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FINITE HOMOGENEOUS STRAIN, FLOW AND RUPTURE OF ROCKS

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CONTENTS.

Phenomena and Plan of Discussion.

Evidences of Movement.—All observers are aware that few rock masses are continuous for any considerable distance. It is seldom that more than a few yards of a rock exposure can be examined without revealing joints, fissures or slickensides. Still more frequently rock masses show slaty or schistose cleavage,* impressed upon them by dynamical causes. In a very great proportion of such cases a little attention also discloses

*Schist and the adjectives derived from it are used in literature in somewhat variable senses. As I use it, schist denotes cleavable rocks which are allied to slates, but in which the cleavage surfaces are not all sensibly parallel to one another as they are in true slate. By no means are schists all crystalline or metamorphic.

the fact that the partings are locally arranged on a definite system. In slaty cleavage the cleavage planes are substantially parallel and very close together; in flags of the slaty class the intervals between cleavage planes are greater; in schists the partings range through small angles, and in these last rocks there are frequently two sets of partings, each cleavage making a small angle with others of the same set, but a large angle with those of the other set. Where the rock is divided by cracks these are often parallel and spaced with a considerable approach to uniformity. Sometimes they occur at a fraction of an inch from one another, while in other instances they are rods apart. In still other cases there are two systems of such cracks crossing one another at right angles, or at angles which approach to right angles. Not infrequently such a double system of fissures is accompanied by a second of like character, at right angles to it, dividing the rock into polyhedral fragments of greater or less size.

Slaty cleavage is at present regarded by most geologists as due to a pressure acting in a direction perpendicular to the planes of cleavage, and this opinion is supposed to be well supported by experiments. Indications are not wanting, however, that many observers are ill satisfied with this explanation. Less attention has been paid to jointing, concerning which there is no consensus of opinion. By some it is considered as due to tensile stresses, while others insist on its intimate association with cleavage. Jointing is also often treated as distinct from faulting and as being unaccompanied by any relative movement of the joint walls. No systematic attempt appears to have been made to elucidate these various structures, which are generally recognized, however, as at least sharing a dynamic origin. Even the experiments on cleavage seem to me not to have been studied with as much care as they deserve.

Scope of the Inquiry.—Orogeny can never be satisfactorily discussed until the dynamic significance of cleavages and cracks is clear. A necessary step toward this end consists in the elucidation of those areas, great or small, throughout which the phenomena are uniform; for, however complex the conditions may be in any body of rock, they may be considered as uniform over a sufficiently small fraction of the whole mass.

Even this seemingly modest step cannot be completed in the present state of science. In the mechanics of artificial structures and machinery it is sufficient to discuss very small deformations, for such only are admissible. In geology this is wholly insufficient, the strains frequently being of enormous amount; so great indeed that laboratory experiments hardly aid one to conceive that they are possible. Yet there is no doubt among geologists that pebbles, even of quartzite, in conglomerates are not infrequently elongated by pressure to double their original length without rupture. Thus in geological mechanics it is absolutely essential to consider finite strains as well as infinitesimal ones.* Now, to discuss such strains completely it would be needful to know the relation between finite strains and the forces which produce them. This relation is not yet known.

One might infer that until it were ascertained discussion would be useless. I hope to show, however, that many relations of finite strain can be elucidated without the assumption of any law connecting stress and strain, and that these relations are of great assistance in the study of orogeny.

The general principles governing finite distortion have, of course, been indicated by natural philosophers; but little attention has been given to their development, because the theory of finite strain is needless for computation of machinery, while this subject will not offer much purely mathematical interest until the stress-strain law is known experimentally. In particular, but little attention has been paid (so far as I am aware) to the planes of maximum strain, which turn out to be those in which geologists have a special interest.†

In the following pages the attempt will be made to develop all the manifestations of uniform or homogeneous finite strain in rock masses regarded as isotropic, exhibiting viscosity and capable of flow, which can be elucidated without assuming a law connecting stress and strain. For this purpose finite strain must first be discussed by itself; then it must be considered just how far the relations of stresses are capable of coördination with those of strain. The influence of viscosity and solid flow must next be shown. Readers willing to assume that these subjects have been logically treated will probably skip them and proceed to the geological applications which follow. Finally, the results will be compared with actually observed phenomena and with the experiments which several investigators have made on slaty structure.

Finite rotational Strain.

Limitations of the Problem.

The mechanical effects short of rupture which force can produce in any mass are translation, rotation, dilation and deformation. The effects of mere translation may be considered separately from the other effects of force, or, in other words, one may consider these other effects relatively to some chosen point of the body itself.

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*I have previously endeavored to show that some fissure systems are satisfactorily explained on the hypothesis of small strains: Bull. Geol. Soc. Am., vol. 2, 1891, p. 49.

†On finite strain consult Thomson and Tait, Nat. Phil., 1879, sec. 181; and Ibbetson, Math. Theory of Elasticity, 1887, p. 69. I am much indebted to both authorities.

If any one point of a body is fixed in space, the mass can be brought from its original orientation into any other orientation by simple rotation about some one axis passing through the fixed point. This is a well known and very fundamental theorem, one of the many which bears Euler's name.

In homogeneous strain each elementary cube of the mass is deformed in the same manner as any other; each straight line in the unstrained mass therefore remains a straight line after strain, being elongated or deflected to the same extent as any of the lines parallel to it, and all lines originally parallel remain parallel. Hence any sphere in the unstrained mass becomes an ellipsoid, and all such ellipsoids are similar.

Irrotational strain is a term applied to a change in form and dimensions unaccompanied by any change in the direction of the axes of the strain ellipsoid. It is manifest that any dilation and any desired ratio between the axes of the strain ellipsoid can be produced without changing the direction of these axes.

Hence if the changes in a homogeneously strained elastic mass are regarded relatively to any one point of it, any change in the relations of its parts may be considered as compounded of a rotation about a single axis into the required orientation and an irrotational strain.

There is no necessary connection between the axes of strain and the axis of rotation, and the latter will not in general coincide with any of the strain axes. The rotation in the general case is resoluble into three partial rotations about the three strain axes.

For the purposes of this paper, it is both necessary and sufficient to examine the conditions affecting the mass in the principal sections of the strain ellipsoid. This is equivalent to selecting any one such section and considering the movements relatively to it. When such a selection is made, the rotations of the plane itself on axes drawn in it are eliminated, and only the rotation of the mass about a line perpendicular to the plane of reference retains its significance.

The first subject of discussion therefore is an ideally elastic mass with one point fixed when subjected to any distortions, however great, which will produce rotation about not more than one axis of the strain ellipsoid.

DISPLACEMENTS.

General Conditions.—Let the center of inertia of a mass remain at rest; let any other point or points of it be moved in planes parallel to the $x\,y$ plane without limitation, provided only that the strain shall be homogeneous, but let every plane originally parallel to that of $x\,y$ remain parallel to it, so that deformation parallel to $o\,z$ shall consist simply of changes of length. Then, if $x\ y$ are the original coördinates of any point and $x'\ y'$ its final coördinates these positions are connected by linear relations,

$$x' = (1 + e)\, x + by;\ y' = ax + (1 + f)\, y;\ z' = (1 + g)\, z;$$

or,

$$x = \frac{(1 + f)\, x' - by'}{(1 + e)\,(1 + f) - ab};\ y = \frac{(1 + e)\, y' - ax'}{(1 + e)\,(1 + f) - ab};\ z = \frac{z'}{1 + g}.$$

Here a, b, e, f, g are absolutely arbitrary and have the same value at all points of the mass.* They are the coördinates after strain of particular points. Denoting $x = 1, y = 1, z = 1$, by $(1, 1, 1)$, points originally at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, are transposed to $(1 + e, a, 0)$, $(b, 1 + f, 0)$, $(0, 0, 1 + g)$.

When the strain is so small that the squares of the displacements are negligible, a, b, e, f, g are to be treated mathematically as infinitesimal; consequently any formula in terms of this notation can be converted into the forms appropriate to small strain simply by neglecting powers of a, b, e, f, g, higher than the first.

Strain Ellipse.—The sphere $x^2 + y^2 + z^2 = 1$ is converted into an ellipsoid, which is found by substituting for x, y and z their values in terms of the accented variables. The section of this ellipsoid by the $x\ y$ plane is an ellipse with semi-axes A and B. Its equation is—

$$\left\{(1 + f)^2 + a^2\right\} x'^2 - 2\left\{b\,(1 + f) + a\,(1 + e)\right\} x'y' + \left\{(1 + e)^2 + b^2\right\} y'^2$$
$$= \left\{(1 + e)\,(1 + f) - ab\right\}^2. \qquad (1)$$

When $b\,(1 + f) + a\,(1 + e)$ is a positive quantity the major axis of this ellipse makes a positive acute angle with $o\,x$. Well-known properties of the ellipse show that its area is the same as that of the circle—

$$x'^2 + y'^2 = (1 + e)\,(1 + f) - ab = A\,B, \qquad (2)$$

and that the axes may be found from the equation—

$$(A \pm B)^2 = \left\{(1 + e) \pm (1 + f)\right\}^2 + (a \mp b)^2. \qquad (3)$$

* The letters e, f and g are used in the same sense as in Thomson and Tait, Natural Philosophy, but I have not found it convenient to use a and b as they are there employed.

The third axis of the ellipsoid is $C = 1 + g$. If η is the length of any one of the axes, A, B and C are the three roots of the cubic—

$$(\eta - A)(\eta - B)(\eta - C) =$$

$$\left\{\eta - (1 + g)\right\} \left\{\eta^2 - \eta \sqrt{(1 + e + 1 + f)^2 + (a - b)^2} + (1 + e)(1 + f) - ab\right\} = 0. \quad (4)$$

The volume assumed after distortion by the unit cube may be called h^3, and—

$$h^3 = A\,B\,C = (1 + g)\left\{(1 + e)(1 + f) - ab\right\}. \quad (5)$$

Rotation.—The limitations of this discussion imply that the plane of $A\ C$ can only revolve about C, so that the position of this plane is determined when the position of A is known. The angle which A makes with $o\ x$ is, say, ν, and this angle can immediately be inferred from (1) by a well-known formula which gives—

$$\tan 2\nu = -2\frac{b(1 + f) + a(1 + e)}{a^2 - b^2 + (1 + f)^2 - (1 + e)^2}.$$

Since the plane $B\ C$ is at right angles to that of $A\ C$, its position follows.

To find the position which the same material lines A and B occupied in the unstrained mass, it is convenient to remember that they must have been at right angles to one another before strain as well as after it; for mere rotation changes no angles, and irrotational strain is by definition a deformation in which the ellipsoidal axes maintain their direction. Hence, if μ was the angle which the fiber A made with $o\ x$ before distortion, its equation was $y/x = \tan \mu$, and by the displacement formulas—

$$\tan \nu = \frac{y'}{x'} = \frac{a + (1 + f)\tan \mu}{(1 + e) + b \tan \mu}.$$

The angle which the other axis made before strain was $\mu + 90°$, so that $\tan(\mu + 90°) = -\cot \mu$, while after strain it becomes $\nu + 90°$. Hence—

$$\tan(\nu + 90°) = \frac{a - (1 + f)\cot \mu}{(1 + e) - b \cot \mu} = -\cot \nu.$$

From these two equations ν can at once be eliminated, since $\tan\nu \cot\nu = 1$. Writing out this equation and reducing, one finds—

$$\tan 2\mu = -2\frac{b(1+e)+a(1+f)}{b^2-a^2+(1+f)^2-(1+e)^2}.$$

The equations for ν and μ can be combined to simpler forms. It will be found on trial that the values already deduced lead to—

$$\tan(\nu+\mu) = \frac{a+b}{(1+e)-(1+f)};\ \tan(\nu-\mu) = \frac{a-b}{(1+e)+(1+f)}. \qquad (6)$$

The angle $\nu - \mu$ is the angle of rotation, so that the condition of no rotation is evidently $a = b$. When the strain is infinitesimal, $a - b$ is infinitesimal, while $1 + e + 1 + f$ approaches 2. Hence $\nu - \mu$ is zero for vanishing strain. If the common limiting value of ν and μ is ν_0, $\tan(\nu + \mu) = \tan 2\nu_0$, or—

$$\tan 2\nu_0 = \frac{a+b}{(1+e)-(1+f)}.$$

Of course this same value is obtained by letting a, b, e and f approach zero in the formulas for $\tan 2\mu$ and $\tan 2\nu$. Thus $\nu - \nu_0 = \nu_0 - \mu$. It is evident that as rotation proceeds new fibers of matter constantly succeed one another in the position of axis, the whole series of fibers in the unstrained mass forming a wedge, $\nu_0 - \mu$ or $\frac{\nu-\mu}{2}$.

Lines of constant Direction.—Lines parallel to $o\,z$ retain their direction relatively to the $x\,y$ plane throughout strain. If the mass were inflexible and subjected to rotation, only these lines would maintain their direction; but when there is strain two other lines may retain their original direction, the two coinciding in the limiting case which separates that of three such lines from that of one.

If $\varkappa$ is the angle which any line in the $x\,y$ plane makes with $o\,x$ before strain and λ the angle which it makes after strain, then—

$$\tan\lambda = \frac{y'}{x'} = \frac{a-(1+f)\tan\varkappa}{1+e+b\tan\varkappa}.$$

If $\lambda = \varkappa$ this gives—

$$\tan\varkappa = \frac{f-e}{2b} \pm \sqrt{\frac{a}{b} + \left(\frac{f-e}{2b}\right)^2}, \qquad (7)$$

which represents two real lines, unless the quantity under the radical is negative. The two coincide when this quantity is zero, or when $4\,ab + (e - f)^2 = 0$. The value of *tan* $\varkappa$ then reduces to $\pm \sqrt{-a/b}$, showing that a and b must have opposite signs. This particular case occurs in the strain often known as shearing motion, as, for example, when a rivet is shorn by tension of the plates which it connects. It will be discussed later.

The condition of no rotation can be derived from *tan* $\varkappa$. The equation represents two lines, and if $\varkappa_1$ and $\varkappa_2$ are the two angles, *tan* $\varkappa_1$ *tan* $\varkappa_2 = -a/b$. If there is no rotation, the axial lines are lines of unchanged direction and *tan* $\varkappa_1$ *tan* $\varkappa_2 = -1$, or $a = b$.*

SIMPLE STRAINS.

Pure Rotation.—If the mass undergoes rotation without strain, each of the axes is equal to unity, and h has the same value. Then by (3), $e = f$ and $a + b = 0$, and by (5), $(1 + e)^2 = 1 - a^2$. Hence $tan\,(\nu - \mu) = a/\sqrt{1 - a^2}$, or $sin\,(\nu - \mu) = a$. This result can also be derived immediately from the displacement formulas.

Dilation.—When the only strain is dilation, $A = B = C = h$, whether or not the displacements cause rotation. Then by (3) $e = f$ and $a + b = 0$. By (2) also $(1 + e)^2 + a^2 = (1 + g)^2$. The rotation is then given by—

$$tan\,(\nu - \mu) = \frac{a}{1 + e} = \frac{a}{\sqrt{h^2 - a^2}}.$$

When there is no rotation, so that the displacements cause pure dilation, $a = b = 0$ and $e = f = g = h - 1$.

In dealing with dilations it is usually convenient to consider h, the ratio of dilation, as greater than unity, excepting when its value is unknown. The volume of a compressed mass is then $1/h^3$, which does not vanish unless the ratio of dilation is infinite.

*The length of the lines of unchanged length exhibits a somewhat remarkable relation. Let k be the length of such a line. Then—

$$\frac{x'}{x} = \frac{y'}{y} = k,$$

and by the displacement formulas—

$$\frac{y}{x} = \frac{k - (1 + e)}{b} = \frac{a}{k - (1 + f)}.$$

This gives—

$$2\,k = 1 + e + 1 + f \pm \sqrt{4\,ab + (e - f)^2}.$$

If k_1 and k_2 are the two values of k, then—

$$k_1\,k_2 = (1 + f)\,(1 + e) - ab,$$

which by (2) is the product of the semi-axes or $A\,B$. Thus the product of these lines remains invariable, whether or not they coincide with the axes.

In any case whatever one may express the axes A and C under the forms $A = h\alpha$, $C = h\beta$ where α and β may be perfectly independent. Then, since $ABC = h^3$, $B = h/\alpha\beta$. The values α, $1/\alpha\beta$ and β are the values which A, B and C would have were there no dilation, and upon the properties of α and β depend those of pure deformation, accompanied by rotation.

Shear.—A shear is the simplest possible deformation. It may be defined as an irrotational strain, unattended by dilation, in which one axis of the strain ellipsoid retains its original length. The unit sphere is thus converted into an ellipsoid, the axes of which are α, 1, $1/\alpha$; and α is called the ratio of shear. It is taken as greater than unity, excepting when it is dealt with as an unknown quantity.

In dealing with shears it is convenient to employ the following abbreviations: *

$$2\,s = \alpha - \alpha^{-1};\ 2\,\sigma = \alpha + \alpha^{-1}.$$

These forms imply that $\sigma^2 - s^2 = 1$.

The displacement formulas for a shear, the contractile axis of which makes an angle ϑ with ox are—

$$x' = x\,(\sigma - s\cos 2\,\vartheta) - ys\sin 2\,\vartheta;\ y' = y\,(\sigma + s\cos 2\,\vartheta) - xs\sin 2\,\vartheta;\ z' = z.$$

To verify this statement consider that $a = b$, so that there is no rotation; $g = 0$ and $(1 + e)\,(1 + f) - ab = 1$, so that there is no dilation; $\tan(\nu + \mu) = \tan 2\,\nu = \tan 2\,\vartheta$, showing that the axes of the strain ellipsoid make angles ϑ and $\vartheta + 90°$ with $o\,x$; finally $b\,(1 + f) + a\,(1 + e)$ is negative, so that the minor axis of the strain ellipsoid makes an acute positive angle with $o\,x$ as required.

When $\vartheta = 90°$ these equations reduce to—

$$x' = x\alpha;\ y' = y\,/\,\alpha;\ z' = z,$$

and when $\vartheta = 45°$, a case of importance,

$$x' = x\sigma - ys;\ y' = y\sigma - xs;\ z' = z.$$

The quantity $2\,s$ is called the *amount* of the shear. There are various aspects of this quantity. One way of looking at it is as the sum of two distortions. The elongation of the major axis is $\alpha - 1$ and the contrac-

* Let $\alpha = \cot \varpi$; then it is easy to see that $\sigma = 1/\sin 2\,\varpi$ and $s = \cot 2\,\varpi$. Here, as will be shown later, $2\,\varpi$ is the acute angle between the circular sections of the strain ellipsoid. The convenience of s and σ depends upon this fact, and the significance of the formulas is increased by bearing it in mind. The quantities s and σ may be regarded as hyperbolic *sine* and hyperbolic *cosine* of an area $\psi = \ln \alpha$; and then $90° - 2\,\varpi$ is the corresponding transcendental angle. This view of the functions, however, is not needful for the purposes of this discussion.

tion of the minor axis is $1 - 1/a$. The sum of the two is $a - a^{-1} = 2s$. While $2s$ measures shear and is not unfitly called the amount of shear, s might equally well have been regarded as the measure of shear; indeed, this would have been more convenient, because it would have accorded with the received nomenclature of stresses.

