phenomena are complicated not only by the transfer of heat from one locality to another through the agency of currents of air and water, but also by the selective absorption of the atmosphere, and it is a question therefore how far the mere receipt of sunshine, or what Humboldt called "the astronomical climate," can be made to explain actual climate. That absorption of radiant energy by the atmosphere affects climate has long been understood. Energy of different wave lengths however is differently absorbed by the same gas, and different gases absorb energy differently; so that the subject is one of great complexity. According to Prof. S. P. Langley,* the temperature of an airless planet, even under a vertical sun, would be little above the freezing point of water; and of course the unilluminated part of such a planet would tend toward the absolute zero. Hence the actual temperature of the earth is determined to a very great extent by selective absorption of the atmosphere. No doubt this has always been the case, and, since the composition of the air must have changed as geological time progressed, it is highly probable that differences in selective absorption have determined, or partly determined the differences in mean tem-

* The Temperature of the Moon. Nat Acad of Sci. vol. iv, part II. 1889, page 193. The conclusion drawn from observations on Mt. Whitney, that an airless planet would fall far short of zero C., is modified in this passage.

perature at different epochs * The glacial epoch is relatively speaking so recent that the composition of dry air was probably much the same then as now ; but even moisture affects absorption of heat rays, greatly increasing it, so that relatively moist air tends to become hotter while dry air tends to sink below the average temperature.

It is notorious that climate is enormously affected by oceanic currents, themselves due to winds of prevalent direction, and it would therefore be in vain to seek any very close relation between temperature and solar radiation in littoral areas. On the other hand, in the interior of the continents, where air currents alone distribute the heat received from the sun, it seems possible that a relation may show itself. Considering the facts presented above it seemed to me that a reasonable trial hypothesis would be this:—The average variation of temperature from a certain mean in purely continental areas during a great season may be nearly proportional to the mean rate at which solar energy is received during that season.

In an ideally continental climate the summers will be hotter and the winters will be colder than in any real climate, since the actual tendency is always toward an equalization of temperature. Hence to test the trial hypothesis it is required to know the lowest latitudes to which mean winter isotherms descend in continental areas, and the highest latitudes which mean summer isotherms reach. In this enquiry the lag of heat effects would have to be taken into consideration, so that the seasons for temperature would not be divided by the equinoxes. I do not know of any isothermal charts suitable for the enquiry in this form. The next best material would be charts for the two extreme months, January and July, for though such charts would give temperatures exceeding the means in intensity, these temperatures would probably be proportional to the means. In other words it is probable that the temperature curve for a representative cold winter is derivable from that of a representative warm winter by projection.

Proceeding on this assumption I collected data from Mr. A. Buchan's January and July charts based on the mean observations for eleven years.† The lowest latitudes to which each isotherm descended in January in Asia and in North America were noted and the highest latitudes to which they ascended in July. They are recorded in the following table.

* It is substantially certain that the sun's own heat has varied. The record of its variation must exist in the rocks, but as a palimpsest. The greater equability of the earlier climates seems to me explicable, at present, only by greater atmospheric absorption.

† Encyc. Brit., 9th ed. Art.: Meteorology.

Deg. F. Isotherm.	Lowest January Latitude. Asia.	N. America.	Highest July Latitude. Asia.	N. America.
—50°	60°	....		
—40	63	....		
—30	59	64		
—20	55	56		
—10	46	52		
0	45	49		
+10	42	47		
20	38	42		
30	36	39		
40	27	34	77	....
50	25	31	72	69
60	21	....	68	66
70	17	....	59	56
80	10	....	48	46
90	....	....	39	43

Some of these temperatures are affected to a greater or less extent by the neighborhood of water as may be seen from the charts. Thus the polar sea must certainly have a cooling effect in Siberia in summer at latitude 77, and 40° must be there somewhat too low a reading for a continental climate in July. Similarly in winter 70° and 80° must be rather higher than the isotherms in winter would be in southern Asia if land continued southward. Again the great elevation of Central Asia and of the Rocky Mountain region must give summer readings lower than would be recorded were these areas at sea-level.

Imperfect as these data are, they seem the best available, and as such they are entered in diagram 2, an arbitrary unit being selected as representing a Fahrenheit degree. Each X represents an Asian observation, and each Y a North American one.