Many of the properties of shear can be inferred in the simplest manner from its definition. Since it involves neither change of volume nor of the area of the strain ellipse, it can consist only in re-arrangement of matter, each fiber perpendicular to the plane of shear, retaining its original thickness, length and direction, though shifted to a new position. Since the major axis of the shear ellipse exceeds unity and the minor axis falls short of unity, there must be four intermediate radii of unit length, and the symmetry of the conditions shows that these four radii form two diameters. Thus there are two diameters which have the same length after strain as before strain. These diameters are the traces on the $x\,y$ plane of planes passing through $o\,z$, and these planes undergo no distortion through strain. In them the circular sections of the strain ellipsoid evidently lie. All planes parallel to these are also, by the properties of homogeneous strain, planes of no distortion. Any two planes of no distortion must stand at the same perpendicular distance apart after strain as before, for were it otherwise the volume of the ellipsoid would be changed.

Thus a shear can consist only in the sliding of planes of no distortion upon one another and in changes of the angles between the two systems of undistorted planes.

The behavior during the straining process of the planes of no distortion is of great geological importance; but as this behavior depends to some extent upon rotation, it appears appropriate to defer its discussion until some of the simpler compound strains have been explained.

COMPOUND STRAINS.

How treated.—For the immediate purposes of this paper it is needful to examine compound strains of several varieties. It seems desirable also to examine the simpler combinations in somewhat more detail than is absolutely essential to the results which will be deduced from them in the subsequent sections in order to give assurance that the geological deductions are not vitiated by the omission of important properties of strain. It is to be hoped also that the treatment here submitted may facilitate the solution of geological problems not touched upon in the present investigation.

Pure Deformation.—Any pure deformation is resoluble into two shears at right angles to one another, one axis being common to the two elementary strains. This will be demonstrated by a proof that any relation whatever between the axes A, B and C of the ellipsoid whose volume is proportional to h^3 can be brought about by two such shears. Let $A = h\alpha$, $B = h\gamma$, and let $C = h\beta$, where A and B are entirely arbitrary. Then since $ABC = h^3 = Bh^2\alpha\beta$, it is evident that $1 / \alpha\beta = \gamma$, or $B = h / \alpha\beta$. Now, if a shear of ratio α is applied axially in the $x\,y$ plane to the sphere, $x^2 + y^2 + z^2 = h^2$, it will reduce this mass to the ellipsoid $x^2 / \alpha^2 + y^2\alpha^2 + z^2 = h^2$. If a second shear of ratio β is applied axially in the $y\,z$ plane it will further reduce the second axis in the ratio β and elongate the third axis in the same ratio. Thus the two shears yield an ellipsoid $x^2 / \alpha^2 + y^2\alpha^2\beta^2 + z^2 / \beta^2 = h^2$, and the axes of this ellipsoid are $h\alpha$, $h / \alpha\beta$ and $h\beta$, or A, B and C.

A converse proposition is also important. Any number of shears applied axially to a sphere can only modify the relations of the axes to values A, B and C, the volume of the mass remaining proportional to $ABC = h^3$. Hence any number of axial shears are reducible to two and not to three, as one might be inclined to surmise. This resolution may take place mathematically with any one of the axes as the common axis of the two shears. In most cases, however, considerations of symmetry point to one of the axes as that common to the two shears.

A simple shear produces relative motion of particles or fibers only in its own plane. Its only effect on fibers in planes at right angles to its own is to elongate them uniformly in one direction without any tendency to the causation of relative motion. Hence the effects of each shear must be considered in its own plane, and the relative motion produced by each of two shears in orthogonal planes is independent.

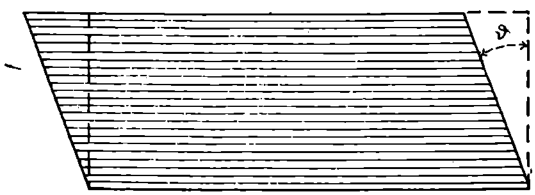

FIGURE 1.—*Scission.*

Shearing Motion or Scission.—A "shearing motion" is the rather ill-chosen designation of a strain nearly corresponding to that which occurs when a bar or plate is shorn by a pair of shears, or when a rivet yields perpendicularly to its axis, say, in a bursting boiler. The term is not happy, because it seems to indicate that there are shears not accompanied by motion. It is, of course, from this strain that the term shear was derived, but this has been transferred to the simpler deformation. The name *scission* would aptly indicate the "shearing-motion" strain, which consists in the relative movement of undistorted material planes, each sheet of infinitesimal thickness remaining in its own mathematical plane, as shown in figure 1. The motion can be well illustrated with a pack of cards.

Scission or shearing motion is that case of strain already referred to in which there is a single line of unchanged direction in the $x\,y$ plane, and it consists of a simple shear compounded with a rotation of the axes of the strain ellipsoid.

The most important case of scission is that in which the direction of the planes of constant direction and no distortion coincide with one of the axes. If this axis is $o\,x$ the displacement formulas may be written simply—

$$x' = x - 2\,ys\,;\; y' = y.^*$$

Here $2\,s = \alpha - \alpha^{-1}$, the amount of the shear involved. The rotation is given by—

$$tan\,(\nu - \mu) = s\,;$$

and since $tan\ 2\,\nu_0 = tan\,(\nu + \mu) = \infty$, the axes of the ellipse at the inception of strain were at 45° to the fixed axes. The quantity $4\,ab + (e - f)^2$ becomes zero by the simultaneous disappearance of its two terms. If ϑ is the angle by which a line originally parallel to $o\,y$ is deflected by the strain,

$$tan\,\vartheta = b = 2s\,,$$

so that the amount of shear may be defined as "The relative motion per unit distance between planes of no distortion." †

Two Shears in the same Plane.—The most frequent combination of two shears in the same plane is that in which the axes of one of these strains makes angles of 45° with those of the other. If the contractile axis of one of the shears makes an angle of 45° with $o\,x$, displacing x to x' and y to y', the ratio of shear being α, and if the contractile axis of the other

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* If the planes of constant direction and no distortion make an angle, ϕ, with $o\,x$, the displacements are given by—

$$x' = x\,(1 + s\,sin\,2\,\phi) - ys\,(1 + cos\,2\,\phi)\,;\; y' = y\,(1 - s\,sin\,2\,\phi) + x\,s\,(1 - cos\,2\,\phi).$$

The product, $a\,b = -\,s^2\,sin^2\,2\,\phi$, is an essentially negative quantity. Hence the signs of a and b are necessarily different. Compare the discussion of formula (7).

† Thomson and Tait, Nat. Phil., sec. 175.

shear coincides with $o\,y$, displacing x' to x'' and y' to y'', the ratio being a_1, then the displacement formulas * are—

$$x'' = x'a_1 = xa_1\sigma - ya_1 s\,;\; y'' = \frac{y'}{a_1} = \frac{y\sigma}{a_1} - \frac{xs}{a_1}.$$

This strain, although the resultant of two irrotational strains, is rotational, since $a - b$ is not zero. It is easy to see that this would probably be the case, for the first shear alters the direction of every line excepting those coinciding with its axes, and the direction of these is changed by the second shear. The rotation is given by—

$$\tan\,(\nu - \mu) = s s_1 / \sigma\sigma_1,$$

where $2\,\sigma_1 = a_1 + a_1^{-1}$ and $2\,s_1 = a_1 - a_1^{-1}$.

It is an important fact that when the shears are of infinitesimal amount this combination becomes irrotational. When a and a_1 differ infinitesimally from unity, $s = e$, $s_1 = e_1$, $\sigma = 1$, $\sigma_1 = 1$ and $\tan\,(\nu - \mu) = ee_1$, an infinitesimal of the second order.†

The two finite shears are equivalent to the rotation stated above and a simple shear of amount—

$$2\sqrt{\sigma^2 s_1^2 + \sigma_1^2 s^2}.$$

Plane undilational Strain.—The most general strain treated in this paper may be considered as a perfectly general undilational strain in one plane, combined with a shear at right angles to this plane and a dilation. The more complex effects are confined to the principal plane in which rotation occurs, and it is therefore desirable to reduce the plane undilational strain to its simplest terms.

One method of resolution consists in regarding the general strain as compounded of elementary strains symmetrically oriented with reference to the fixed axes, namely, an axial shear; a shear at 45° to $o\,x$; and a scission, the unchanged direction of which coincides with one of the axes.

It is somewhat easier to test the results of analysis in this case than to

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* When the first shear makes an angle ϑ with $o\,x$ the formulas are—

$$x'' = xa_1\,(\sigma - s\cos 2\,\vartheta) - ya_1 s\sin 2\,\vartheta\,;\; y'' = \frac{y}{a_1}\,(\sigma + s\cos 2\,\vartheta) - \frac{x}{a_1}\,s\sin 2\,\vartheta.$$

Here $ab = s^2\sin^2 2\,\vartheta$, and is essentially positive.

† When the shears make an angle ϑ and the strain is infinitesimal, $\tan\,(\nu - \mu) = ee_1\sin 2\,\vartheta$, which is also an infinitesimal of the second order, so that any two shears, and therefore any number of shears of infinitesimal amount, combine to an irrotational strain.

analyze the general strain. To begin with, changes of notation are convenient. The expression—

$$-2a(1+e) \pm \sqrt{1+4a^2(1+e)^2}$$

represents two values, one of which is minus the reciprocal of the other. Let the positive value be α_2^2, so that the negative value is $-\alpha_2^{-2}$. Then, if $2\sigma_2 = \alpha_2 + \alpha_2^{-1}$ and $2s_2 = \alpha_2 - \alpha_2^{-1}$, it is easy to see that—

$$-a(1+e) = \sigma_2 s_2.$$

Call the value of s_2/a minus α_3. Then—

$$a = \frac{s_2}{\alpha_3} \text{ and } 1+e = \sigma_2 \alpha_3;$$

and if one denotes—

$$\frac{\alpha_3(1+f) - \sigma_2}{2s_2} \text{ by } s_1,$$

$$1+f = \frac{\sigma_2 + 2s_2 s_1}{\alpha_3}.$$

Thus far only changes of notation have been introduced. To find the value of b in terms of this notation and for this case, consider that the sole condition of plane undilational strain is the invariability of the area of the strain ellipse. This is expressed by—

$$(1+e)(1+f) - ab = 1 \text{ or } b = \frac{(1+e)(1+f)-1}{a}.$$

Introducing the new notation into this expression—

$$b = -\alpha_3(2s_1\sigma_2 + s_2).$$

To interpret these values, suppose the final position of x and y to be x''' and y''', so that—

$$x''' = (1+e)x + by = \alpha_3\left\{(x - 2s_1 y)\sigma_2 - ys_2\right\};$$

$$y''' = (1+f)y + ax = \frac{y\sigma_2}{\alpha_3} - \frac{x - 2s_1 y}{\alpha_3} s_2.$$

This evidently expresses a simple axial shear of ratio α_3 combined with a compound strain. If x'' and y'' are the displacement values for this last—

$$x'' = (x - 2s_1 y)\sigma_2 - ys_2; \; y'' = y\sigma_2 - (x - 2ys_1)s_2.$$

By substituting $x' = x - 2s_1y$ and $y' = y$ these become the equations of a simple shear at an angle of 45° to ox. Finally—

$$x' = x - 2s_1y, \quad y' = y,$$

are the equations of a simple scission.

Thus the most general plane undilational strain is resoluble into an axial shear, a shear at 45° to ox, and a scission in the direction of one of the axes.

When $a = b$ this general strain reduces to a single shear. If $b/a = (1+e)/(1+f) = a_3^2$ the strain reduces to two shears or, in other words, the scission vanishes. If $a = 0$ and $(1+e)/(1+f) = a_3^2$ the strain is an axial shear combined with a scission.*

* *A second Resolution.*—The above method of resolution is the most convenient for computation, but it fails to disclose a relation of much geological significance. It is a fact that any plane undilational strain is resoluble either into two shears at an angle ϑ or into a shear and a scission at an angle ϕ. The significant difference between these two combinations is that the two shears cause a relatively small rotation which is an infinitesimal of the second order when the strain is infinitesimal, while the shear and scission produce a large rotation which is of the same order as the strain when this is infinitesimal. The criterion discriminating the two classes of strains is exceedingly simple. When a and b have the same sign the strain is invariably equivalent to two shears. When a and b have opposite signs the strain is invariably equivalent to a shear and a scission. As in the case of the other resolution, it is easiest to discriminate changes of notation from equations of condition synthetically.

Let a and b have the same sign. Then to show that the strain is compounded of two shears one may proceed as follows: Adopt the notation—

$$\sigma_2 = b\frac{(1+f)+a(1+e)}{2\sqrt{ab}}; \quad \sin 2\vartheta = \frac{-\sqrt{ba}}{s_2}; \quad a_3^2 = \frac{b}{a}.$$

Each of these expressions is possible whenever a and b have the same signs, and then only. In addition, the condition of plane undilational strain is $(1+e)(1+f) - ab = 1$. Here, then, is a number of equations just sufficient to determine a, b, e and f. Remembering that $1+e$ and $1+f$ are necessarily positive, they give—

$$a = \frac{s_2 \sin 2\vartheta}{a_3}; \quad b = -a_3s_2 \sin 2\vartheta; \quad 1+e = a_3(\sigma_2 - s_2 \cos 2\vartheta); \quad 1+f = \frac{\sigma_2 + s_2 \cos 2\vartheta}{a_3}.$$

Is is easily seen that these values answer to an axial shear of ratio a_3 and a second shear of ratio a_2 at an angle ϑ with ox.

Let a and b have opposite signs. This is implied in the expression—

$$a_3 = \frac{1+\sqrt{-ab}}{(1+f)},$$

and the condition of plane undilational strain is $(1+e)(1+f) - ab = 1$. Purely notative are the following:

$$2s_1 = a_3a - \frac{b}{a_3}; \quad \sin 2\phi = \frac{1+e-a_3}{a_3s_1}.$$

These four equations give—

$$a = s_1\frac{(1 \mp \cos 2\phi)}{a_3}; \quad b = -a_3s_1(1 \pm \cos 2\phi); \quad 1+e = a_3(1 + s_1 \sin 2\phi); \quad 1+f = \frac{1 - s_1 \sin 2\phi}{a_3}.$$

These values answer to an axial shear of ratio a_3 and a scission of ratio a_1. The direction of the scission makes an angle ϕ with ox if the given values of a and b are satisfied by choosing the upper sign in these expressions. In the opposite case the direction of the scission makes an angle ϕ with oy.

The foregoing synthesis shows how a plane undilational strain may be resolved when the displacements are given. Cases also arise in which it is desirable to find the displacements a, b, e and f from a known shear and values of ν and μ. If the ratio of the shear is α, the values of σ and s can be derived from it, and these two values, together with the values of $\nu + \mu$ and $\nu - \mu$ constitute four equations from which a, b, e and f can be deduced. They give—

$$a = s \sin(\nu + \mu) + \sigma \sin(\nu - \mu);\ 1 + e = \cos(\nu - \mu) + s \cos(\nu + \mu);$$

$$b = s \sin(\nu + \mu) - \sigma \sin(\nu - \mu);\ 1 + f = \cos(\nu - \mu) - s \cos(\nu + \mu). \quad (8)$$

These values, substituted in the formulas of preceding paragraphs, show to what simplest strain system a given rotation and shear are referable.

Strain due to Pressure.—For the sake of keeping the discussion of strains together, it may be assumed here by anticipation that a pressure produces a cubical compression of ratio h* and two equal shears of ratio α at right angles to one another. For brevity, let—

$$2\,t = \alpha - \alpha^{-2};\ 2\,\tau = \alpha + \alpha^{-2}.$$

Then the displacement formulas for a strain due to a pressure in the direction ϑ are—

$$x' = \frac{x}{h}(\tau - t \cos 2\,\vartheta) - \frac{y}{h} t \sin 2\,\vartheta;\ y' = \frac{y}{h}(\tau + t \cos 2\,\vartheta) - \frac{x}{h} t \sin 2\,\vartheta;$$

$$z' = \frac{z}{h}(\tau + t).$$

It will be observed that these formulas are analogous to those for simple shear.

When the pressure is vertical, so that $\vartheta = 90°$,

$$x' = \frac{x\alpha}{h};\ y' = \frac{y}{\alpha^2 h};\ z' = \frac{z\alpha}{h}.$$

If a vertical strain of this kind is combined with a scission or shearing motion in a horizontal direction, the values of x only will be modified by the second strain. If x'' is the final value of x and $2\,s_1$ is the amount of the shear produced by the scission—

$$x'' = \frac{x\alpha}{h} - \frac{2\,y s_1}{\alpha^2 h};\ y'' = y';\ z'' = z';\ \text{and } \tan(\nu - \mu) = \frac{s_1}{\tau}(\tau - t).$$

*Here h is taken greater than unity, and is the reciprocal of the value which in a given case would satisfy (5).

Here the rotation is of the same order as the strain and is not negligible when the strain is small.

If the strain produced by vertical pressure is combined with a shear at 45°, the value of z will be unchanged. If σ_2 and s_2 are the values of σ and s for this added shear, and if x''' and y''' are the final displacements for this case—

$$x''' = \frac{x a \sigma_2}{h} - \frac{y s_2}{a^2 h}; \; y''' = \frac{y \sigma_2}{a^2 h} - \frac{x a s_2}{h}; \; z''' = z'; \; tan\ (\nu - \mu) = \frac{s_2 t}{\sigma_2 \tau}.$$

In this case, when the strain is infinitesimal, the rotation is an infinitesimal of the second order.

Elongation.—Simple elongation (unattended by changes in the area of the section perpendicular to the direction of elongation) is sometimes regarded as a simple strain. It may as well or better be considered as compounded of two shears and a dilation. In discussing dilation it was pointed out that the three axes of the strain ellipsoid may be written $A = h\alpha$, $B = h / \alpha\beta$, $C = h\beta$. When the strain is simple elongation in the direction of B, $h\alpha = 1$, $h\beta = 1$ and $B = h^3 / AC = h^3$. Thus elongation consists of two shears each of ratio h and a cubical dilation h.

In the case of contraction or negative elongation a value h_1 is to be substituted for h and $h_1 = 1 / h$. Thus contraction is compounded of cubical compression $1 / h$ and two shears. If h is the same in the two cases, the same shears are involved in each strain but differently combined. In elongation the tensile axes of the shears coincide, while in contraction the contractile axes coincide.

The same two shears which without dilation would stretch a mass to an infinite length, when differently combined would reduce it to an infinitesimal thickness without cubical compression.

PLANES OF MAXIMUM TAGENTIAL STRAIN.

Position of undistorted Planes.—Attention has already been called to the fact that in a simple shear the circular sections of the strain ellipsoid are undistorted planes parallel to which relative motion takes place, and further inquiry into them is essential to a full elucidation of this strain. In the other plane undilational strains there are similar planes, though their behavior is modified in essential respects. In tri-dimensional strain the corresponding planes are no longer undistorted, but nevertheless influence the character of the deformation. It seems most logical to begin with a discussion of the case of simple shear and afterwards to modify the results for complex strains.

The circular sections of the shear ellipsoid for which the ratio is a make an angle with the major axis whose cotangent is a.* If this angle is called ϖ, the amount of shear is—

$$2\,s = a - a^{-1} = cot\,\varpi - tan\,\varpi = 2\,cot\,2\,\varpi = 2\,tan\,(90^\circ - 2\,\varpi).$$

Here s, or half the so-called amount of shear, appears as measured by the divergence from 90° of the angle $2\,\varpi$ between the circular sections of the shear ellipsoid. A right angle is the value which $2\,\varpi$ assumes when the strain is infinitesimal.

The original position of the particles constituting the planes of no distortion, relatively to the fibers which coincide with the axes of the ellipse, bears a simple relation to ϖ. Suppose the shear to be axial and that the sphere $x_1^2 + y_1^2 + z_1^2 = h^2$ is converted into the ellipsoid $x^2 / a^2 + y^2 a^2 + z^2 = h^2$, so that $y_1 / x_1 = a^2 y / x$; then the original position of the material plane forming the circular section of the shear ellipsoid was $a^2\,tan\,\varpi = a = tan\,(90^\circ - \varpi)$.

Thus these material planes made before shear the same angle with the minor axis of the ellipsoid which they make after strain with the major axis.

Planes of maximum Strain.—It is instructive to regard the planes of no distortion from another point of view. Consider any two very thin plane layers in the unstrained mass which include between them the axis $o\,z$, and let the angle which they make with $o\,x$ be φ. After strain these planes will still be planes; they will make an angle φ' with $o\,x$ and $tan\,\varphi = a^2\,tan\,\varphi'$ or

$$tan\,(\varphi - \varphi') = \frac{(a^2 - 1)\,tan\,\varphi}{a^2 + tan^2\,\varphi}.$$

The greater the angle $\varphi - \varphi'$ becomes, the greater must be the tangential strain. Now this angle and its tangent are greatest when $tan\,\varphi = a$ or when $tan\,\varphi' = 1 / a = tan\,\varpi$. Thus the undistorted planes are those for which tangential strain is a maximum. For the axes, on the other hand, $\varphi - \varphi' = 0$, and there is no tangential strain.

Angular Range of undistorted Planes.—Though at the end of a shear or other plane strain there are planes which have the same dimensions as before strain, it is not true that these planes have undergone no distortion. On the contrary, there is but one strain in which any lines escape

* The intersections of the shear ellipse with the circle of equal area are points in these sections, since the radii of the ellipse retain their original length, say unity. These intersections are given by—

$$\frac{x^2}{a^2} + a^2 y^2 = 1 = x^2 + y^2,$$

whence $a = \pm x / y$.

temporary distortion. In general, the circular sections of the shear ellipsoid consist of different particles when the strain begins from those which occupy the circular sections when the strain ends. In other words, these geometrical planes sweep through a certain angle, coinciding successively with all the particles in a wedge of the mass bounded by limiting material planes. Futhermore, one of the circular sections sweeps in general through a different angle from that over which the other ranges, so that the rate of movement relatively to the particles is different. This difference of rate is a matter of much importance when the mass possesses viscosity, as all real matter seems to do.

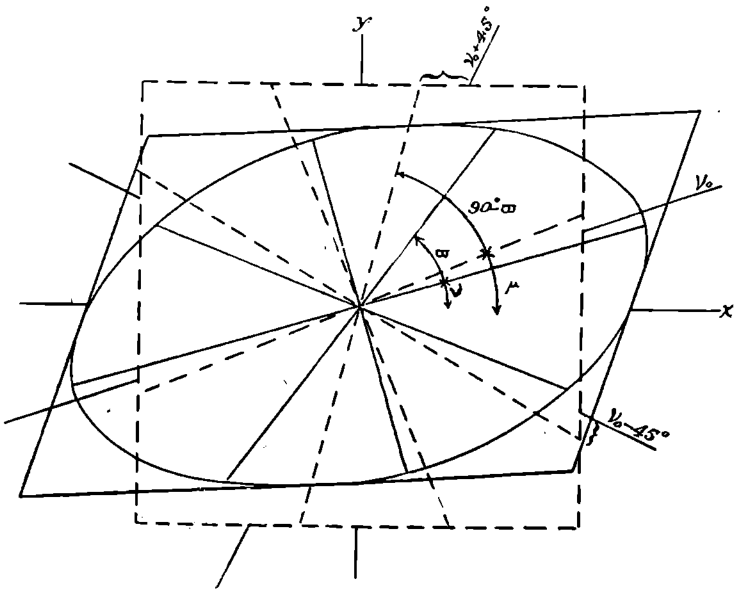

FIGURE 2.—*Range of circular Sections.*

The square of broken lines is strained to the rhomb in full lines. The full lines intersecting at the center are the final axes and lines of no distortion. The broken lines intersecting at the center show the positions which these same lines occupied before strain. The lines ν_0 and $\nu_0 \pm 45°$, which are drawn only to the outside of the square, indicate the position of the fibers which at the inception of strain coincided with the major axis and the lines of no distortion. The { { mark the wedges in the unstrained solid over which the geometrical planes of no distortion sweep. For the displacements see example, p. 34.

The range of the circular sections must therefore be determined, and it is most easily discussed by examining in the unstrained mass the limiting angles between which the circular sections will vary when strain of assigned amount takes place. The general formulas afford the means for such a determination.

When strain begins the major axis of the shear ellipse makes an angle, ν_0, with $o\,x$, and the undistorted planes then make an angle of 45° with the major axis or angles $\nu_0 \pm 45°$ with $o\,x$. When the strain is complete the major axis makes an angle, ν, with $o\,x$, and the undistorted planes make angles ϖ with this axis. But before strain began this last axial fiber made an angle, μ, with $o\,x$, and the particles constituting the last undistorted plane then made an angle, $90° - \varpi$, with μ. Thus in the undistorted mass the angles bounding the wedge through which the circular sections will sweep are $\nu_0 \pm 45°$ and $\mu \pm (90° - \varpi)$.

On the side of the minor axis toward which rotation takes place this range is therefore—

$$\nu_0 + 45° - \left\{ \mu + 90° - \varpi \right\} = \varpi - 45° + \frac{\nu - \mu}{2},$$

and on the opposite side of the minor axis the range is—

$$\left\{ \mu - (90° - \varpi) \right\} - (\nu_0 - 45°) = \varpi - 45° - \frac{\nu - \mu}{2}.$$

The difference of range is thus the angle of rotation, and is actual whenever the strain is a rotational one.

In a simple shear, then, there is no difference in range, and the range on each side is $\varpi - 45°$. In the case of scission or shearing motion it is easy to see that $2\,(\varpi - 45°) = \nu - \mu$, so that the range is zero on the side from which rotation takes place, and one and the same set of fibers are exposed to maximum tangential strain throughout the process of strain, while the other circular section sweeps through the maximum possible angle. In any case of plane strain the difference in range is at once assigned by the angle of rotation, so that for two shears in the same plane at an angle of 45° the difference is measured by $tan\,(\nu - \mu) = ss_1/\sigma\sigma_1$.

For plane strains the value of ϖ may be simply expressed in terms of the displacement coëfficients. It is easy to see that—

$$\frac{1}{a^2} = tan^2\,\varpi = B/A.$$

Hence also—

$$tan^2\,2\,\varpi = \frac{4\,AB}{(A - B)^2} = 4\,\frac{(1 + e)\,(1 + f) - ab}{(e - f)^2 + (a + b)^2}. \tag{9}$$

Case of Strain in three Dimensions.—It has been pointed out already that the relative motions of the particles in the $x\,y$ plane due to a shear α are unaffected by an axial shear β in the $B\,C$ plane. The sole effect of the second shear, so far as the $x\,y$ plane is concerned, is to change the length of all lines parallel to the common axis of the shears uniformly in the ratio β. Hence if before the imposition of the β-shear a line made an angle ϖ with A, this shear will alter the angle ϖ to, say, ω, and—

$$tan\ \omega = \beta^{-1}\ tan\ \varpi = 1 / \alpha\beta = B / h. \tag{10}$$

Lines making the angle ω with A will not be undistorted when β differs from unity, but they will be lines of maximum tangential strain whatever may be the value of β.

The value of ω cannot easily be determined immediately from the displacement coëfficients. It can be expressed in terms of the axes for $tan^3\ \omega = B^2 / AC$, but the value of B / C is a complicated one, on account of the inclination of the plane $B\ C$.

Rotation is supposed to be confined to the axis $o\ z$, and is therefore unaffected by the shear β. Hence for strain in three dimensions, as well as in plane strain, the difference of range of the planes of maximum strain measured in the unstrained solid is the angle of rotation, $\nu - \mu$.

Numerical Example of Strain.—The application of the formulas developed may be illustrated by an example. Let—

$$a = 0.1;\ b = 0.3;\ 1 + e = 1.2;\ 1 + f = 0.7;\ 1 + g = 1.1.$$

This is a rotational strain, since $b > a$. Equations (6) also show that $\nu + \mu = 38° 40'$ and $\nu - \mu = -6° 1'$. If the displacements constituted a pure rotation, $sin\ (\nu - \mu)$ would equal a. As this is not the case, there is strain. Formula (5) gives $h = 0.962$, so that the strain is a compressive one. If deformation were confined to the $x\,y$ plane, $1 + g$ would equal h. Hence there are two shears. To find them it is most convenient to determine the axes of the ellipsoid from (3), which gives $A = 1.275$, $B = 0.635$, $C = 1.1$. Then also $\alpha = A / h = 1.325$, $\beta = C / h = 1.143$. Equation (1) shows that the major axis makes a positive acute angle with $o\ x$. The rotation, dilation and the ratios of the two shears are now known.

To resolve the rotation and the α shear into component, plane, undilational strains, let a_1, b_1, e_1 and f_1 be the displacements which would produce only the α shear and the rotation. Then formula (8) leads to these values—

$$a_1 = 0.0695;\ b_1 = 0.2872;\ 1 + e_1 = 1.2572;\ 1 + f_1 = 0.8113,$$

which give for the elementary plane strains—

$$\alpha_2 = 0.9168;\ \alpha_3 = 1.2524;\ s_1 = 0.0708.$$

The α shear with the rotation is therefore equivalent to a shear with its contractile axis coinciding with $o\ y$ of ratio 1.2524, together with a shear the *tensile* axis of which makes a positive angle of 45° with $o\ x$, its ratio being $1/a_2 = 1.0908$; and lastly, a scission for which $s_1 = 0.0708$. Since a_1 and b_1 have the same sign, the plane undilational strain might have been regarded as due to the combination of two shears without any scission, but these shears would not be at 45° to one another.

The value of ϖ is given by $\tan \varpi = 1/a = 0.7545$, so that $\varpi = 37° 2'$. Had only a_1, b_1, e_1 and f_1 been given, ϖ could have been obtained from (9), which, of course, gives the same angle.

The first fiber to occupy the position of major axis at the inception of strain made an angle with $o\,x$, which was $\nu_0 = (\nu + \mu)/2 = 19° 20'$, and at this same time the positions of the lines of maximum strain were at $\nu \pm 45°$; *i. e.*, at 64° 20′ or — 25° 40′. The original position of the fiber which eventually constitutes the final major axis was at an angle μ or 20° 20½′ to $o\,x$. The original position of the fibers which at the end of the strain undergo maximum strain was at $\mu \pm (90° - \varpi)$; *i. e.*, 75° 18½′ and — 30° 37½′. The angles in the unstrained mass bounding the fibers which subsequently undergo maximum strain on the side from which rotation takes place are thus, $\mu + 90° - \varpi$ and $\nu_0 + 45°$, and these differ by 10° 58½′. On the other side the limiting angles are $\nu_0 - 45°$ and $\mu - (90° - \varpi)$, which differ by only 4° 57½′. Thus the fibers on the positive side of the major axis pass through the condition of maximum strain more than twice as rapidly as do those on the negative side of the major axis. If the resistance which the mass offers to deformation varies with the rapidity of deformation (as is the case with real substances), this difference will somewhat affect the results. Had a and b different signs, this difference would be far greater.

The angle ω for this example is by formula (10) 33° 25′, so that the β shear changes the direction of the lines of maximum strain by some 3½ degrees, though without tending to produce any further relative motion upon them.

Figure 2 is drawn for the displacements a_1, b_1, e_1 and f_1, and illustrates the range of planes of maximum strain for this example.

FINITE STRESS.

RELATIONS OF STRESS AND STRAIN.

In the foregoing discussion the geometrical properties of homogeneous strain due to given displacements as exhibited on any principal plane of a strain ellipsoid have been developed, and I am aware of no important property of such strain which has been omitted. If the relations of displacement to stress (or force per unit area) could be as fully developed, we should have a substantial basis for a theory of finite distortion, since however heterogeneous a strain may be, any infinitesimal portion of the mass is homogeneously strained.

The relations between finite stress and displacement lack satisfactory experimental basis and cannot therefore be fully developed, but it is desirable to show just where knowledge ends and ignorance begins.

Stresses in a Shear.—From the discussion of the properties of shear, it follows that the undistorted planes are necessarily subjected to purely tangential stresses; for they are neither elongated nor drawn apart during strain, while normal forces acting upon them would produce such effects.

The stress phenomena in a shear can be examined as a case of equilibrium, and such an examination reveals the somewhat important fact that the planes of maximum tangential stress do not coincide with the planes of maximum tangential strain.* It also teaches how the two component forces involved in a finite shear are related, and thus, in spite of ignorance of the direct relations between stress and strain, the inquiry is by no means fruitless.

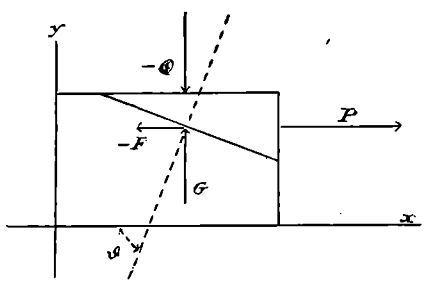

FIGURE 3.—*Stresses in finite Shear.*

Let the rectangle $o\,b$ represent one-quarter of a strained cube and let $-Q$ and P be the stresses (or forces per unit area) holding it in this state of strain. Then it is easy to find the stress on any plane cutting the $x\,y$ plane at right angles along the line $a\,c$. Let the normal to the plane make an angle ϑ with $o\,x$. Then—

$$ab = ac \sin \vartheta; \; bc = ac \cos \vartheta.$$

If F and G are the component stresses on $a\,c$ parallel to $o\,x$ and $o\,y$, these components must hold the stresses on $a\,b$ and $a\,c$ in equilibrium. Now, the total force on $a\,c$ in the direction of $o\,x$ is $-F\,a\,c$ and the whole force on $b\,c$ is $P\,b\,c$. Q and G are similarly related, so that—

$$-Fac = Pbc = Pac \cos \vartheta,$$

$$Gac = -Qab = -Qac \sin \vartheta,$$

or—

$$-F = P \cos \vartheta; \; G = -Q \sin \vartheta.$$

*In at least some treatises on elasticity and geological mechanics it seems to have been assumed that these planes do coincide.

The position of the plane remaining constant, it is permissible to combine F and G like simple forces to a tangential component, T, acting in the direction of $a\,c$, and a normal component, N, acting perpendicularly across $a\,c$. Evidently, if P and Q are considered as in general positive quantities—

$$T = -F \sin \vartheta - G \cos \vartheta = (P + Q) \sin \vartheta \cos \vartheta,$$

$$N = -F \cos \vartheta - G \sin \vartheta = P \cos^2 \vartheta + Q \sin^2 \vartheta,$$

and T will be a maximum with reference to ϑ when—

$$\cos^2 \vartheta = \sin^2 \vartheta \text{ or } \vartheta = \pm 45°.$$

Although the tangential stress is greatest for this angle, one has no right to infer that the maximum tangential strain is at 45°, because there is a normal stress on the plane at this angle amounting to $(P + Q)/2$. On the contrary, it was shown above (page 34) that the maximum tangential strain in a shear occurs for planes which make an angle with $o\,x$ the tangent of which is $1/\alpha$, or the normal to which is given by $\tan \vartheta = \alpha$. The conditions of this plane are also such that there can be no normal stress acting upon it, and hence $N = 0$, so that one of the stresses must have a negative value and—

$$\tan^2 \vartheta = \frac{P}{-Q} = \alpha^2.$$

This relation enables one to determine the forces which produce a finite shear. The area on which the stress Q acts is α, and the force acting on the distorted cube in this direction is minus $Q\,\alpha$. The area on which P acts is $1/\alpha$, and the lateral force is therefore P/α; but by the last equation $-Q\,\alpha = P/\alpha$, so that a finite shear, as well as an infinitesimal one, results from the action of two equal forces acting at right angles to one another in opposite senses.*

Simple Pressure.—Knowing the composition of a shear enables one to pass synthetically to the case of simple pressure or traction. If two equal shears at right angles to one another are combined, the contractile axes coinciding, each must produce the same effect as the other if the mass is isotropic. Each must also produce the same effect as if it acted alone. This statement does not imply a relation between stress and strain, for the shear in the $x\,y$ plane leaves the mass unstrained in the $y\,z$ plane. Hence two equal shears, each of ratio α, reduce the unit cube

* I have met with no demonstration of this relation between finite shearing stress and strain, but I am not prepared to state that none has been published.

to a thickness $1/a^2$ any of the sides of the mass having a length a. The upper surface has an area a^2 and the side an area $1/a$.

The tensile stress on sides of the mass is P in each direction, so that the two tensile forces are each P/a. When only one shear acted on the mass the contractile stress was Q, but the second shear increased each unit area to a, so that the contractile stress of the first shear was thereby reduced to Q/a. The stress due to the second shear is of precisely the same amount, so that the total contractile stress becomes $2\ Q/a$ on an area a^2. Thus the total force acting on this surface is $2\ Q a$, which, as has been shown, is equal to $2\ P/a$ in absolute value.