It will be observed that the points representing temperatures lie remarkably close to the curves representing the present rate of receipt of solar energy, and thus seem strongly confirmatory of the trial hypothesis which, it may be well to state, was not framed to fit the temperature observations. A still closer agreement would be obtained by using a somewhat larger unit for temperature. The climate represented by the curves is intentionally shown as somewhat more extreme than the observed climate because such would be the case if the air were immobile. It might be easy to lay too much stress on the close agreement of the curves and temperatures. I desire to draw but one conclusion from it, viz: the mean rate at which sunshine is received per unit area in continental climates is so important an element in determining the seasonal deviation from the mean temperature, that the curves representing mean rate afford a sound basis for comparison between continental climates for different values of the eccentricity of the earth's orbit or for different values of obliquity of the ecliptic, the composition of the atmosphere and the intensity of solar radiation being supposed constant. This conclusion seems to me fully justified by diagram 2, and it will be assumed in what follows.

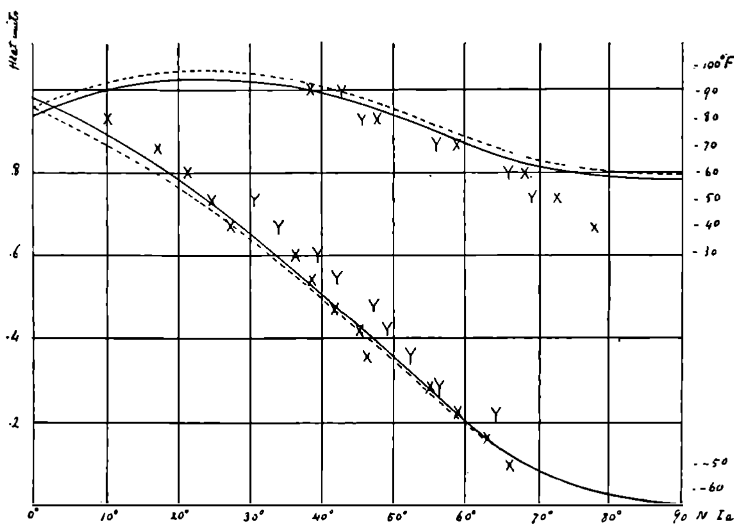

FIGURE 2.

Heat rates for present conditions in each of the great seasons. The rates for zero eccentricity and present obliquity are shown by dotted lines. Points marked X are Asian mean temperatures in January and July. Points marked Y are North American mean temperatures for the same months.

In considering the effect of secular variations of the earth's orbit on the accumulation of ice, certain criteria of climate must be fixed upon. So far as I can see, the conditions prevailing in a glaciated hemisphere should be as follows. The torrid and lower temperate zone, in which evaporation chiefly takes place, should be as warm as is consistent with other conditions; for it must be remembered that the tension of aqueous

Heat-rates for Northern Summer.

Lat.	P	N	X	B
0°	0·9397	0·9592	1·0626	0·9552
10	0·9945	1·0150	1·1245	1·0144
20	1·0216	1·0427	1·1552	1·0457
30	1·0208	1·0419	1·1543	1·0489
40	0·9930	1·0135	1·1229	1·0245
50	0·9411	0·9605	1·0642	0·9762
60	0·8721	0·8901	0·9862	0·9111
A	0·8270	0·8441	0·9352	0·8768
70	0·8115	0·8283	0·9176	0·8587
80	0·7869	0·8032	0·8898	0·8381
90	0·7798	0·7959	0·8818	0·8326

Heat-rates for Northern Winter.

Lat.	p	n	x	b
0°	0·9797	0·9592	0·8785	0·9552
10	0·8955	0·8768	0·8030	0·8698
20	0·7869	0·7705	0·7057	0·7610
30	0·6577	0·6439	0·5898	0·6326
40	0·5127	0·5019	0·4597	0·4893
50	0·3583	0·3508	0·3213	0·3384
60	0·2051	0·2009	0·1840	0·1901
A	0·1164	0·1140	0·1044	0·1198
70	0·0821	0·0804	0·0736	0·0764
80	0·0198	0·0194	0·0177	0·0182

P and p stand for present obliquity, eccentricity and length of seasons.

N and n stand for present obliquity, 23° 27′, and zero eccentricity.

X and x stand for present obliquity, eccentricity = 0·0745, and greatest difference of seasons.

B and b stand for obliquity = 24°36′ and zero eccentricity.