Let the mass thus strained be subjected to an hydrostatic pressure equal to P/a. Then the tensile forces would be balanced and the pressure on the upper surface would become $3\ Q a$.

Thus, two equal shears combined with an hydrostatic pressure equal to either component of either shear, applied to the unit cube, reduce to a simple pressure acting on one surface of the cube. Had the shears been so combined that their tensile axes coincided, a dilational stress equal to either component of either shear would have been needful to reduce the system to a simple traction.

Conversely, it is evident that a finite traction or pressure is resoluble into a dilational stress (positive or negative) and two shearing stresses, just one-third of the force being employed in each of the three component stresses. It is well known that precisely this resolution takes place for infinitesimal tractions, but the analysis of such tractions is usually stated as if the conclusions were true only for the limiting case of infinitesimal forces.

These results seem to exhaust what can be known of the relations of finite stress and strain without a further knowledge of the actual value of a in terms of Q. No two different pressures or different shears or dilations can be compared without a law relating to stress and strain.

Meaning of Hooke's Law.—It was to fill this gap that the famous law of Hooke was proposed. This is *Ut tensio sic vis*, which is now translated, Strain is proportional to stress. The brevity of Hooke's law has often been admired. The fact is that it is too brief fully to express the meaning really attached to it. It does not appear in this form of the law whether the stress (or pressure per unit area) is to be reckoned for the solid in an unstrained state or after the mass has reached a condition of equilibrium under the action of the external forces tending to deform it. But since the purpose of the mathematical theory of elasticity is to find equations expressing equilibrium of elastic masses, it is clear that this equilibrium must be supposed established before one can reason on the system of stresses which will maintain it. As a matter of fact, the fundamental equations are always derived in this way, and the stress is taken primarily as the force per unit area of the mass in a state of equilibrium. Thus, a less ambiguous statement of this law would be: Stress in an elastic mass which has reached a condition of equilibrium is proportional to the strain which the mass has undergone.

It is a curious fact that this is not the law which Hooke intended to express. Hooke's words are, "*Ut tensio sic vis:* That is, the Power of any Spring is in the same proportion with the tension thereof: That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward."* Thus Hooke's law as he meant it is clearly *load* is proportional to strain, and he had no idea of confining his law to infinitesimal deformations.

When the stresses and strains are infinitesimal it is easy to show that the two assertions, *stress* is proportional to strain and *load* is proportional to strain, are really equivalent; but for finite deformations they lead to very different results.

Let a unit cube be extended to a length $1 + e$ by a load L, and let the reduced area of the cross-section be A. Then the tension per unit area or the stress P is given by—

$$L = AP,$$

and if stress is proportional to strain,

$$P = Me, \text{ or } L = AMe,$$

where M is the constant, called Young's modulus and sometimes (though improperly) *the* modulus of elasticity. As was shown above, exactly one-third of the load is employed in producing dilation, however great L may be. Hence if k is the modulus of compressibility, the volume of the distorted cube is $1 + L/3\,k$. The volume is also the area of the distorted mass multiplied by its length, or $A\,(1 + e)$. Thus—

$$A = \frac{1 + L/3\,k}{1 + e}.$$

Substituting this value in the last equation gives an equation between load and strain, viz:

$$L - Me + Le\,\frac{3\,k - M}{3\,k} = 0,$$

*Quoted by P. G. Tait, "Properties of Matter," 1890, p. 204, from Hooke's lectures "de Potentia Restitutiva."

which is an hyperbola in L and e asymptotic to—

$$e = \frac{-3k}{3k - M}, \text{ and } L = \frac{3kM}{3k - M}.$$

Thus the fundamental assumption really made in the theory of elasticity is that the load-strain curve is an hyperbola instead of the straight line which Hooke supposed to represent the relation. The difference, however, as already remarked, is without consequence, so long as deductions from it are confined to very minute deformations.*

Stress System.—Any force acting on one face of a cube may be resolved into a normal component and two tangential components acting in the directions of the edges of the face. Hence the most general system of forces of constant direction acting on a cube is resoluble into six normal components and twelve tangential ones. If the center of inertia of the cube is at rest, the normal forces on opposite faces must be equal, and

* *Nature of the Proof of Hooke's Law.*—Hooke's law holds good, or, in other words, there is a linear relation involving a finite parameter between small stresses and strains, provided the stress-strain curve fulfills two conditions, viz., that the curve is continuous both in form and value, and that the tangent of the angle which it makes with the axes at the origin is finite. It seems to me that some discussion and even some confusion would have been avoided if elasticians had taken this geometrical view of the functions rather than a purely algebraical one. Thus Green simply assumed that the stress-strain function was developable, and that the development contained a term in which only the first power of the variable appeared, while Clebsch seems to have looked upon this algebraical relation as a mathematical necessity. This it certainly is not, for there are many continuous functions the development of which contains no term in the first power of the variable. These all represent curves which coincide with one of the axes at the origin; *e. g.*, the hyperbola referred to the vertex as origin.—Mr J. W. Ibbetson, in his excellent Mathematical Theory of Elasticity, makes an attempt to demonstrate Hooke's law by pure reason, independently of experiment. He expressly assumes, however, that the curve is continuous, and he states, without any attempt at proof, that the rate of variation of any traction component with any strain coördinate can never change sign or vanish. This last is equivalent to asserting that the curve cannot coincide with either axis at the origin. These two assumptions together cover the whole ground of Hooke's law, and really leave nothing to be proved.—Saint-Venant, in his edition of Clebsch, p. 40, attempted to show that if the internal stresses of an elastic mass depend in any continuous manner on the mutual distances of the molecules, Hooke's law follows. He points out that continuity involves a linear relation between the differentials of a function and the *corresponding* differentials of any variable. He then shows that on the assumption made corresponding small stresses and strains are corresponding differentials, and deduces the conclusion stated above. This argument does not satisfy me at all, for though one may undoubtedly write $d\,f(x) = A\,d\,x$, where A is constant and the relation is therefore linear, yet A may have and often does have the values zero or infinity. Saint-Venant made no attempt in the passage referred to to show that A must be finite in the case of elastic strains, and seems to have overlooked the necessity for such a proof.

In the same work, page 39, this great elastician forcibly remarks: "Generally and philosophically no purely mathematical consideration can reveal the manner in which the forces acting on the elements of a body and the geometrical changes which they produce depend upon one another." Experiment alone, and only somewhat refined experiment, betrays the fact that even the hardest substances yield somewhat to the smallest pressures, and that the stress-strain curve is continuous in form as well as value from positive to negative strains. One set of experiments is needful to show that a fly lighting on the end of a steel bar which is clamped at the center distorts it, and another set is required to show that the distortion is of the same absolute amount whether the fly settles on the upper or the lower end of the mass.

the twelve tangential forces must consist of six couples, each tending to produce rotation.

In this paper consideration is confined to those cases in which there is a tendency to rotation only about the line $o\,z$, and this limitation eliminates four of the couples. Thus the case to be considered here consists of three pairs of normal forces and two unequal couples tending to produce rotation in opposite directions. This force system is shown in the following diagram:

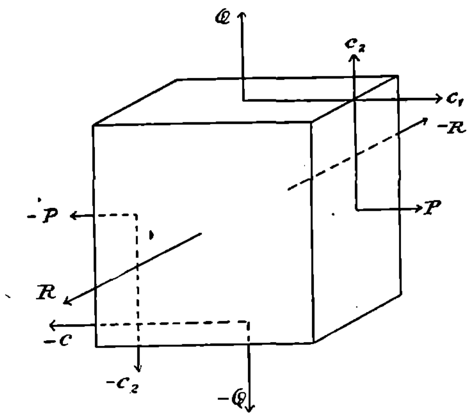

FIGURE 4.—*System of Forces.*

It has already been shown that any normal force, whether finite or infinitesimal, is resoluble into a dilation and two shears, exactly one-third of the force producing dilation, and the remainder producing two equal shears at right angles to one another. Analyzing each of the normal forces P, Q, R separately, it will appear that the action of all of them may be tabulated as two shearing stresses and a dilation—thus:

Axes of	x	y	z
Dilation...........	$\frac{1}{3}(P+Q+R)$	$\frac{1}{3}(P+Q+R)$	$\frac{1}{3}(P+Q+R)$
Shear.............	$-\frac{1}{3}(Q+R-2P)$	$\frac{1}{3}(Q+R-2P)$	0
Shear.............	0	$\frac{1}{3}(Q+P-2R)$	$-\frac{1}{3}(Q+P-2R)$
Sum..............	P	Q	R

Turning now to the couples C_1 and C_2, and supposing $C_1 > C_2$, their combination is equivalent to two equal and opposite couples, each equal to C_2, and a single unbalanced couple, $C_1 - C_2$. The combination of two equal and opposed couples is easily shown to be equivalent to a shear, the axes of which bisect the angles made by the component forces.* Here, therefore, the balanced couples are equal to a shear at 45° to P or Q.

There now remains a single unbalanced couple tending to produce rotation of the mass about $o\,z$. Unless still other external forces are introduced, this couple will merely rotate the mass without strain. If, however, one of the faces of the cube is compelled to coincide with a fixed plane having the same direction as the forces of the couple, as if the mass rested on or against an inflexible frictionless support, this couple, together with the resistance, will effect distortion and will convert the square section on the $x\,y$ plane into a rhomb with two of its sides parallel to the fixed plane. The distortion thus produced will consist merely in a tangential shifting of planes parallel to the support and will involve no change of volume. In short, the strain is shearing motion or scission.

No system of forces of constant direction and constant intensity will produce scission. The combination of a couple and an inflexible resistance is equivalent to a stress system like that of a simple shear, but which undergoes rotation relatively to the fixed axes of reference during strain. The dynamic origin of a scission thus differs essentially from that of a shear.

If a cube resting upon an inflexible support coinciding in direction with $o\,x$ were subjected to the force system of figure 4, the couple C_2 would be inoperative and the stress system would reduce to dilation, axial shears, and the rotational shearing stress which produces scission. This last may be called scissive stress.

No support is absolutely inflexible, and in real cases of supported masses the strains produced will be of a character intermediate between those produced when there is no support and when the support is ideally rigid. Such strains evidently involve both scission and a shear at 45° to the axes.

On the whole, then, the entire force system, including a resistance to rotation, produces a dilation, a shear in the $y\,z$ plane, two shears in the $x\,y$ plane, one of them at 45° to the axes, and a shearing motion in the $x\,y$ plane. The most general strain discussed in preceding pages corresponds to any combination of these strains, each of which has been treated in detail. It has also been shown that a general strain of the type here treated is resoluble into just these components.

*See an elementary proof of this proposition in Bull. Geol. Soc. Am., vol. 2, 1891, p. 55.

LINES OF UNALTERED DIRECTION.

It was shown above that, in general, three diameters of the strain ellipsoid have the same direction after strain as before strain.* It is usual to assume that these same lines retain their direction during the process of strain,† but this appears to be true only under certain limitations.

If the displacements a and b are connected by the equation $a = mb$; formula (7), which assigns the position of the lines of unchanged direction in the xy plane, becomes:

$$tan\ \chi = \frac{f-e}{2b} \pm \sqrt{m + \left(\frac{f-e}{2b}\right)^2},$$

and the position of the axes of the principal ellipse at the inception of strain is given by—

$$tan\ 2\nu_0 = \frac{(m+1)\,b}{e-f}.$$

Hence one may write—

$$tan\ \chi = -\frac{m+1}{2}\left\{cot\ 2\nu_0 \pm \sqrt{\frac{4m}{(m+1)^2} + cot^2\ 2\nu_0}\right\}.$$

In this formula ν_0 depends solely upon the direction of the external force relatively to the resistance and not upon its intensity. Consequently, if the $tan\ \chi$ is to preserve its initial values throughout the straining process, m must be constant. Now, the displacements may be such that a or b is zero throughout deformation, and m is then constantly zero or infinity. It may also happen that $a = b$, so that $m = 1$, and this case also involves no hypothesis as to a relation between stress and strain in homogeneous matter; but if m is a finite quantity differing from unity, the assumption that m is constant is equivalent to the hypothesis that the ratio of the displacements bears a constant relation to the ratio of the stress components which produce them. This hypothesis is only justifiable when the strain is very small.

When there is no rotation, or when $a = b$, the elastic cube acts as if it rested upon an inflexible support and were affected by stresses axially disposed. When one of the displacements a or b disappears, the strain involves only axial deformations and scission. This again implies the presence of an inflexible support or an equivalent rotating system of forces. Hence the lines which have the same direction after strain as

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* Two of these diameters may coincide and both of these may become imaginary.

† Thomson and Tait speak of these lines as unaltered in direction *during* the change of strain, but they may have meant *by* rather than during. Nat. Phil., section 181.

before strain will keep this direction during strain only when the mass acts as if it rested on or against an inflexible support.

If this support is parallel to $o\,x$, either $tan\ \varkappa = 0$ or:

$$tan\ \varkappa = \frac{f - e}{b} = tan\ (90^\circ + 2\ \nu_o).$$

Properties of Matter.

Viscosity.—The ideal elastic substance is one which requires a perfectly difinite stress to hold it permanently in any given state of strain at a given temperature. This stress is wholly independent of previous states of strain or rates of straining. Real substances fulfill this definition only under certain conditions, and careful experiments always show that the more rapidly deformation is produced, the greater is the resistance to be overcome. Thus a spring, suddenly stretched by a given weight, yields rapidly to a certain extent and may seem to become stationary; but careful observation shows that is continues to yield slowly to the traction for a time, though it ultimately comes to rest. If the material were ideally elastic, it would immediately assume this ultimate state of strain, and the fact that the attainment of equilibrium is gradual proves that the original resistance is a function of the rate of deformation. Fluids show similar phenomena.

Viscosity is that property in virtue of which matter presents to stress a resistance into which the rate of deformation enters as a factor. Viscosity and shear are inseparable, and mere dilation is unattended by viscous phenomena.* The coëfficient of viscosity of a substance is *ceteris paribus*, the shearing stress required to produce the unit shear in the unit time. The degree of viscosity is considered as increasing with this coëfficient, so that sealing wax and tar are more viscous than water, and steel is more viscous than lead or copper.

Substances which yield indefinitely though slowly to stresses, however small, are now known as viscous fluids. Those which in the course of time reach statical equilibrium under the action of deforming stress, such as tallow and steel, are called viscous solids.

If stress is applied very slowly (or rather infinitely slowly) viscosity does not come into play. Thus, a viscous solid or fluid in permanent

* Viscous resistance is often likened to friction. Each is a dissipative resistance to tangential motion, but there are marked differences between them. Friction exists only where there is normal pressure, and is therefore wholly absent on the planes of maximum tangential strain in a shear. Friction also has its maximum value when the surfaces between which it exists are at rest. Viscous resistance opposes relative motion of surfaces between which there is no normal pressure when the rate of motion is finite, but vanishes when this rate is infinitesimal. Thus there is rather an analogy than a similarity between viscosity and friction.

statical equilibrium acts like an ideally elastic or ideally fluid mass. Under these conditions the resistance which a solid offers to deformation is due entirely to its "rigidity," this term being defined in the theory of elasticity as the degree of resistance which a solid in permanent equilibrium opposes to stresses tending to change its shape.* Under this definition india-rubber and tallow possess rigidity as well as cast iron, but the modulus of rigidity of the metal is greater than that of the gum or the fat. In short, rigidity is an essential property of solids.

A highly viscous fluid subjected to a stress of brief duration presents great resistance to deformation. Thus, if the earth were substantially a mass of sufficiently ultra-viscous fluids, it would behave to the attractions of the sun and moon sensibly like an infinitely rigid body, because of the rapid change in the direction of these attractions. There are valid grounds, however, for the belief that the earth is really solid.

The viscosity of rocks often controls the directions in which they yield to stress. When two equal stresses acting on the same rock-mass change their directions at different rates, that stress which rotates at the smaller rate will encounter the smaller resistance and will produce the greater effect. It has been shown in the earlier part of this paper that all rotational strains are accompanied by relative tangential motion on two sets of mathematical planes which rotate relatively to the mass at different rates. The difference of their effects due to viscosity will be discussed under the head of geological applications.

Flow.—At least some solids in the so-called "state of ease" (freedom from internal partial constraint) almost completely recover their original form after small strains when time is allowed to overcome the viscosity. It is apparently true of all bodies, however, that when strained beyond a certain limit short of rupture, they are permanently deformed. The process by which this deformation is effected is termed flow, and the limit at which a substance initially in a state of ease begins to flow is called the limit of solidity. When the limit of solidity differs but little from the ultimate strength, the substance is known as brittle. When the limit of solidity is a fixed quantity, so that any excess of stress produces continuous flow, the mass is said to be plastic. When a continuously increasing stress is needful to produce continuous flow, the substance is said to be ductile, and in this case a "hardening" of the mass attends the flow, as, for example, in the manufacture of wire.

Plastic flow thus differs from ductile flow. I am not aware of any phenomena which point decisively to the existence of ductility and the attendant hardening among rock masses, but it cannot be amiss to call

*The word rigidity, as used in the theory of elasticity, has nearly the same meaning as stiffness in common parlance.

attention to this property, which possibly plays some part in the interior of the earth if not near the surface.

Plastic flow certainly plays an important part in geological mechanics. The motion of glaciers is known to be in part ascribable to it, and it is clearly evinced in the details of rock structure. At great depths below the surface a partial gradual relief of strain in any rock mass will bring to bear a gradual increase of stress difference, which may be considered entirely indefinite in amount. Granting, then, that there is no infinitely brittle rock or no rock in which the ultimate strength falls short of the limit of solidity, flow must ensue at great depths whenever a sufficient relief of strain occurs. No geologist needs to be reminded of the instances pointing to such flow. They are innumerable and most various.

If a mass capable of plastic flow is suddenly subjected to a definite load greater than it can bear without flowing, one-third of the load will immediately be employed in compression and the process of flow will produce no further modification of the volume. Flow is thus continuous shear.

The shearing process must take place along certain lines, and these must be the lines which are first strained beyond the limit of solidity. In other words, flow must take place along the lines of maximum tangential strain discussed in a former part of this paper, and which by (10) stand at an angle $90° - \omega$ to the line of a simple, direct pressure. When the load is of fixed amount, the stress will gradually diminish as the mass flattens out; so that the last lines of flow will make a smaller angle with the line of force than the earlier ones. A greater amount of flow would occur along the earliest lines affected. If the mass were of such a character as to show evidences of the relative motion after equilibrium had been reached, a cross-section of it would reveal a structure at least comparable with schistosity, the flatter lines being more pronounced than the steeper ones.

Relation of plastic Solids to Fluids.—Let S be the resistance which a plastic solid opposes to distorting stress at the elastic limit, and let n be the stress which would be required to produce the unit shear if the mass were perfectly elastic (or, in other words, the modulus of rigidity); then if stress is proportional to strain, S/n is the shearing strain which the mass experiences at the elastic limit, and any greater strain would be accompanied by flow. If the mass continues to flow as long as the stress is maintained above the fixed limit S/n, the substance is known as perfectly plastic.

If S is infinitesimal, the mass will yield to any shearing stress, however small. Such a mass, resting on a level surface, would spread out to a layer of infinitesimal thickness, much like a fluid. It does not follow, however, that because S is very small, n is also small. The rigidity of a mass seems quite independent of its elastic limit. Thus wrought iron and cast steel have nearly the same modulus of rigidity, though the elastic limit is very different for the two substances. A material, then, may have a very low elastic limit and yet oppose great resistance to deformation within that limit.

If the rigidity of a mass is great, the lines of maximum tangential strain under pressure will make angles of little more than 45° with the line of pressure. If such a mass is prevented from undergoing relative motion in these directions, a much greater force will be necessary to compel it to move in any other direction. Fancy a cube of matter of low elastic limit, but great rigidity, placed in a shallow tray just wide enough to receive it; and let a small, uniformly distributed pressure be applied to the upper surface of the cube. Then, above the edge of the tray, the mass would break down at angles of about 45°, but the laminæ standing

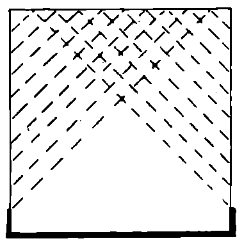

FIGURE 5.—*Plastic Solid under Pressure.*

at 45° and supported by the tray could not move sensibly. The result would be that a pyramidal mass would remain in the tray, forming an angle of 45° with the line of pressure.

This is substantially the way in which a body of solid, discrete particles would act. A cube of such material released in a tray would resolve itself into a pyramid, sloping at the angle of rest. It is also easy to show that the maximum value of this angle is 45°.* A mass of very fine powder composed of frictionless spheres would be perfectly plastic, inasmuch as it would yield to any shearing stress, however slight, which were not resisted by external constraint. The elastic limit would also be zero. Its rigidity could be displayed only when flow were prevented by constraint in the direction in which flow tends to take place. It would then evince rigidity by its ability to retain a pyramidal shape. In short, a mass resembling shot of infinite fineness appears to represent the case of a perfectly plastic solid with infinitesimal elastic limit.

*The angle of rest is, say, ρ, and $\tan \rho = R/N$, where N is the normal pressure, and R the frictional resistance due to this pressure. This resistance cannot exceed the pressure to which it is due, and R/N cannot exceed 1, the tangent of 45°.

Consider now the case in which n is very small and S great. This case also bears some resemblance to a fluid. A cube of material with these qualities would yield to the slightest pressure, and the strain ellipsoids would be flattened to infinitely thin disks. The lines of maximum tangential strain would therefore be perpendicular to the line of pressure. To convert this solid into a liquid the elastic limit and the rigidity must both disappear; but this is not of itself sufficient. The flow of a liquid takes place perpendicularly to the direction of pressure; consequently, in the solid which approaches infinitely near to the liquid state, the strain ellipsoids must be infinitely flattened before flow begins. This relation is secured if S is infinitesimal and n is an infinitesimal of the second order.

In the discussion of strains it was shown that the lines of maximum tangential strain, or the lines on which flow must take place, make an angle with $o\,x$, which has a certain value, ω. It appears from the above that this angle has a value of 45° for an infinitely rigid solid, even if this solid is perfectly plastic and has no elastic limit, so that it is reduced to molecular powder. For fluids, on the other hand, this angle is zero, and the rigidity is an infinitesimal of the second order. Intermediate values of ω answer to solids of moderate rigidity.

Rupture.—In a homogeneous mass under pressure, rupture must take place on the lines of maximum tangential strain: for rupture is strain carried to such an intensity that cohesion is overcome. A mass in which flow has preceded rupture cannot be regarded as homogeneous, since in the direction in which flow occurs the strength of the mass may be and perhaps must be weakened. In the case of pressure this makes no difference, the tendency to flow and to rupture being in the same direction.

Tensile stresses produce ruptures by a different method. One can conceive of a mass breaking up by mere dilation or without any relative tangential motion, while purely compressive forces cannot be imagined as leading to rupture. In tensile strains shears coöperate with dilation. Thus, if a bar under tension is homogeneous, the tension will be relieved by the smallest possible fracture, which is in a direction perpendicular to the axis of the bar. If, however, the bar has undergone flow along the surfaces of maximum tangential strain and has thus been sensibly weakened in these directions, it may split diagonally to the axis or irregularly along some other path of least resistance. Thus, a rubber band when suddenly stretched almost always breaks as straight across as if cut with scissors, but a bar of mild steel gradually stretched to the breaking point often splits diagonally, while a wooden bar gives a most irregular surface of fracture.

In rocks, tensile rupture and fracture by pressure can often be distinguished. Granites, and even conglomerates, often break under pressure in extraordinarily smooth, continuous, plane surfaces. Under tension the rupture of granite would follow an irregular surface of least resistance, leaving projecting crystals on each side; and in conglomerates few pebbles would be broken, nearly every one adhering either to one fragment or the other. Stratified rocks under tension would behave much like a wooden bar. Only unusually uniform rocks could give smooth surfaces of rupture under tension. Such surfaces do occur in the case of columnar eruptives, and these columns can be shown to be produced by tension in the cooling mass. Even when tension produces surfaces of rupture which are smooth, they are apt to be curved or broken. In a word, tension tears masses asunder; pressure cuts them to pieces.

Geological Applications.

Cases to be considered.—It is probable that pure dilation and pure irrotational shear are strains of rare occurrence in rock masses. One of these requires two, the other three pairs of forces acting at right angles to one another with identical intensity. Simple pressure, on the other hand, is common, especially where disturbances are not in progress. During orogenic changes inclined pressures must be frequent. The most important stress systems are therefore direct pressures and inclined pressures. The last includes two cases, in one of which the mass suffering pressure rests upon or against an unyielding support, while in the other the mass rests upon or against materials which yield readily. In the former of these cases the stress system reduces to a simple pressure, compounded with a scissive stress; in the latter to a pressure and a shearing stress.

In dealing with each strain viscosity and a tendency to flow or rupture must be considered, the aim being to relate actual phenomena to their immediate causes and to enable the geologist, in some measure at least, to judge of the local direction of the forces the effects of which he observes.

When gravity acts upon a mass homogeneous strain is, strictly speaking, impossible, excepting within infinitesimal limits of space, each level surface being subjected to greater pressure than the next above it. On the other hand, the forces involved in the deformation and fracture of rocks are very great, except in some extreme instances, such as that of moist clay. For ordinary firm rocks the ultimate strength is such that a column of from one to several thousand feet in height would be needful to produce at its base a pressure sufficient to induce rupture. Consequently, in masses of such material from a few score of feet to a few hundred feet in thickness, gravity plays but a small part compared with rupturing stress; and portions of the rock having dimensions of this order may often properly be regarded as homogeneously stressed. When large masses are similarly strained, gravity may determine in which of several directions, all equally stressed by external pressure, rupture will take place. Cases of such determination I have discussed in a former paper.*

Effects of direct Pressure.—A direct, uniformly distributed pressure of sufficient intensity, applied to an elastic brittle mass presenting great resistance to deformation, would induce fracture. The ruptures would take place along those lines subject to the greatest tangential strain, since these are the directions in which the material would first be strained beyond endurance. These lines would stand at 45° to the line of force if the mass presented infinite resistance to deformation. If this resistance is not infinite, they will stand at greater angles to the line of force. The angle which the normal to the direction of rupture makes with the line of force is called ω in the discussion of the strains (see p. 34).

There will generally be more than one direction of rupture, and in masses the thickness of which in the direction of pressure is considerably smaller than the lateral extension, there will often be four systems of parallel fissures, two systems answering to each of the two equal shears arising from simple pressure. If, however, there is any inequality of resistance in the plane perpendicular to the line of pressure, whether this is due to the character of the mass under pressure or to inequalities in the support which this mass receives from its surroundings, the strain ellipsoid will have three unequal axes, and rupture will take place only in the plane of the greatest and the least of these axes. In this very common case the mass will be divided into columns, with angles depending upon the strain. When the mass is large and the pressure is horizontal, gravity opposes the tendency of the vertical axis of the strain ellipsoid to elongate, and rupture will tend to take place by relative motion in horizontal planes, separating the rock into vertical columns. The constraint of surrounding masses may outweigh this tendency.

Something can be said of the spacing of the fissures thus formed, but this subject can be most conveniently discussed under the head of inclined pressure.

If the pressure continues after rupture has occurred, the blocks or columns will grind against one another producing slickensides, and sometimes further ruptures, of which the discussion will also be deferred.

Many rocks under the action of direct pressures rapidly applied behave approximately as highly elastic brittle masses of great rigidity, and in these cases the range of the planes of maximum strain is practically *nil.* Consequently, systems of fissures at sensibly right angles to one another

* Bull. Geol. Soc. Am., vol. 2, 1891, p. 62.

are not infrequent, nor is it very unusual to find such a pair of systems of fissures accompanied by a second similar pair in a plane at right angles to the first. The residual blocks are then bounded by from four to eight planes. In the last case four of the planes are parallel to the other four.*

In many cases the rock does not rupture without previous deformation of considerable amount. When this happens the lines of rupture make an angle of more than 45° with the line of force. The normal to the direction of rupture then makes an angle ω with the line of force, and this angle decreases with the deformation. If the deformation were very great, as it would be with a mass of india-rubber, ω would approach zero. If the direct pressure were relieved by rupture and the rock were perfectly elastic, the residual fragments would recover their original shape, and their acute angles would then lie in the line of force.

Thus when rocks show fissures cutting one another at acute angles it is certain that finite deformation has taken place. If the mass has remained under tension, the line of force when direct bisects the obtuse angles. If the mass has been relieved of pressure and the rocks have acted as elastic masses, the line of force bisects the acute angles.

It is usually possible from general conditions to judge which of two rectangular directions is the more probably that from which a rupturing force has acted. I have, however, never yet met an instance in which it seemed to me that the line of force bisected the acute angles of fissure systems. Orogenic forces are commonly very persistent, and even if a mass behaved as substantially elastic up to the moment of rupture, it is improbable that the residual blocks would continue capable of regaining their original shape after the lapse of, say, even a few years. In many cases it is quite clear that deformation has become permanent. Thus I have examined very numerous pebbles in conglomerates, some of which had been much flattened by pressure and others also much fractured. The direction of flattening was then a certain indication of the direction of force, and this direction bisected the obtuse angles between the fissure systems intersecting the pebbles. In other cases the character of slickensides and accompanying faults shows that no reversal of motion has taken place, and that the residual masses must have lost the elasticity which they seem to have exhibited up to the moment of rupture.

Observations on artificial structures seem to confirm this opinion. It has been pointed out by Mr. Clarence King and others that slabs of marble supported at their ends or corners gradually sag toward the center. So, too, in old buildings, such as the Alhambra, I have seen slabs of rock very much bent by end pressures acting for hundreds of

* When a rock fragment is bounded by planes with more than four differently directed normals, it must have undergone successive ruptures.

years. This does not imply that there is no true elastic limit, but only that it is lower than brief laboratory experiments would lead one to suppose. Were there no elastic limit, it seems to me that we should find, for example, quartz crystals in vugs among the more ancient rocks sensibly distorted by their own weight.

Usually then the line of a simple, direct pressure which has produced two or four systems of fractures in large rock masses, or in the pebbles of conglomerates, will be found to bisect the obtuse angles between the fissures as the mass now stands. In any case, where it is suspected that the line of force bisects the acute angles between fissures, the slickensides should be minutely examined to ascertain whether they show reversal of motion, and all the attendant phenomena should be investigated.

When a simple pressure on a rock mass increases very gradually, it will for some period exceed the elastic limit of the rock and fall short of the ultimate strength. Flow must then take place. The only feature of this flow which will reveal itself to observation will be the relative movements of adjoining particles. Hence, although the path in space of each particle will be hyperbolic,* the evidence of movement will indicate relative transfer of adjoining particles in opposite directions along lines of maximum tangential strain. The energy of this relative movement will evidently increase with the excess of the pressure above the limit at which flow begins, sometimes called the limit of solidity.

Thus, if one supposes the pressure suddenly to surpass the limit of solidity and then to be kept constant, the mechanical effects of the relative motion (and the chemical effects attending the expenditure of energy) will be very pronounced on the lines on which flow begins. As the process continues and the stress diminishes with the increase of the area of the mass, the lines first affected will make an increasing angle with the line of force, while the new fibers of the material which are forced into the direction of maximum strain will be less and less affected.

The result will at least resemble schistose structure and will be marked by the presence of lines of relative movement intersecting one another at very acute angles. In the case of direct uniform pressure there will be four such sets, each set at a large angle to all the others.

If the load were to increase in the same proportion as the area of the loaded mass, so that the stress would be kept uniform, an indefinite amount of flow might be produced, provided that the rock is not hardened

* During flow there is no progressive change of volume. Hence, a point for which at the inception of flow $x = 1$, $y = 1$, will be moved to a point x', y', and $x'^2 y' = 1$. The curves of this form are sometimes called the lines of flow. They would be more aptly called lines of *absolute* movement. They should carefully be discriminated from the lines of *relative* movement, which are straight. The latter are the only ones of which the deformed mass can give direct evidence. In the case of simple shear the lines of absolute movement are simple hyperbolas asymptotic to the axes.

like drawn wire. If the flow were very great (literally infinite) the lines along which relative movement took place at the inception of strain would become horizontal. The schistose partings would then in each set range through the angle ω.

Relative motion, in a mass subjected to direct uniformly distributed pressure, can only take place perpendicularly to the line of pressure when the strain ellipsoids are infinitely thin discs or when the rigidity is zero. In other words, only liquids, viscous or otherwise, can act in this manner. The behavior of semi-fluid material, like wet clay, approximates closely to that of a viscous fluid.

Rigid Disc in resisting Medium.—The behavior of an elastic mass under simple pressure leads to an extremely simple method of proving a dynamical proposition of much importance to geologists. A simple pressure acting against a resistance converts any sphere of unstrained matter into an oblate ellipsoid of revolution, the minor axis of which is in the direction of the pressure. If the constant pressure were to exceed the constant resistance, the mass would move in the direction of the pressure and of the minor axis of the oblate ellipsoid. Now, it is a well-known fact that the whole or any portion of an elastic mass which is in equilibrium, whether at rest or in motion, may be supposed to become infinitely rigid without disturbing the equilibrium. This is an almost self-evident proposition, for a mass is in equilibrium only when there is no influence tending to change its form, and it therefore makes no difference whether this form is capable of change or not. Hence in the present case the strain ellipsoid may be supposed replaced by a rigid mass. Consequently a rigid ellipsoid of revolution moving under the influence of a pressure against a resistance will be in equilibrium when it opposes its greatest surface to the resistance.

Similarly an elastic sphere under tension becomes a prolate ellipsoid, and consequently a rigid prolate ellipsoid moving under the influence of tension against resistance will be in equilibrium when its longest axis coincides in direction with the tension.

If a cube were circumscribed about either of these spheres, with four of its edges in the direction of the force, it would become a rectangular parallelopiped with sides parallel to the axes of the ellipsoid. Any plate or rod may be made up of a single layer or row of such flattened or elongated cubes. Hence any rigid disc or rod moving against a resistance under the influence of pressure will be in equilibrium when its smallest dimension is in the direction of pressure. If it moves under the influence of traction, its longest axis will fall into the line of traction.

If a flattened pebble is dropped into a running stream, the water will exert a pressure upon the stone until its inertia is overcome, and during this time the pebble will tend to swing across the current so as to present its greatest area to the pressure. As soon as the resistance due to its inertia is overcome, the pebble will sink through the water as if the fluid were at rest till its edge touches the bottom, and it will then tip down stream till it meets support. In rapid steams irregularities in the bottom cause local upward currents, which project pebbles into the main current much as if they had been dropped into it. These pebbles sink to the bottom again where the movement of the water is more uniform. Many pebbles thus deposited will, with few exceptions, be inclined down stream and will rest against one another, like overlapping tiles.

This relation explains the fact that both in modern streams and in the ancient river channels containing the auriferous gravels, many of which have been tilted since their deposition, the pebbles, as miners say, "shingle up stream," or, as zoologists would express it, "imbricate" toward the source. Elongated, rod-like pebbles are usually found lying across the channel. The indication afforded by this behavior of pebbles seems entirely trustworthy so far as the local current is concerned. In applying it, however, it must be remembered that powerful streams are often accompanied near shore or close to obstructions by local "back currents," in which the pebbles would be arranged in a direction opposite to that of the main stream.

If a flat pebble or a mica scale is allowed to subside in relatively quiet water, the fluid may be considered as exerting a pressure on the lower side against a resistance due to the action of gravity on the stone. The disc will then tend to assume a horizontal position. It is for this reason that allothigenetic mica scales in sandstones or other rocks usually follow the direction of the bedding. In massive sandstones this is an assistance in determining the true stratification.

A very familiar illustration of the action of the strain ellipsoid moving against resistance is afforded by a bubble of gas rising through still water. The spherical bubble is compressed to an ellipsoid, which might be replaced by a rigid mass of the same density, and it rises with its equator in an almost perfectly horizontal plane.

On beaches pebbles are sometimes imbricated for a few feet in one or another direction and sometimes lie nearly flat. The constant reversal of the currents due to breaking and retreating waves prevents any extensive methodical arrangement, and this fact is of assistance in discriminating marine gravels from river deposits.

There are also instances of the almost self-evident fact that a rod-like mass moving under the influence of traction, like a vessel under tow, will move end on. In glassy rocks, such as many rhyolites and andesites, the mass often shows a banded structure, marked by the presence of microlites, most of which are parallel to the banding. These microlites are no doubt of greater density than the glass, but, on account of the viscosity of the melted glass and the enormous surface per unit volume which the microscopic prisms expose, they cannot be supposed to attain their actual arrangement as a result of gravity like the mica scales in sandstone. On the other hand, if one supposes an irregular orientation of the microlites in the glass, and that tangential motion has been set up between adjacent layers of the viscous mass, every microlite standing across the direction of relative motion would be swung into the line of relative motion by the opposite traction exerted on its two ends by the moving layers. It appears to me, therefore, that "rhyolitic structure" indicates "shearing motion" or, as I have called it, *scission* in the direction of the banding.*

Inclined Pressure and yielding Medium.—An inclined pressure acting on a tabular mass of rock is equivalent to a direct pressure and a tangential force. This last, with the resistance necessary to keep the center of inertia of the rock at rest, forms a couple. If the rock is surrounded by masses of comparatively feeble resistance, it will then rotate until the couple is exactly balanced by the resistance to rotation. The rock is thus subjected to the action of a simple pressure and two balanced couples; constituting a simple shear, neither of the axes of which coincides with the line of pressure.