N, n, B and b are computed from the formulas developed in the note appended to this paper which P, p, X and x are derived from N. and n as explained in the text. The rates for the southern hemisphere, shown in diagram 3, are identical with those in the northern hemisphere except in the case of greatest eccentricity when they are multiples of those in corresponding northern latitudes.

A represents the latitude of the arctic circle, or 66° 33′ in all cases excepting those of B and b for which A = 65° 24′.

vapor increases much more rapidly than temperature* and so also must the rate of evaporation. On the other hand the cold in high latitudes must be great to promote condensation in the form of snow; besides which the temperature gradient should be high or steep because the energy available for wind, and for water currents due to winds, is in direct proportion to the difference of temperature. The great foe to glaciation in summer is rather warm rain than sunshine, for warm rain represents heat transferred from lower latitudes to higher ones. A cer-

* The tension is a function of the temperature, and this function is not linear.

tain amount of sunshine in high latitudes will not seriously diminish the accumulation of névé; for a great part of the winter snowfall has a temperature far below freezing; and in summer, water resulting from superficial melting will freeze again as it percolates through subjacent snow until the entire accumulation of the past winter is raised to the melting point. Such a process is apparently essential to the formation of glacier ice. While a portion of the direct sunshine is harmlessly employed in converting snow to ice, another and very large part will be reflected from the névé fields. Hence it seems to me that the features of a summer climate in a glaciated hemisphere which are most favorable to ice accumulation are cool tropics and a low temperature gradient toward the pole, even if the direct sunshine in very high latitudes must be increased to bring about a dry climate.

It seems proper to begin a comparison between various climates by considering the conditions in opposite hemispheres at any time when the difference of seasons is considerable. Since the total amount of heat received by a hemisphere between equinoxes is wholly independent of the duration of this interval, one hemisphere will then have long cold winters with short hot summers, and the other will have long cool summers and short warm winters. A comparison of curves representing such conditions shows further, that in the genial hemisphere in winter the temperature gradient will be very high and the zone of evaporation very hot, so that the weather will be very wet as well as relatively warm. The summer in the genial hemisphere on the contrary will be cool and not particularly wet. The high winter temperature, and the corresponding brevity of the season in the genial hemisphere would certainly preclude glaciation with the present mean temperature of the globe. Hence in discussing the conditions favorable to glaciation it is needless to consider those in which the winter is shorter than the summer. This is a conclusion upon which, so far as I know, every one is agreed.

A comparison may now be made between the present climates and that which would prevail in both hemispheres were the eccentricity zero, the apparent obliquity of the ecliptic maintaining its present value. The seasons would then be of equal length, and the climate would be intermediate between that of the present time in the two hemispheres. Five or six thousand years ago the seasons were of equal length, and the eccentricity is now and was then so small as to affect the amount of heat received by an insignificant fraction. In fact the earth now receives 1·00014 times as much heat as it would do if the eccentricity were zero. The rates given in the table and the curves shown in figures 2 and 3, enable one to judge sufficiently of the climate of zero eccentricity. The winter in the northern hemisphere would be a little cooler throughout and the temperature gradient a little smaller. Other things being

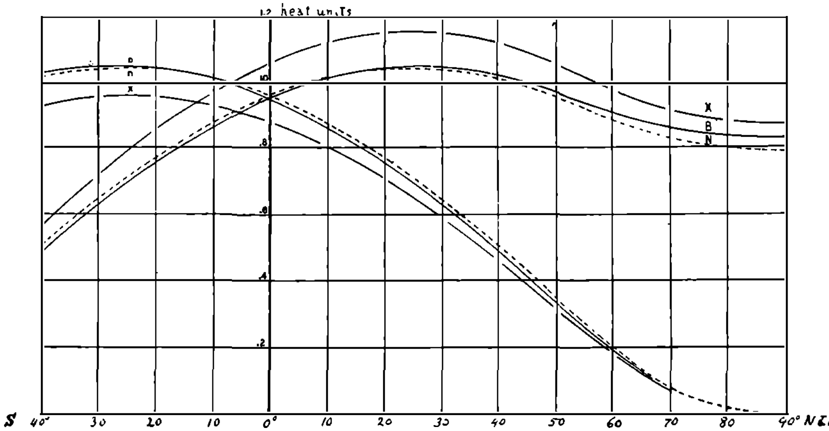

FIGURE 3.