As has been shown above, the strain produced by a pressure and a shearing stress is a rotational one, the amount of rotation, however, being small as compared with that involved in some other strains. One of the directions of maximum tangential strain will therefore sweep over a greater range of material particles than the other, or will affect a given set of particles for a shorter time. That set of planes of maximum strain which shifts its position more rapidly will encounter greater resistance from viscosity and will produce the smaller effect.

If the mass is strained beyond the elastic limit, but not to the point of rupture, a schistose structure will result, but one set of schistose partings will be confined to a somewhat smaller angle than the other and the more pronounced partings will be associated with the smaller angle.

If the pressure is intense enough to produce rupture, fracture will take place chiefly along the partings which have the smaller range.

The axes of the strain ellipsoid will bisect the angles which the last schistose partings make with one another, and the minor axis of the

* The above discussion is incomplete. A full treatment would of course assign a definite value to the couple which resists the tilting of a disc moving in a fluid. The reader will find the subject more fully developed in Thomson and Tait, Nat. Phil., sections 320–325, with interesting instances. That discussion is decidedly difficult, while the main points in which geologists are interested seem to be adequately demonstrated by the exceedingly elementary method here presented.

strain ellipsoid, or the direction of maximum compression, will lie between the line of pressure and the compressive axis of the additional shear.

When the rock is ruptured without sensible deformation the strain under discussion will not be rotational and will be indistinguishable from that which would result from a simple shear; for in the $x\,y$ plane one of the shears arising from direct pressure coöperates with the shear resulting from the preliminary rotation, and their combined effect will be greater than that of the second shear in the $y\,z$ plane arising from the direct pressure component.

The character of the finite strain is best seen by an illustration such as that in the diagram, figure 6, *A*.

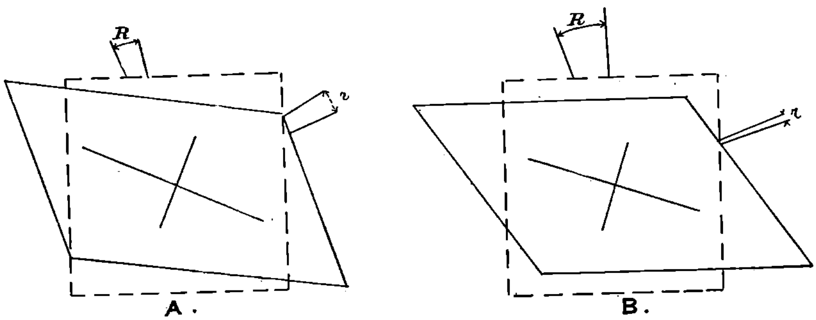

FIGURE 6.—*Strained Cubes.*

The dotted squares are strained to the rhombs, drawn with full lines. *A* results from two shears at 45° to one another, the ratio of each shear being 5 / 4. *B* results from a shear and a scission, the ratio of each of the two shears involved being also 5 / 4. The central crosses mark the direction of the ellipse axes. The angle *R* is the material angle through which one set of planes of maximum tangential strain sweeps, and *r* is the other corresponding angle. In *A*, $R - r = \nu - \mu = 2° 45'$. In *B*, $R - r = 15° 21'$.

Inclined Pressure and unyielding Resistance.—When a tabular mass of rock subjected to inclined pressure rests against a mass which does not yield considerably, the free couple which results from the tangential component of the pressure and the resistance of the supporting mass can only be equilibrated by strain in the rock itself. The strain will therefore include as one component a shearing motion or scission.

This strain is rotational, the angle of rotation being far greater in this case than in that of a yielding support. The rotation is here of the same order as the strain. Consequently one set of planes of maximum tangential strain will sweep through the mass much more rapidly than the other, and the difference in their action will be very pronounced. The nature of the distortion is seen in figure 6, *B*.

Partial Theory of the Spacing of Fissures.—When a slab of rock resting broadside against an inflexible support ruptures under the influence of a pressure inclined to the supporting plane, it is easy to see that the pressure can be relieved only by several cracks, which must divide the mass into sheets bounded by planes of maximum tangential strain. Such a division is extremely common on a large scale in granite and other relatively homogeneous masses; on a small scale it is frequent in the pebbles of conglomerates which have been subjected to pressure. It is therefore very desirable to ascertain the conditions which determine the thickness of the sheets.

A slab of rock must evidently rupture in such a manner as to relieve the pressure upon it, and this relief must be accompanied by a readjustment of the fragments. This consideration at once assigns a superior limit to the spacing of the cracks. Suppose, for example, that in a case illustrated by the following diagram cracks were to form only at *a* and *c*, then, since a perpendicular from the upper end of *a* falls between *a* and *c*,

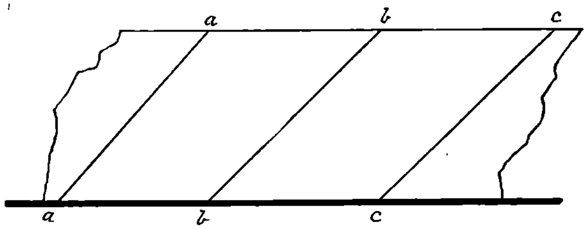

FIGURE 7.—*Widest possible Spacing of Fissures.*

the fragment *a a c c* cannot rotate without increasing its vertical dimension, and the pressure cannot be in any way relieved by the ruptures. But if a third crack, *b b*, is so placed that its lower end is perpendicularly below the upper end of *a*, the fragment *a a b b* can be rotated so as to decrease the vertical dimension, and thus to relieve pressure. Hence the cracks must be at least so near to one another that the terminations of adjacent cracks are in vertical lines, and the higher the angle which the cracks make with the fixed plane, the nearer must they be to one another. This, however, is an extreme case; for an infinitesimal rotation of the vertical line *a b* about any point of it would not diminish the thickness of the mass. The actual distance between fissures must therefore be less than that assigned by this limit.

When the process of straining is so slow that the mass can fully adjust itself at each instant to the external forces (an important limitation) it seems impossible to avoid the conclusion that the actual spacing will be such as to depotentialize the greatest possible amount of energy for a given length of fissure. In other words, the cracks will be so disposed as to "do the most good." If so, the spacing can be determined if one can succeed in expressing in exact terms the depotentialization of energy per unit length of crack.

The lines of maximum tangential strain make angles ω with the major axis of the strain ellipse. When the strain is due to a pressure at a positive, acute angle φ with the fixed plane (parallel to $o\,x$), the major axis makes an acute negative angle ν with $o\,x$. That set of planes of maximum tangential strain which have the smaller range during the process of strain, and upon which there is the least viscous resistance, then makes an angle $\omega + \nu$ with $o\,x$.

Let w be the thickness of one of the sheets into which rupture may divide the slab of rock, and let ϑ be the angle which the diagonal between the obtuse angles of the sheet makes with $o\,x$, as indicated in the following diagram, figure 8. Then, if the thickness of the slab at the moment of rupture is $2\,l$, a little consideration shows that—

$$w = 2\,l \cos(\nu + \omega) \left\{ 1 - \tan(\nu + \omega) \cot \vartheta \right\}.$$

It is evident that the total length of the cracks is inversely proportional to the thickness of one sheet or to w.

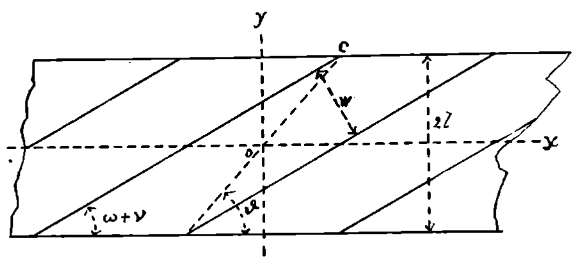

FIGURE 8.—*Spacing of Fissures.*

To determine the relief of pressure it is convenient to begin by considering a mere change of strain. Suppose a slab to be in equilibrium under the action of a simple, direct pressure. Let the mass now undergo a small change in physical properties, such that it yields by a small additional amount to the pressure. Then the potential energy of the system is diminished in proportion to the amount of this secondary yielding, measured in the direction of the force. Only one fiber passing through the center of the mass, however, will move solely in the same direction in which the pressure acts, all other particles moving on hyperbolic lines.

If the pressure were inclined to the surface at an angle φ, the depotentialization of energy under similar circumstances would also be measured by the movement of particles in the direction of the pressure, irrespective of the movements which they execute in other directions.

The rupturing of the slab into sheets may be regarded as a change in its physical properties such as is contemplated above.

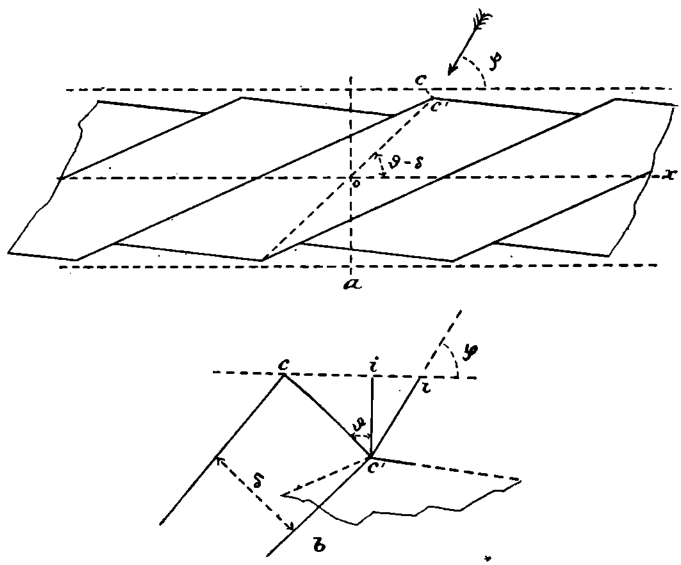

FIGURE 9.

Cut *a* shows the same mass as figure 8, the sheets now being rotated through a small angle δ by the pressure acting in the direction ϕ. Cut *b* represents the corner c' of one sheet on an enlarged scale, together with the original position of this corner at c.

The line $c' r$ is the distance measured in the direction of the force through which the point c has moved in passing from c to c', so that $c' r$ is proportional to the depotentialization of energy. The angle of rotation being small and arbitrary, say, δ, the angle $cc' i = \vartheta$ and—

$$(c' r) = (cc') \frac{\cos \vartheta}{\sin \varphi}.$$

Then, too, $(cc') = (oc)\, \delta = l\, \delta / \sin \vartheta$, so that the relief of pressure varies with the line—

$$(c' r) = \frac{l\, \delta}{\sin \varphi} \cot \vartheta.$$

The relief of pressure per unit length of crack therefore varies with—

$$w\, (c' r) = \frac{4\, l^2\, \delta}{\sin \varphi} \cos (\nu + \omega) \left\{ 1 - \frac{\tan (\nu + \omega)}{\tan \vartheta} \right\} \cot \vartheta;$$

and to find the distribution of fissures which will ensure the greatest depotentialization of energy per unit length of crack, it is only necessary to determine the value of ϑ, which will give the last expression a maximum value. This problem leads at once to—

$$2 \tan(\nu + \omega) = \tan \vartheta,$$

or—

$$w = l \cos(\nu + \omega),$$

a result of very convenient simplicity.

Thus far it has been assumed that only one set of fissures forms in the slab. This case is of frequent occurrence in rocks, but it is not much more frequent than the appearance of two sets of fissures forming large angles to one another. Under conditions such that viscosity does not essentially modify the results, there should be as great a tendency to form fissures on the other set of planes of maximum tangential strain, making an angle $\omega - \nu$ with $o\,x$, as on those planes discussed above.

When two sets of fissures form at large angles to one another they must seemingly develop simultaneously; for, if a single set of sheets were to form first, and secondary fissures were to be produced by the grinding of the sheets against one another, it is easy to see that the secondary fissures would make but a small angle with the primary divisions, and there would be more evidence of movement on the first fissures than on the subsidiary cracks. These cracks would also not in general pass from one primary sheet to the next.

When the two sets of joints form simultaneously, each set must form under similar conditions, and I can think of no reason to suppose that they do not form independently. Hence the theory already developed seems to apply also to the second set of fissures, the only change needful being the substitution of $\omega - \nu$ for $\omega + \nu$.

The results derivable from this theory of division certainly accord with some well known facts. Thus, if a tough mass is acted upon by a shearing tool, it is a matter of daily experience that the mass undergoes a single cut. For this case viscosity comes into play, and by the theory only one set of fissures will be developed, $\varphi = 0$, $\omega + \nu = 0$ and $w = l$, which means that only one fissure will intersect the mass. Again, if one attempts to cut a brittle substance like glass with a shearing engine, the mass, according to experience, shatters instead of simply dividing. By the theory as applied to this case the elastic deformation is extremely small, neither set of planes of maximum tangential strain has a sensible range, and $\omega + \nu = 0$, while $\omega - \nu = 90°$, nearly. Hence only a single fissure will tend to form in the direction $\omega + \nu$, but the mass will be divided into scales of almost infinitesimal thickness in the direction $\omega - \nu$. In other words, the theory substantially accounts for the facts.

In theory as in practice, only masses capable of considerable deformation under the system of external stresses can be divided by a single clean cut.

This conclusion seems to throw some light upon a general feature of geological fractures. In the laboratory rocks are very brittle substances, and every geologist has experienced a feeling of surprise that in natural rock-exposures clean cuts in a single direction are so frequent. It now appears that cuts of this description can occur only when the stresses are such as to produce a considerable elastic or plastic deformation of the mass. There is, of course, abundant other evidence that such stress systems really accompany orogenic movements.

Examples of inclined Pressure.—According to a famous theory developed by Navier and Poisson the ideal isotropic solid is characterized by the property that in a simple elongation of small amount the linear lateral contraction is just one-fourth of the increment of length. Though most elasticians refuse to acknowledge the theoretical basis of this conclusion (viz, that the action between two molecules is reducible to a single force acting between the centers of mass), there is no doubt that several substances, and especially glass, behave sensibly as this theory demands. As some rocks are glasses, it is certainly legitimate to assume, for the purpose of illustrating the theory of rupture developed above, that the relation $1/4$ holds true.*

Let a pressure F act upon a slab of a rock fulfilling Poisson's ideal at an angle of 30° and let the mass rest against a rigid support. Then if U and Q are the horizontal and vertical components—

$$U = -F\cos 30° = -\frac{F\sqrt{3}}{2};\ Q = -F\sin 30° = -\frac{F}{2}.$$

If also n is the modulus of rigidity, it is easy to show and is well known † that—

$$e = g = -\frac{Q}{10n};\ f = \frac{4Q}{10n} = -4e;\ b = \frac{U}{n}.$$

* *Possible Test of Poisson's Hypothesis.*—One of the obstacles attending the discussion of Poisson's solid and the question whether or not the coëfficients of rigidity and compressibility for isotropic solids are independent consists in the fact that it is difficult to determine Young's modulus and the modulus of rigidity for the same body with sufficient accuracy to justify theoretical conclusions. There seems to be a method of direct comparison which would test the question if the experimental difficulties should not prove too great. If M is Young's modulus,

$$f = -F\sin\phi\,/\,M \text{ and } b = -F\cos\phi\,/\,n,$$

or—

$$\frac{b\tan\phi}{f} = \frac{M}{n}.$$

If, then, a testing machine were so adapted as to produce a pressure at 45° to a stationary plane, the deformation of a cube subjected to its action would give b/f and M/n. If Poisson's hypothesis is verified, $M/n = 10/4$.

† Compare Thomson and Tait, Nat. Phil., section 683. For Poisson's solid $3k = 5n$.

Thus one may express e, f and g in terms of b—

$$e = g = \frac{-b}{10\sqrt{3}}; \; f = \frac{4b}{10\sqrt{3}}.$$

The line of unaltered direction is then given by—

$$\tan \varkappa = \frac{f - e}{b} = \frac{1}{2\sqrt{3}} = \tan 16^\circ \, 4'.$$

In the case of finite strain it has been shown that this angle preserves its initial value unchanged: so that if $b = -1, f - e = -0.2887$. Since also $e = -f/4$, the following is a consistent set of displacements for $\varphi = 30^\circ$:

$$b = -1; \; e = 0.0577; \; f = -0.2308; \; g = e.$$

Knowing the displacements, the corresponding values of ν and ω may be determined as has been shown in the earlier part of this paper. If these angles only are required they may be found from the following formulas. For any value of b when Poisson's ratio obtains—

$$\tan \omega = \frac{\sqrt{(2 - 3e)^2 + b^2} - \sqrt{25e^2 + b^2}}{2\sqrt[3]{(1 + e)^2 (1 - 4e)}};$$

$$\tan 2\nu = -\frac{2b(1 - 4e)}{(1 - 4e)^2 - (1 + e)^2 - b^2}.$$

For the present displacements the formulas give—

$$\omega = 28^\circ \, 43'; \; \nu = -22^\circ \, 37'.$$

For the spacing of the two possible sets of fissures, therefore—

$$w = l \cos(\omega \pm \nu) = (1 - 4e) \cos(\omega \pm \nu),$$

which for this set of displacements gives 0.765 and 0.481.

Supposing that both systems of fissures form, the following diagram (figure 10) shows their disposition at the moment of rupture.*

* *Example for* $\phi = 60^\circ$.—It may be of interest to some readers to give a second example of a similar strain. In the diagram, figure 11, the force is supposed to act at 60° to the plane of support. If, also, $b = -\frac{1}{2}$, the following values result: $e = 0.0866$; $f = -0.3464$; $g = e$; $\varpi = 36^\circ \, 35'$; $\omega = 32^\circ \, 4'$; $\nu = -16^\circ \, 32'$; $\mu = -32^\circ \, 34'$; $h = 0.9172$; $A = 1.236$; $B = 0.575$; $C = 1.087$; $D = od = 0.869$; $w =$ 0.630 or 0.432. The range for one set of planes of maximum tangential strain is $0^\circ \, 24'$, and for the other set $16^\circ \, 26'$.