Heat rates for each of the great seasons under various conditions. The curves N and *n* represent the present obliquity and zero eccentricity. They also appear in figure 2. The curves X, and *x* show the rates for greatest eccentricity and winter of maximum length in the northern hemisphere. The curves B and *b* display the rates for an obliquity of 24° 36′ with zero eccentricity.

the same, the winter precipitation would be somewhat smaller, but more of it would fall as snow. In summer the July temperature would be two or three degrees F. higher than it now is, and the heat gradient would almost imperceptibly exceed the present. On the whole the normal weather would probably be within the range of present experience or, in other words, the winter would be what is now considered a cold one, and the summer such as is now thought unusually warm in the northern hemisphere; but the seasons would not be so extreme as they would now be in the southern hemisphere were the distribution of continental areas there the same as it is in the northern hemisphere.

Having established a climate of zero eccentricity it may be compared with the extreme climate of highest eccentricity. It appears from diagram 3 and the table that the winter of the eccentric period in the rigorous hemisphere would be intensely cold as compared with that of the period of zero eccentricity, but it is important to observe that the difference would be most marked in the tropics. It is not needful to depend on my formulas or tabulated values of heat rates to assure oneself of the general truth of this result. For an increase in the length of the winter diminishes all heat-rates or temperatures in the same proportion; and since the rates are highest in the tropics, the greatest decrease must also take place in the torrid belt. The indications of the diagram are that the temperature of January in the eccentric period would be greatest at the tropic of Capricorn, and that it would there be no warmer than it now is in July at 45° of north latitude. It follows that the evaporation would be very small and that the snowfall in high northern latitudes would be small. The heat gradient during the eccentric winter is also less steep than it is possible to make it by any other combination of conditions, and therefore the winter of this period in the northern hemisphere would be the calmest, dryest and coldest possible.

The summer of the eccentric period in the hemisphere of rigorous climate will be the hottest possible; nearly 20° F. hotter, it would seem, than that of the present time in temperate latitudes. The evaporation would of course be immense. The heat gradient toward the pole is also considerably greater than it now is, or than it would be at the time of zero eccentricity. Hence the summer would be wet as well as hot. It seems to me, then, that the period of greatest eccentricity would be most unfavorable to glaciation, the snowfall being the smallest, and the summer rainfall the largest which can occur with the present obliquity. It seems much less favorable than the period of zero eccentricity when the winter cold is great enough to preclude much rain in the higher portion of the temperate zone, while the temperature in the tropics is great enough to produce active evaporation. It would be manifestly absurd to suppose equality of seasons sufficient to produce an ice age; but I am forced to the conclusion that, so far as eccentricity is concerned in the matter at all, the smaller the eccentricity the more favorable are the conditions for glaciation.

Thus far only a slight reference has been made to the variation of the obliquity. Its influence on glaciation has been considered by various authors, and Dr. Croll attributes to a greater obliquity a considerable influence on the temperature of the arctic circle. The greatest possible obliquity of the apparent ecliptic is believed to be 24° 36′, and I have adopted this value for computation.* The heat-rates are recorded in the table, and the corresponding curves are shown in figure 3.

At first sight the winter curve of the period of greatest obliquity seems practically undistinguishable from that for the present obliquity, the eccentricities being in both cases zero.

* Stockwell gives as the greatest obliquity 24° 35′ 57″·53 for the values of the masses of the planets which he adopts. Smiths. Cont., vol. xviii, 1873, p. 175.

There is a difference however of considerable importance. In either case the maximum mean rate of receipt of sunshine, or the maximum temperature during the northern winter, is at the Tropic of Capricorn and it is from the maximum that the winds laden with moisture will blow toward either pole.* Now when the obliquity was 1° 9′ greater than it now is, the Tropic of Capricorn was so much further south, and the area to the north of this tropic was about 1,800,000 square miles greater than that north of the corresponding parallel to-day. This area is somewhat greater than the sum of the areas of the Mediterranean Sea and the Gulf of Mexico. The gradient northward in winter was almost exactly what it is to-day, the temperature was lower throughout, though not more than a few degrees; but this was compensated for, more or less completely, by the increased area of evaporation supplying precipitation to high northern latitudes. It would appear therefore that the precipitation may have been as great as it now is, but that a larger part of the precipitation must have been snow.

In summer at high obliquity the zone of evaporation was 1,800,000 square miles less than it now is, and the temperature gradient toward the pole was smaller than any other, smaller even than that of the summer in the genial hemisphere of the period of highest eccentricity. Hence it was perhaps the dryest possible summer. Its temperature was somewhat below that of the present time in the southern hemisphere from the equator to latitude 45°. Beyond this point it was a little higher but, as has been pointed out, dry summer heat in very high latitudes cannot greatly diminish the accumulation of snow.

I began this enquiry without the remotest idea as to what conclusion would be reached. At the end of it I feel compelled to assert that the combination of low eccentricity and high obliquity will promote the accumulation of glacial ice in high latitudes more than any other set of circumstances pertaining to the earth's orbit. It seems to me that the glacial age may be due to these conditions in combination with a favorable disposition of land and water. This theory implies, or rather does not exclude simultaneous glaciation in both hemispheres. It does not imply that the ice age should last only ten or twelve thousand years. If the conditions here suggested are correct, variations in the disposition of land and water may have determined intervals of glaciation, not necessarily the same ones in New England and the basin of the Mississippi; and there may have been considerable time dif-

* It is well known that the July and January winds blow across the equator. This tendency is strongest in July because of the greator land area of the northern hemisphere.

ferences in the inception or the cessation of glaciation in various regions. It is not needful to assume that the glaciation of the Sierra Nevada either began or ended synchronously with the ice age in New England.

The date at which a minimum of eccentricity last coincided with a maximum of obliquity can almost certainly be determined. According to Stockwell the obliquity has been diminishing for the past 8000 years and was within 21 minutes of its maximum value at the beginning of that time. According to Leverrier the eccentricity passed through a minimum 40,000 years ago, the value being then about two-thirds of the present one. So far as I know the obliquity has not been computed beyond 8000. This can of course be done for Stockwell's value of the masses of the planets, or for newer or better ones. All the indications seem to be that within thirty or forty thousand years conditions have occurred and have persisted for a considerable number of thousand years which would favor glaciation on the theory of this paper. It is conceivable that very remote coincidences of high obliquity and low eccentricity might be determined, answering perhaps to glaciation in the Paleozoic; but until some simple law governing the periodicity of secular variations is discovered such a result is not to be looked for; it is at present practicable to formulate the variations only by omitting terms above a certain order and exterpolation beyond a few score thousand years is consequently untrustworthy.

Calculation of Sunshine per unit area.

The following note explains the method of finding the amount of solar energy received per unit area of the earth's surface in any latitude between equinoxes, the ellipticity of the earth being ignored. Mr. L. W. Meech has already solved the very similar problem of finding the heat received per unit area for the entire year, and it is possible to develop from his formulas those applicable to the present purpose; but this would take nearly as much space as a fresh presentation. Mr. Meech's method of dealing with the subject is also quite different from that here presented, which I worked out before making acquaintance with his admirable memoir.*

When the sun's declination is δ, any parallel in latitude λ will be projected onto the circle of illumination as an elliptical arc dividing the circle into two unequal portions. If λ and δ have opposite signs, the smaller of these areas will be a crescent which I shall call W. To find the area W shown in diagram 1, let a be the earth's radius, a and b the semiaxes of

* Smithsonian Contr. to Knowledge, vol. ix, 1857.

the ellipse into which the parallel of λ is projected, x and z the coördinates of the point t at which the ellipse meets the great circle in the projection, c the center of the ellipse, e the center of the small circle which is projected into the ellipse. Then it is easily seen from the diagram that

$$a = \alpha \cos\lambda;\; b = \alpha\cos\lambda\sin\delta;\; c = e\cos\delta = \alpha\sin\lambda\cos\delta;$$

$$z = \frac{\alpha\sin\lambda}{\cos\delta};\; x = \sqrt{\alpha^2 - z^2} = \frac{\alpha\cos\lambda}{\cos\delta}\sqrt{1 - \frac{\sin^2\delta}{\cos^2\lambda}}$$

The area of the circular segment r, z, t is $\alpha^2 \sin^{[-1]}\frac{x}{\alpha} - xz$ and that of the elliptical segment r, s, t is $ab\sin^{[-1]}\frac{x}{a} - xz + xc$. The difference is the area sought or

$$W = \alpha^2 \sin^{[-1]}\frac{x}{\alpha} - ab\sin^{[-1]}\frac{x}{a} - xc$$

The arc whose sine is x/a is the sun's semidiurnal arc. Its value in the cold season may be called for brevity X. In the warm season at the same point its value will be π–X. If ϑ is the earth's longitude in its orbit and if ε is the apparent obliquity of the ecliptic it is easy to see and perfectly well known that

$$\sin\delta = \sin\varepsilon\sin\vartheta$$

and this value is to be substituted in that of W. It is also convenient to employ the abbreviation

$$\varDelta(\vartheta) = \sqrt{1 - \frac{\sin^2\varepsilon}{\cos^2\lambda}\sin^2\vartheta},$$

so that the semidiurnal arc is

$$X = \text{arc}\sin\frac{\varDelta(\vartheta)}{\sqrt{1 - \sin^2\varepsilon\sin^2\vartheta}}.$$