It is apparent from the formulas that e and b fully determine ω, ν and ϖ. It is also true that if the two values of w and the angles between the

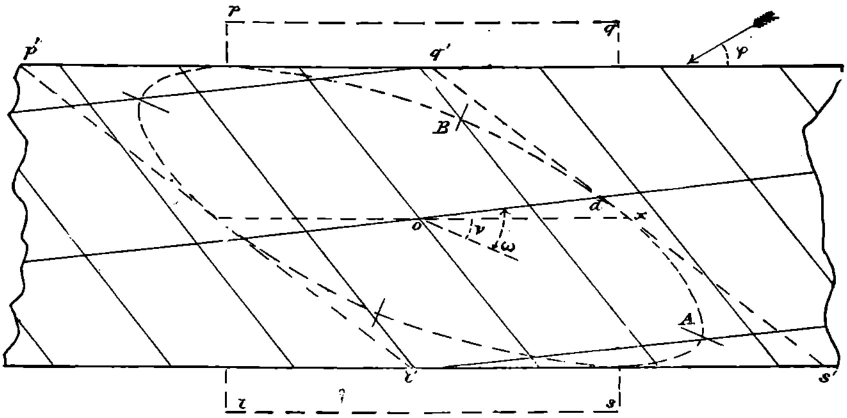

FIGURE 10.—*Results of Rupture by Pressure at 30° to fixed Plane.*

It is assumed that $f = -4e$ and that $b = -1$. Thus $e = 0.0577 = g$ and $f = -0.2309$; $\varpi = 31°\,20'$; $\omega = 28°\,42'$; $\nu = -22°\,37'$; $\mu = -51°\,19'$; $h = 0.951$; $A = 1.562$; $B = 0.521$; $C = 1.058$; $D = 0.926$; $w = 0.765$ or 0.481. The range for one set of planes of maximum tangential strain is 0° 41′ and for the other 28°.

cracks (or 2ω) were known by observation, the displacements and the value of φ could be deduced. The two values of w would in such a case enable one to find ν and e, while, when these quantities are ascertained,

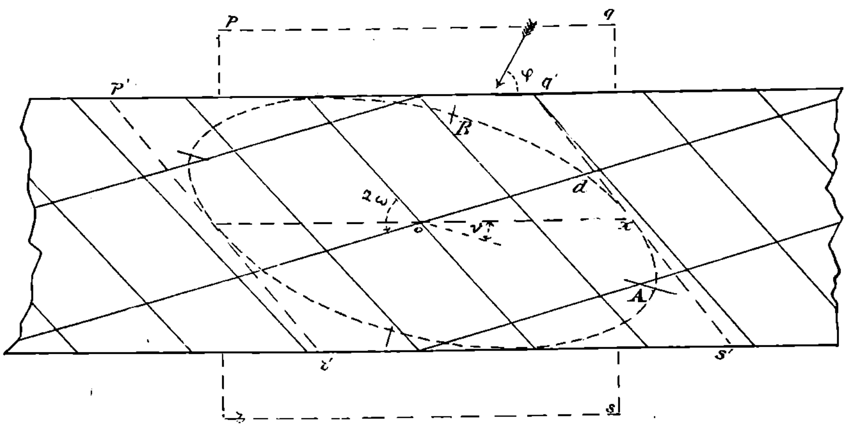

FIGURE 11.—*Results of Rupture by Pressure at 60° to fixed Plane.*

the formula for $\tan 2\nu$ will give b. Finally $\tan \varphi = b/10\,e$. But to determine the direction of the force in this manner it must be shown that the mass was rigidly supported and that it was subject to no lateral constraint. Thus great caution must be exercised in applying this or any similar method to geological occurrences.*

Distortion on Planes of maximum Strain.—It has already been pointed out that the planes of maximum strain are not in general undistorted planes. Consider one of the planes making an angle $\omega + \nu$ with the fixed plane and inquire what ellipse on this plane would answer to a circle of unit radius in the unstrained solid. Prior to strain this material plane made an angle ϖ with that radius of the sphere which ultimately forms the minor axis of the ellipsoid. This radius also originally made an angle $-\mu$ with $o\,x$. Hence it is easily seen that the original angle of the plane to $o\,x$ is $90° - (\varpi - \mu)$.

If y is the original position of the extremity of the unit radius drawn on the intersection of this plane with that of $x\,y$—

$$y = \cos\,(\varpi - \mu).$$

If y' is the corresponding ordinate after strain—

$$y' = (1 + f)\cos\,(\varpi - \mu).$$

If the altered length of this radius is denoted by D, its value is given by—

$$D = y' / \sin\,(\omega + \nu) = \frac{(1+f)\cos\,(\varpi - \mu)}{\sin\,(\omega + \nu)}.$$

It is easy to see that D is in general less than unity. Were the strain plane and undilational, D would be unity, and this case is realized in simple scission (or shearing motion), which may be tolerably frequent among rocks. For a simple shear D would also be unity, but this is a strain probably seldom realized. Whenever a compressive strain is accompanied by two shears the radius in question undergoes contraction and is less than unity.

On the other hand, the unit radius parallel to $o\,z$ is elongated to $1 + g = 1 + e = C$ by an inclined force.

Thus the ellipse on a plane making an angle $\omega + \nu$ with $o\,x$, whose major axis is C and whose minor axis is D, corresponds to a circle in the original mass. The strain involves a compression in the direction of $\omega + \nu$ and an elongation in the direction $o\,z$.

* *Lateral Constraint.*—If a mass were not only supported on a rigid foundation but confined by rigid walls perpendicular to the fixed foundation and parallel to the horizontal component of the force, the strain is also easily calculated on Poisson's hypothesis. Evidently $g = 0$, and it is easy to see that $f = -3e$. Of course, b retains the same value as if there were no lateral constraint. I am not aware that any particularly interesting results arise in this case, which differs from plain undilational strain only in the fact that there is cubical compression. It applies to Mr Sharpe's theory of slaty cleavage.

Various Results of Strain.—In the foregoing pages a theory of slow rupture has been presented, which will be supplemented a little later by considering the possible effect of vibration in those cases in which the rupture is sudden. Observation seems to indicate that many rocks have been strained to the breaking point so gradually that the theory developed is applicable.*

In applying the results reached to the elucidation of geological phenomena the physical character of the rock must be carefully considered. Some rocks when strained with moderate rapidity approach in behavior the ideal, elastic, brittle, non-viscous solid. In such cases an inclined pressure will produce two systems of cracks such as those illustrated in figures 10 and 11. If the mass is held in the strained state, so that the fragments have no opportunity to recoil, the direction of the force may then be inferred approximately by mere inspection. The plane in which it lies will be perpendicular to the systems of fissures. Its direction will intersect the obtuse angle made by the fissures, and it will make a smaller angle with the short side of the parallelogram of cracks bounding a column of the rock than with the long side. The direction of the force can be calculated exactly from the lengths of the sides of the parallelogram and the angle between them, if Poisson's hypothesis is assumed.

If the rock is viscous but not plastic (or if it is strained under such conditions as to bring the viscosity into play, but not to keep the rock for a considerable time in a state of strain exceeding the elastic limit and falling short of the ultimate strength), the effect of the viscosity on the long sides of the parallelogram will be far greater than on the short sides, because of the difference in range of the two planes of maximum tangential strain. Hence fissures will form only in the directions of the short sides of the parallelogram and the rock will be divided into sheets.

If the conditions are such as to develop both the viscosity and the plasticity of the rock-mass, flow will tend to take place parallel to the short side of the parallelogram because of the inferior viscous resistance. If the plasticity is sufficiently great, the strain will not manifest itself as rupture in this direction, but merely as plastic deformation. If the plasticity is not sufficient to prevent all rupture, it will at least diminish the amount of rupture needful to relieve the strain, and there will be mingled effects of deformation and rupture.

These mingled effects might consist either in a wider spacing of this set of fissures or in the distribution of short cracks through the mass. Of these the latter seems the more probable. The area corresponding to

*Striking instances of the rupture of cast iron blocks are depicted in the frontispiece of Todhunter's History of Elasticity. In a general way they accord with the theory developed in the text; but the blocks employed were too slender to give the full system of fissures demanded by the theory for slabs of moderate thickness and great area.

one parallelogram of the figures must receive a certain amount of relief, and if this is not entirely accomplished by flow it must be completed by rupture; but a rupture at each end of the parallelogram would relieve the strain without the help of flow. Thus it appears most logical to suppose that in such cases short "close joints" will be distributed through the plastically deformed mass, excessively minute variations in the resistance of the material determining their precise disposition.

The set of planes corresponding to the long side of the parallelogram cannot behave in the same way as those already discussed. This set sweeps through the mass so rapidly that there is no time for flow of considerable amount to take place. Hence, if they receive expression at all, it must be as sharply cut fissures or as "master joints."

Theory of Slaty Cleavage.—In considering what properties would be exhibited by a plastic, viscous rock which had been rigidly supported and subjected to a pressure inclined at a moderate angle to the plane of support, it is difficult to see how the mass would differ from true slate. The relative tangential motion along the set of planes which eventually makes an angle $\omega + \nu$ with the plane of support would inevitably manifest itself as a cleavage, alternating in some cases with close jointing. In the direction of $o\,z$, or perpendicular to the plane of the figures 10 and 11, this cleavage would be invariable. In the direction $\omega + \nu$ the cleavage would be confined to a very small angle, less than one degree in the examples given above. Thus the mass would cleave very sharply along lines parallel to $o\,z$, less sharply along $\omega + \nu$. Expansion would take place parallel to $o\,z$, while contraction would take place in the direction $\omega + \nu$. This contraction might be accompanied by a puckering of the cleavage surfaces, because the cleavage planes formed at the inception of strain would be still further contracted as strain progressed. The amount of relative distortion in the directions $o\,z$ and $\omega + \nu$ would vary with the direction of the force and the intensity of the strain. The only case in which there would be no distortion on the cleavage plane occurs when the force is parallel to the fixed support. All of these peculiarities of this strain are characteristic of slate, and they seem to cover all of the principal properties of that much-debated rock. I shall return to the comparison of properties in a later portion of this paper.

Influence of Shock.—Although the preceding discussion shows how sheets of rock may be produced by the action of orogenic forces, I am not satisfied that all fractures are produced in this way. There seem, in fact, to be instances in which the spacing of more or less nearly rectangular fissure systems is closer than can be accounted for on the assumption that the fissuring is of minimum amount.

If pressure is applied so rapidly that a considerable shock attends rupture, a corresponding quantity of energy will remain in the fragmental mass in the form of vibrations. These vibrations will take place along the lines of unaltered direction, making an angle $\varkappa$ with $o\,x$. In the extreme case of scission this direction is also that of the lines of maximum tangential strain. In every other case the vibrations will occur at an acute angle to the planes of maximum strain, and in no instance will they be perpendicular to these planes.

At the instant when the rupture takes place the whole mass is strained, in one or more directions, to the limit of endurance. Rupture and the inception of waves of compression are simultaneous, and these waves are propagated from the surfaces of primary rupture, but not perpendicularly to them. The waves must interfere and, where they intensify one another, there must be resultant shearing couples in the direction of the planes of maximum tangential strain. These waves must be propagated at the same rate that relief of pressure takes place, a rate dependent upon the properties of the mass. If, then, the waves are of considerable amplitude, it appears to me that on those surfaces at which they reinforce one another, they must intensify the strain beyond the limit of endurance.

Thus there seems sufficient reason to believe that a pressure very rapidly applied, producing primary ruptures attended by shock, will be immediately followed by secondary ruptures in the same direction as the others at intervals dependent upon the wave length of the impulse. In much the same way a high explosive shatters a rock far more than black powder.

A phenomenon of which no explanation has been offered in this paper is that of thick slates and of those flags which are to be considered as very thick slates. These, though cleavable to a certain thinness, are not capable of further splitting. Such rocks indicate a flow which is not uniformly distributed through the mass, but, on the contrary, passes through maxima at intervals corresponding to the thickness of a slate or flag. It is possible that at the inception of strain such masses were in a state of tremor so intense that the interference of waves determined surfaces along which flow began. These surfaces would be weakened by the flow, and further strain would be distributed among them rather than over the intervening solid sheets. Effects of a similar kind are produced on a pile of sheets of paper, such as "library slips," resting on an inclined, cloth-covered table which is jarred by rapid blows.

The question would seem to be one of the direction and intensity of the vibrations rather than of their existence. The tendency of rectilinear motion to pass over into molar vibrations of rapidly decreasing period is so strong as to make it most improbable that such a distortion as is involved in the formation of slate should ever be unattended by vibrations of sensible amplitude.*

Though this hypothesis of thick slates seems probable enough, I am not able to offer a detailed explanation of the process or to show under what conditions thick slates would form rather than thin ones.

Secondary Action on ruptured Rock.—It often happens that the pressure which causes such systems of fissures as have been treated above is not fully relieved by these ruptures and the small relative movements inseparable from rupture. If pressure continues on the divided mass the results will vary with circumstances. When the pressure is oblique and the mass is divided only into sheets, faults of considerable throw may take place. I have previously discussed the distribution of motion in such a system.† It appears from the present investigation that a simple fault arises from pure scissive stress, while distributed faults are due to pressure combined with scissive stress, or, in other words, to oblique pressure. Thus a distributed fault is much the more general case and, in my observation, much the more common. A solitary fault is an extreme and very special case of a distributed fault.

At the instant of rupture there is very little normal pressure between the sheets or blocks of rock, but as rotation progresses the pressure tends to increase. Extensive movements are therefore accompanied by a forcible grinding of the fragments against one another. At first the stresses called into play are but little inclined to the surfaces, and the result is often to produce slaty structure close to the surfaces of the fragments. I have observed very many cases in which large blocks of granite showed slaty cleavage close to the bounding surfaces which had evidently been produced in this way. The cleavage was certainly in part due to close jointing, but in most cases the cleavage faded out at some little distance from the edge of the mass, and the inner portion of the slaty selvedge must therefore have arisen from strain without rupture (Heim's Umformung ohne Bruch). Such slaty selvedges thus furnish important evidence as to the manner in which slate is formed in nature, evidence entirely accordant with that afforded by experiment.

When the secondary pressure becomes more nearly perpendicular to the faces of the sheets of rock, these may themselves be divided into secondary sheets, and a continuance of the process will reduce the rock to a confused rubble.

Effect of tensile Stresses.—Jointing has been referred to tensile stresses by several authors. It is therefore desirable to examine what effects tensile stress can have upon homogeneous substances. To give abstract

*See Am. Journ. Sci., vol. xxxi, 1886, page 115.

†Geology of the Comstock Lode, U. S. Geol. Surv., monograph iii, chapter 4.

ideas a concrete form, suppose that a hot cube of homogeneous matter were to be cooled from one side only, and that the cooled surface underwent contraction. This contraction would produce tension throughout the cooling surface, excepting at the edges, so that the surface would assume the form of a very shallow dish, as illustrated in the following diagram:

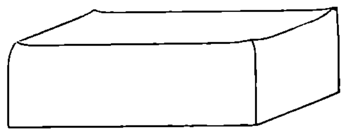

FIGURE 12.—*Contraction of Mass cooling from one Side.*

Since the tension would be zero at the edge, it would clearly be greatest at the center, and here rupture would take place in the surface film when the tension reached the limiting value. The tension at the center might be relieved by cracks of various characters. A single straight crack would relieve it only in one direction, and would, in fact, tend to increase tension in the direction of the crack, because the crack must gape, and its edges would therefore slightly exceed its median line in length. This form of rupture is therefore impossible under symmetrical conditions. The same objection applies to two cracks forming a letter T. Complete relief at the center would be afforded either by an X-shaped crack or by one in the form of a Y. Of these, the latter has the smaller total length for a given intersected area. Now, the cracks will clearly form in such a manner as to afford the greatest amount of relief per unit length of crack, and hence the rupture will take the shape of the letter Y. This will afford total relief at the center and partial relief at surrounding points. This relief, under symmetrical conditions, must be equally distributed, and therefore the three cracks must make with one another angles of 120°.

This simple inference is confirmed by observation. Thus, if one allows a vessel containing melted wax to cool slowly, an excessively thin transparent film forms on the surface. Then a minute opaque Y becomes visible near the center. This is due to the cracking of the film and the great acceleration of the process of solidification on the sharp exposed edges. So, too, a slight blow on glass often produces cracks in the same shape. Cracks in asphalt pavements frequently show a tendency to the same form; so do those in drying mud, and many of the divisional surfaces in columnar lava meet one another at angles approaching 120°.

As the cooling progresses the cracks must extend from the center; but the longer they are the less is the relief which they afford in the circle circumscribing their extremities and, unless the cooling area is small, other cracks must form. Either, then, rupture will take place at new centers or the original cracks will branch at their ends. A greater amount of relief for a given length of crack will be afforded by the latter method. The branches will be thrown off at angles of 120° for the same reason that the first cracks formed this angle. As the process continues the branching will be repeated, and it is evident that the surface will be divided into regular hexagons. The more slowly the cooling progresses the smaller will be the tension in the exposed surface and the larger will the hexagons be.

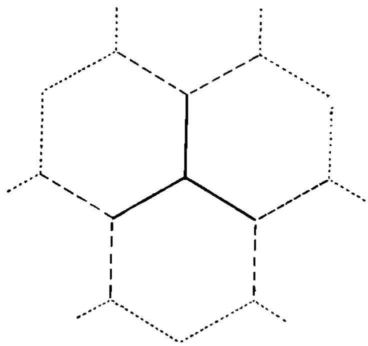

FIGURE 13.—*Primary tension Cracks.*

In some cases tension may accumulate in one of the hexagons after division to such an extent that a secondary rupturing takes place. This too will begin by three radiating cracks at the center, and these must be perpendicular to the greatest tensions or to the lines joining opposite angles of the hexagon. They will thus divide the hexagon into inequilateral pentagons, and the secondary fissures will be at right angles to the sides of the hexagon.

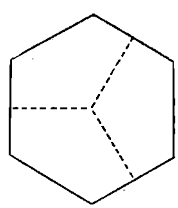

FIGURE 14.—*Secondary tension Cracks.*

The tensions are due to stresses acting at the isothermal surfaces and tangential to them. Hence, if the mass cools faster at one side than at the other, the resulting columns will be curved and will everywhere be perpendicular to the isotherms for which the tensions reach the limit of cohesion.

Each of the columns cools as a separate body, and if the following figure represents a vertical section of one of them, the dotted lines approximately represent the position of the isotherms. In the separate column the tension will be greatest at the edges, because these, exposing a great surface per unit volume, will chill most rapidly. When the column has reached a sufficient independent length, the tension on the edges will be so great that they will rupture perpendicularly to the isotherms. These ruptures will cut inward from the sharp edges and divide the columns laterally by cup-shaped surfaces. The interval between these vertical subdivisions of the column depends on the amount of tension which the mass can stand without rupture, and will evidently be of the same order as the diameters of the columns themselves.