The value of W may now be written

$$W = \alpha^2\left\{\sin^{[-1]}(\cos\lambda\sin X) - \cos^2\lambda\sin\varepsilon\sin\vartheta . X - \sin\lambda\cos\lambda\,\varDelta(\vartheta)\right\},$$

and when λ increases W decreases; so that the rate of increase of W, or the area on the circle of illumination occupied by the projection of a zone of unit width in the cold season, is represented by

$$-\frac{dW}{d\lambda} = \alpha^2\sin 2\lambda\,\left\{\cot\lambda\,\varDelta(\vartheta) - \sin\varepsilon\sin\vartheta . X\right\}.$$

Let the amount of sunshine received upon the unit area at the unit distance from the sun in the unit of time be H. Then in an instant of time the amount received upon the zone of unit width at a distance r from the sun would be

$$\frac{-dW}{d\lambda}\cdot\frac{H}{r^2}\,dt.$$

By the principle of the conservation of the moment of momenta, the radius vector of an unperturbed planet sweeps over equal areas in equal times. If h is a constant this, which is one of Kepler's laws, is expressed by $r^2 d\vartheta = h dt$. By substitution in the last expression this gives the following measure of the receipt of solar energy for a small change in the earth's longitude,

$$-\frac{dW}{d\lambda}\frac{H}{h}d\vartheta = \frac{H\alpha^2 \sin 2\lambda}{h}\{\cot\lambda\,\Delta(\vartheta)-\sin\varepsilon\sin\vartheta.\,X\}\,d\vartheta \quad (1)$$

and the whole heat received between the autumnal and the vernal equinoxes on a zone of unit width will be proportional to the integral of this quantity from $\vartheta = 0$ to $\vartheta = \pi$, or to twice the integral from zero to $\pi/2$, since the conditions are symmetrical. To find the corresponding value for the summer interval it is only necessary to substitute π–X for X.

To facilitate integration it may be noted that

$$\sin\vartheta\,.\,X\,.\,d\vartheta = \cos\vartheta\,dX - d(X.\cos\vartheta)$$

and

$$\frac{dX}{d\vartheta} = -\frac{\sin\varepsilon\cos\vartheta\tan\lambda}{(1-\sin^2\varepsilon\sin^2\vartheta)\Delta(\vartheta)}$$

These values reduce (1) to

$$-\frac{dW}{d\lambda}\frac{H}{h}d\vartheta = \frac{H\alpha^2}{h}\sin 2\lambda\Big\{\cot\lambda\,\Delta(\vartheta) +$$

$$\tan\lambda\left(\frac{1}{\Delta(\vartheta)} - \frac{\cos^2\varepsilon}{(1-\sin^2\varepsilon\sin^2\vartheta)\Delta(\vartheta)}\right) + \frac{d}{d\vartheta}\left(X.\cos\vartheta\,\sin\varepsilon\right)\Big\}d\vartheta. \quad (2)$$

For values of λ from 0 to $\pi/2-\varepsilon$, or from the equator to the polar circle, this is integrable term by term. If $\sin\varepsilon/\cos\lambda = \varkappa$ and if $E^1(\varkappa)$, $F^1(\varkappa)$, $\Pi^1(\varkappa)$ denote complete elliptic integrals of the three classes for the modulus $\varkappa$; and if Z is the integral of (2), being the solar radiation received between equinoxes on the zone of unit width,

$$Z = \frac{2H\alpha^2}{h}\sin 2\lambda\Big\{\cot\lambda\,E^1(\varkappa) + \tan\lambda[F^1(\varkappa) - \cos^2\varepsilon\,\Pi^1(\varkappa)]$$

$$\pm\frac{\pi}{2}\sin\varepsilon\Big\} \quad (3)$$

the plus sign giving the value for summer and the minus sign that for winter.

At the equator, or for $\lambda = 0$, (3) reduces to the term containing $E^1(\varkappa)$. At the polar circle Z ceases to be doubly periodic; for then $\varDelta(\vartheta)$ degenerates into $\cos\vartheta$ and $\cos\lambda = \sin\varepsilon$ so that

$$\frac{dX}{d\vartheta} = -\frac{\cos\varepsilon}{1 - \sin^2\varepsilon\sin^2\vartheta}.$$

These relations also reduce (1) to an integrable form and give for this one latitude

$$Z = 2\frac{H\alpha^2}{h}\sin 2\varepsilon\left\{\tan\varepsilon + \cos\varepsilon\, ln \cot\frac{\pi/2 - \varepsilon}{2} \pm \frac{\pi}{2}\sin 2\right\}$$

Within the polar circle the limits of integration change, because for a part of the time there is no illumination. Furthermore $\sin\varepsilon/\cos\lambda > 1$, and a transformation is needful to reduce the functions to standard forms. To effect this let

$$\frac{\sin\varepsilon\sin\vartheta}{\cos\lambda} = \sin\varphi, \text{ and } \frac{\cos\lambda}{\sin\varepsilon} = \mu < 1.$$

Then

$$\varkappa = 1/\mu;\ \varDelta(\vartheta) = \cos\varphi;\ \cos\vartheta = \sqrt{1 - \mu^2\sin^2\varphi} = \varDelta(\varphi);$$

$$d\vartheta = \frac{\mu\cos\varphi\, d\varphi}{\varDelta(\varphi)}.$$

The superior limit of integration is given by

$$\varDelta(\vartheta) = 0 \text{ or } \varphi = \pi/2.$$

By these substitutions (2) becomes*

$$-\frac{dW}{d\lambda}\frac{H}{h}d\vartheta = \frac{H\alpha^2}{h}\sin 2\lambda\left\{\frac{\sin\varepsilon}{\sin\lambda}\varDelta(\varphi) + \frac{\cos\varepsilon}{\tan\varepsilon\sin\lambda}.\frac{1}{\varDelta(\varphi)}\right.$$
$$\left.-\sin\lambda\frac{\cos\varepsilon}{\tan\varepsilon}\frac{1}{(1-\cos^2\lambda\sin^2\varphi)\,\varDelta(\varphi)} + \frac{d}{d\varphi}(\sin\varepsilon\,\varDelta(\varphi).X)\right\}d\varphi,$$

and this when integrated from zero to π is

$$Z = 2\frac{H\alpha^2}{h}\sin 2\lambda\left\{\frac{\sin\varepsilon}{\sin\lambda}E^1(\mu) + \frac{\cos\varepsilon F^1(\mu)}{\tan\varepsilon\sin\lambda} - \frac{\sin\lambda\cos\varepsilon}{\tan\varepsilon}\Pi^1(\mu)\right.$$
$$\left.\pm\frac{\pi}{2}\sin\varepsilon\right\}.$$

Knowing the zonal receipt of solar energy for any and every latitude, the heat per unit area is within reach; for the length of a parallel of latitude is $2\pi a\cos\lambda$, and if u is the sunshine per unit area for the interequinoctial period

* In testing the truth of this formula it is convenient to remember that
$$\frac{\cos^2\phi}{\Delta(\phi)} = \frac{\Delta(\phi) - (1-\mu^2)/\Delta(\phi).}{\mu^2}$$

$$u = Z / 2\pi\alpha \cos\lambda.$$

At the equator this becomes

$$u = 2\,\frac{H\alpha}{\pi h}\,E^1(\varkappa);$$

$E^1(\varkappa)$ being in this case the quadrant of an ellipse the numerical eccentricity of which is $\sin\varepsilon$. The quantity h is a function of the sum of the masses of the sun and earth, say m, the major semiaxis of the earth's orbit, A, and the eccentricity, e, in fact

$$h^2 = Am\,(1 - e^2).$$

Now the major axis of a planet's orbit has until lately been assumed to be absolutely constant. In 1879, Mr. D. Eginitis showed that a surmise of Tisserand's was correct, and that the major axis is subject to a secular inequality of the third order with respect to the masses.* For most purposes, however, this minute change is negligible as well as the increment of mass due to the accumulation of meteoric matter. Hence e may be regarded as the only variable in the value of h. If the obliquity were zero and the eccentricity were zero, so that the orbit would be a circle of radius A, its plane also coinciding with that of the equator, $E^1(\varkappa)$ would reduce to $\pi/2$, and the solar energy received between equinoxes at the equator would be

$$u = \frac{H\alpha}{\sqrt{Am}}.$$

Now the unit in which radiant energy is measured is arbitrary; this last value is a convenient unit and I have adopted it.

For computation it is desirable to reduce the elliptic integral of the third class to integrals of the first and second classes. This is possible by a well known theorem which applied to $\Pi^1(\varkappa)$ and $\Pi^1(\mu)$ gives

$$\Pi^1(\varkappa) = \Pi\left(\varkappa, \sin\varepsilon, \frac{\pi}{2}\right) = F^1(\varkappa) + \frac{\cot\lambda}{\cos\varepsilon}\left\{F^1(\varkappa)\,E\left(\varkappa, \frac{\pi - 2\varepsilon}{2}\right) - E^1(\varkappa)\,F\left(\varkappa, \frac{\pi - 2\varepsilon}{2}\right)\right\}$$

E and F denoting integrals of amplitude less than $\pi/2$, and

$$\Pi^1(\mu) = \Pi\left(\mu, \cos\lambda, \frac{\pi}{2}\right) = F^1(\mu) + \frac{\tan\varepsilon}{\sin\lambda}\left\{F^1(\mu)\,E(\mu, \varepsilon) - E^1(\mu)\,F(\mu, \varepsilon)\right\}.$$

* Ann. Observ. Paris. Mém. vol. xix, 1889, paper H.

Substitution of these values in $Z/2\pi a \cos \lambda$ gives the following expressions for actual use:—

At the equator the modulus is $\sin \varepsilon$ and for either season alike

$$u = \frac{2\,E^1}{\pi\sqrt{1-e^2}}.$$

From the equator to the polar circle the modulus is $\sin \varepsilon / \cos \lambda$ and the sunshine per unit area between equinoxes is

$$u = 2\frac{\sin \lambda}{\pi\sqrt{1-e^2}}\left\{ E^1 \cos \varepsilon \left[\frac{\cot \lambda}{\cos \varepsilon} + F\,(90^\circ - \lambda)\right] - F^1 \cos \varepsilon \left[E\,(90^\circ - \lambda) - \tan \lambda \tan \varepsilon \sin \varepsilon\right] \pm \frac{\pi}{2} \sin \varepsilon \right\}.$$

At the polar circle

$$u = 2\frac{\cos \varepsilon}{\pi\sqrt{1-e^2}}\left\{ \tan \varepsilon + \cos \varepsilon\, ln \cot \frac{90^\circ - \varepsilon}{2} \pm \frac{\pi}{2} \sin \varepsilon \right\}$$

From the polar circle to the pole the modulus is $\cos \lambda / \sin \varepsilon$ and

$$u = 2\frac{\sin \lambda}{\pi\sqrt{1-e^2}}\left\{ E^1 \cos \varepsilon \left[F(\varepsilon) + \frac{\tan \varepsilon}{\sin \lambda}\right] - F^1 \cos \varepsilon \left[E(\varepsilon) - \frac{\cos \lambda}{\tan \varepsilon \tan \lambda}\right] \pm \frac{\pi}{2} \sin \varepsilon \right\};$$

and at the pole this reduces to

$$u = \frac{1}{\sqrt{1-e^2}}\left\{ \sin \varepsilon \pm \sin \varepsilon \right\}$$

These formulas can be computed from known series or by the help of Legendre's tables. For $\varepsilon = 23^\circ\ 27'$ I have computed the values tabulated in both ways with coincident results. For $\varepsilon = 24^\circ\ 36'$ I used the tables and checked by differences with the other series. The radiant energy received during the entire year is of course the sum of the energy received during the two seasons. The quantities thus found and multiplied by a constant coincide with Meech's values to four significant figures. Meech took the apparent obliquity at $23^\circ\ 28'$, and the sine of this angle is 1·0007 the sine of $23^\circ\ 27'$, so that a closer correspondence could not be expected.* I have stated my results to four decimals, though for the purposes of this paper three would have been ample.

It need scarcely be mentioned that the effect of the real ellipticity of the meridian is to reduce the receipt of solar energy towards the the poles by a trifling amount.

Washington, D. C., June, 1894.

* The obliquity was $23^\circ\ 27'\ 30''$ in 1852, or after Meech had begun his investigation. It is now about $23^\circ\ 27'\ 11''$.