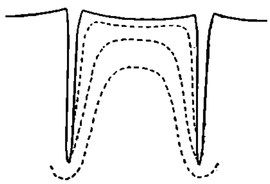

FIGURE 15.—*Cooling of Columns.*

The foregoing deductions all correspond very closely to the phenomena of columnar structure in massive rocks. Basalts, diabases, and the like, however, are far from being homogeneous, and it is surprising that the surfaces of the columns should be so smooth. If one were to cut a suitable bar of diabase and break it by tension in a testing machine the fracture would certainly be much rougher than the side of a diabase column. This seems to be accounted for, at least in part, by the fact that rupture probably takes place immediately after solidification of the groundmass and before any difference in rate of cooling between the embedded crystals and the groundmass has locally weakened the cohesion of the latter. When masses of mud in drying out split into columnar fragments, the torn surfaces are less smooth and the divisions less regular.*

REVIEW OF THEORIES OF SLATY CLEAVAGE.

Why Needful.—So little attention has been paid by geologists to systems of faulted fissures that the field may be said to be a new one. I know

* There is an intimate connection between the problem of columnar structure and that of the division of space with minimum partitional area. See an investigation of the latter subject by Sir William Thomson (Lord Kelvin), Mittag-Lefflers Acta Math, vol. 11, 1887–'88, p. 121, and Plateau, Statique des Liquids, vol. 1.

of no serious attempt to deal with the mechanics of the subject prior to a paper already cited, in which I discussed those cases of rupture in which the deformation could properly be regarded as infinitesimal. The results there reached have been born out by further observation. In this paper the investigation has been extended to cases of faulted fissures in which deformation of the rock is finite, and the conclusions certainly accord with very numerous observations which I have made in the Sierra Nevada of California and elsewhere, nor are any facts known to me which seem in conflict with the theory presented.

On the other hand, jointing and slaty cleavage have been much discussed. Some authorities regard them as closely allied, while others refer them to radically different causes. Many experiments have been made on slaty cleavage and various theories have been propounded to account for it. The theory here advanced is new, and I may say that it is a surprise to myself. I have long felt that the theory which refers slaty structure to a pressure perpendicular to a fixed plane of resistance and parallel to two lateral constraining planes was unsatisfactory. The combination seems too artificial. The chances against its occurrence seem too large when the frequency of slaty structure is considered. I did not anticipate, however, that analysis would show so large a range of conditions under which slaty structure might result, and I entertained the idea that if a slanting force produced slaty cleavage the force would slant in the direction of the grain of the slate. I have been led by purely algebraical reasoning to believe that the force may be inclined to the fixed plane within very wide limits, covering at least 60°, and that in all cases the plane of the force is at right angles to the grain of the slate.

Under these circumstances it is absolutely necessary to pass in review the principal theories hitherto advanced and to compare the new theory with the results of experiment and observation.

Origin of Jointing.—Joints are commonly nearly plane surfaces dividing rock masses and arranged in groups, the members of which are parallel to one another. Fault fissures are also frequently arranged in similar groups, and show similarly flat surfaces, the term joint being employed when the amount of relative motion on the divisional surfaces is imperceptible or is regarded as negligible. Jointing has been referred to tensile stresses by distinguished authorities, including W. Hopkins, but the correctness of this reference has been questioned. Thus, Professor W. King,* after special investigation, stated his opinion that, in their original condition, the walls of joints were in close contact, and protested against the classification of columnar structure with jointing. Mr G. K. Gil-

* Trans. R. Irish Acad., vol. 25, 1875, p. 605.

bert* also draws a contrast between the divisions of a mass due to shrinkage cracks and those known as jointing.

There can be no question that jointing is due to mechanical causes; for joint planes cut through conglomerates with almost the same regularity that they divide the most homogeneous rocks. They must, therefore, be due either to tensile stresses or to compressive stresses. In the former case the joints must gape when first formed. I am fully satisfied that Professor King was correct in asserting that the surfaces are usually in close contact immediately after rupture. In very many cases dislocation has taken place on jointed surfaces and, where this has occurred, any irregularity in the surface will force the walls asunder.

It has been shown in the preceding pages that the columnar structure of lava is easily accounted for, but I know of no way in which such a system of divisions as occurs in jointing can be accounted for by tension.†

On the other hand, it is not difficult to show that most of the phenomena of joints are fully accounted for by pressure, direct or inclined; but to this statement there is one exception. If joints are produced by pressure, they are due to a tendency of the rock to move in opposite directions along the joint plane; or, in other words, to a tendency to faulting. Hence, if pressure is the cause, there is no distinction, excepting one of degree, between joints and faults.

There is no doubt whatever that faults are often met with on joint planes, yet this association is no proof that the two phenomena are not, as they have often been assumed to be, independent of one another. But the study of many thousand divisional planes which would certainly be regarded as joints by almost every geologist has led me to the conclusion that the jointing and the faulting are concomitant. The faults are often extremely small, but it is very rarely that in a system of joint planes throws of an eighth of an inch or less cannot be detected; and where the rock is hard, slickensided surfaces will be found even when the relative movement is much less than an eighth of an inch. Few geologists have been in the habit of looking for faults of such tiny dimensions, and I believe that the distinction between faults and joints has arisen from this omission. Partings due to tension would be free from slickensides.

It might be thought that a refutation of this conclusion is to be found

*Am. Jour. Sci., vol. 24, 1882, p. 50.

†Divisional surfaces produced by pressure differ from those produced by tension in a manner which is distinguished even in ordinary parlance. Rupture under tension is only another name for *tearing*, while division under pressure always involves as an essential feature that sort of *cutting* which is performed with scissors or shears. Thus the very similes employed in describing rock fractures often indicate an instinctive perception of the mechanical processes involved, even when an attempt is made to reconcile phenomena with a less natural theory.

in the fact that joints frequently die out; but faulted fissures, even those carrying important ore deposits or considerable dikes, also die out. Nevertheless, the dying out of joints shows that the movements involved must at some points be microscopic, and indeed submicroscopic.

M Daubrée has succeeded in reproducing jointed structure in the most striking manner by pressure on mixtures of beeswax and resin. The following cut is copied from his experimental geology and explains itself:

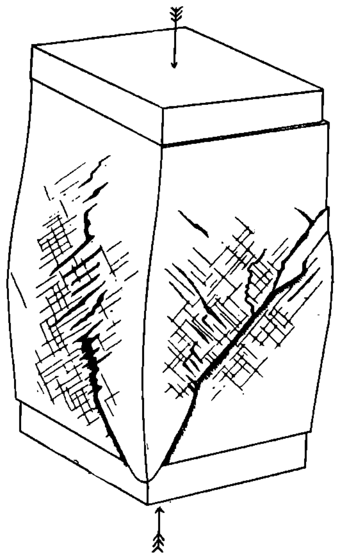

FIGURE 16.—*Daubrée's Experiment on Crushing.*

Here, as in nature, there are joints which die out, but they are associated with faults of measurable throw. The system of divisions is precisely that deduced for a direct pressure in the earlier portions of this paper from the theory of strain.

Still another lesson can be learned from this experiment. The sides of the crushed column bulge in such a manner as to show that plastic deformation has taken place as well as rupture. Now since these ruptures can be conceived only as relative tangential movements pushed to the limit of cohesion, it seems to me clear that the plastic deformation also must consist in relative tangential movement, and, indeed, in the same directions as the joints, but not reaching the limit of cohesion. If one inquires what is the effect of this plastic movement on the structure of the mass, one can only reply that it must be something very analogous to schistosity.

Others have made experiments with similar results. It is well known that cast iron, building stone and similar substances, crushed in testing machines, do not yield on planes parallel to the support, but at angles approximating to 45°, when the slabs are broader than they are thick. Not all of the cracks pass through the masses experimented upon. The slabs are somewhat deformed, and, in short, the phenomena are strictly comparable with those of M Daubrée's experiment, though less brilliantly illustrated.

Jointing and Cleavage.—Many geologists have been struck by the intimate manner in which jointing and cleavage (whether schistose or slaty) are associated, and there seems to have been a growing tendency to assert or imply a relationship between them, even in spite of an assumed theoretical difference in origin. Professor William King advanced the hypothesis in 1857 that slaty cleavage was derived from jointing, the jointed surfaces having been welded under pressure. This conclusion, indeed, has not, to my knowledge, been adopted by any other observer; but rejection of the conclusion does not imply rejection of the facts upon which it was based—viz., that dislocated jointing occurs "developed to a degree of fineness bordering on that of mineral cleavage," as at Carragrian, near Galway, and the occasional alternation of joints with parallel slaty cleavage.* Professor A. Heim distinguishes cleavage due to microscopic dislocated joints from cleavages unattended by joints, or true slaty cleavages. Of these he makes two classes: "micro-cleavage," consisting in a flattening of the component grains of the rock, and cleavage due to the re-arrangement by pressure of previously existing scales in positions more and more nearly perpendicular to the line of pressure. All three varieties are associated so intimately, according to Heim, as to be found in one and the same thin section. Even in cases of pure micro-cleavage relative movement without fracture in adjoining cleavage planes may be detected.† M Daubrée speaks of "the surfaces of slipping which produce schistosity;"‡ Dr H. C. Sorby has described cases of discontinuity on a microscopic scale which lead to cleavage; Messrs Geikie, Peach and Horne describe fluxion structure and shearing as productive of schistosity and highly cleaved rocks, the planes of cleavage being parallel to the thrust plane,§ and other similar observations could be cited to show that relative tangential motion and slaty cleavage are at least most intimately associated in nature.

Phenomena of slaty Cleavage.—Workers in slate distinguish not only the cleavage faces, but also "side" and "end." Most slates can be split only

* Trans. R. Irish Acad., vol. 25, 1875, p. 612, *et passim.*

† Mech. der Gebirgsbildung, vol. 2, 1878, pp. 54–56.

‡ Études Synthétiques, 1879, p. 321.

§ Nature, vol. 31, 1884, pp. 29–35.

in one direction, which appears to be usually that of the dip of the slate in its original position. A block of slate thus bears some analogy to a block of wood, so far as its fissility is concerned. In some cases, however, slates are said to split equally well from any edge of a block. Fossils occurring in slate are usually distorted, and numerous measurements of these fossils have been made for the purpose of ascertaining in which direction the greatest elongation and contraction have taken place. As might be expected, these measurements do not accord very closely, for it is difficult to expose a fossil in such a manner that all its dimensions are accessible without obscuring the relations of these dimensions to the dip and strike of the slate. Sometimes there seems to be no relative distortion in the plane of the cleavage. In other cases the fossils are greatly distorted in the cleavage plane, the longer axis coinciding with the dip.

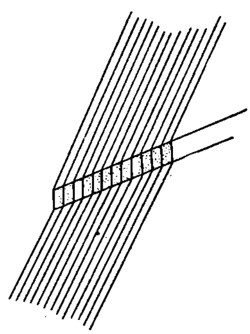

FIGURE 17.—*Steps in Slate.*

It is frequently asserted that the greatest elongation of the fossils is always in the direction of the grain of the slate, and the greatest contraction perpendicular to this direction. This implies that there has been no tangential movement among the laminæ, or that there is no fluxion structure and no close jointing or "Ausweichungsclivage" in the rock; for in any such case the axes of the strain ellipsoid must fail to coincide with the dip, the strike, and the perpendicular to the cleavage. Now these structures are known to be frequent in slaty rocks and distinguishable from true, slaty cleavage only under the microscope. The deductions from the measurements of the fossils can therefore be only approximately true. I have myself seen fossils in slate in which fluxion structure was plainly manifested, in my opinion, and in that of an eminent paleontologist whom I consulted. Slaty developments of crystalline rocks are by no means unknown, and these are closely allied to schistose rocks, in which crystals have certainly undergone distortions involving fluxion phenomena.

In slate quarries there are usually "steps" produced by the presence in the slate of strata differing in lithological character from the bulk of the rock. The cleavage is deflected by these strata, and when they are sharply defined the deflection is also sharp. When the cleavage is at right angles to the stratification the deflection is nearly or quite imperceptible, and it seems to be maximum when the angle approaches 45°. The illustration on the opposite page is taken from Mr Alfred Harker's admirable memoir on slaty cleavage.*

All of the above phenomena must of course be accounted for in any satisfactory theory of slaty cleavage.

Theories of slaty Cleavage.—The earlier geologists naturally associated slaty cleavage and mineral cleavage, and ascribed both to the same or similar causes. Professor John Phillips was the first to offer a mechanical explanation.† In doing so he was prudently indefinite. He described the distortion as a "creeping movement among the particles of the rock, the effect of which was to roll them forward in a direction always uniform over the same tract of country." This language has been interpreted as equivalent to the hypothesis of a simple "shearing motion," but it will by no means bear this limited construction. Phillips had in mind a rotational strain and a fluxional structure, but his paper contains nothing to indicate the absence of forces acting perpendicularly to the cleavage planes. He neither denies nor asserts the coöperation of such forces. He also says nothing to indicate that his theory was applicable only to heterogeneous matter, and it is fair to conclude that he supposed that slate might be produced from homogeneous substances.

Mr D. Sharpe explained the structure as due to the contraction of rock in the line of pressure and a partially compensating elongation at right angles to it. This strain is one of two dimensions, and consists of a simple shear (not a shearing motion) with a cubical compression. The fissility produced he referred to the fact that a fracture perpendicular to the direction of pressure would run along the flattest faces of the component grains and meet the smallest number of them. This explanation implies that the mass is heterogeneous, and that the adhesion between the component particles is smaller than the cohesion within the particles.‡

Dr H. C. Sorby, to whom geology owes so great a debt for the introduction of the microscope as an instrument of lithological research, natur-

* Brit. Assoc. Ad. Sci., 1885, p. 813. Mr Harker's paper contains very full citations of the literature of slate, and the reader who cares to pursue the subject is advised to consult it. No attempt is made in the present paper to give a full bibliography.

† Brit. Assoc. Rep., 1843, p. 61.

‡ Q. Jour Geol. Soc., vol. 5, 1849, p. 128.

ally attacked the question from a microscopical standpoint. He found that the mica of the slates was largely concordant with the cleavage and referred the fissility to the effect of direct pressure in deflecting mica scales toward a direction at right angles to the line of pressure.*

This theory is supplementary to that of Mr Sharpe, and it is to the united effect of the flattening and deflection that slaty cleavage is now usually ascribed.

Mr A. Laugel assumed on the authority of Sharpe that the strain consists of a simple shear. He pointed out the fact that in a simple shear in a homogeneous mass the planes of least resistance (or, as I have called them, of maximum tangential strain) stand at an angle with the axes of the shear dependent upon the deformation. In the notation of this paper † he reached the result $tan^2 \varpi = B/A$. He gave no proof of this, however, and did not explain how the double cleavage implied in this equation of the second degree could be reduced to the simple cleavage of slate.‡ In my opinion he was on the right path to a sufficient explanation, but he certainly did not achieve it.

Professor John Tyndall's famous experiments on slaty cleavage in wax in a direction perpendicular to the pressure were published in 1856.§ He dissented from Sorby's theory, regarding his wax as homogeneous, and finding that the intermixture of scales rather interfered with than promoted cleavage. Dr Sorby replied to Tyndall, citing experiments of his own on clay mixed with mica scales and pointing out that wax contains prismatic crystals; so that, in his opinion, the wax must be considered as composed of elongated elements capable of re-arrangement by pressure, according to his theory.||

Mr Daubrée found that clay without mica scales when extruded through a small opening assumes a schistose structure, the lamination being close in proportion as the material is more finely divided.¶ He also obtained evidence of schistose structure in flint glass, softened by heat and forced through an opening. In this case at least there could be no question that the resultant structure was independent of heterogeneous particles.

Dr Sorby made an addition to his theory of slaty cleavage in 1880. In his original theory it was assumed that the mica before compression was distributed through the mass without any order. As a matter of fact, the mica scales in shale are, for the most part, parallel to the bedding.

* Ed. New Phil. Mag., vol. 55, 1853, p. 137.
† See formula (9), p. 33.
‡ Comptes Rendus, vol. 40, 1855, p. 978.
§ Phil. Mag., vol. 12, 1856, p. 37.
|| Phil. Mag., vol. 12, 1856, p. 127.
¶ Géol. Exp., 1879, p. 413.

In certain cases, however, he observed that the bedding was almost obliterated by the disturbances due to the pressure. The supplementary hypothesis is that the preliminary effect of pressure is to give the mica an irregular distribution, the final effect being to reärrange the mica scales in the planes of cleavage.*

Objections to the Hypothesis of Heterogeneity—In my opinion, there are the gravest objections to the hypothesis that slaty cleavage is due to the lack of homogeneity of a rock mass which has been subjected to the action of force. Neither Tyndall nor Daubrée found that the presence of scales promoted schistosity, but just the reverse. The wax employed by Tyndall may have consisted largely of prismatic bodies; but before pressing his wax he softened it, making these bodies, as well as their groundmass, very plastic. He also kneaded the mass, so that the component particles must have welded. Even if every one of the prisms had assumed a horizontal position, there is no reason to suppose that the cohesion between them and the groundmass of the wax was feebler than that between the different portions of any one prism, or that any schistosity, at all approaching slaty cleavage, would have resulted. Similar remarks apply to Daubrée's experiments on clay.

Dr Sorby's supplementary hypothesis is suggestive in the same connection. All geologists will grant that disturbances are sometimes such as nearly or quite to obliterate the bedding of shales, but none will assert that this is a condition of slaty cleavage. We all know that the bedding is often most distinctly preserved in masses of roofing slate, and that the lamination is not infrequently fairly regular. In such cases it seems to me impossible to contend that the mica scales originally concordant with the bedding have been stirred up in such a manner as to be distributed at all angles through the mass. Again, there are many somewhat indurated shales not affected by slaty cleavage in which there are countless mica scales, nearly all of them concordant with the bedding. If the distribution of mica scales constituted the fissility called slaty cleavage, such beds should split like slate in the planes of bedding. Such beds are sometimes fissile to a certain extent, but cases in which this fissility could be mistaken for slaty cleavage are very rare, if, indeed, any are known. When rocks split along their lamination at all like slate, geologists expect to find, and usually do find, that the rock possesses true slaty cleavage coinciding locally in direction with the planes of bedding, but superinduced upon and independent of bedding.

Similar objections apply to Mr Sharpe's theory of the flattening of the rock components. It affords no explanation of Professor Tyndall's experiments, and were it correct some fine-grained sandstones, at any rate,

* Q. Jour. Geol. Soc., vol. xxxvi, 1880, p. 73.

would cleave along the bedding exactly like slate, which does not accord with observation.*

It appears to me, therefore, that no theory of slaty cleavage will be satisfactory which does not apply to the case of homogeneous matter.

Analysis of Experiments.—Slaty cleavage has been produced artificially in several different ways. Plastic substances compressed between rigid masses exhibit such cleavage; so too do plastic masses extruded through small openings; poor qualities of iron or brass when drawn to wire often show thin splinters, indicating the presence of cleavage; metals, pastry and clay rolled out into sheets show similar fissility, and, as Professor E. Reyer has pointed out, the bruise produced on soft rocks by a slanting blow with a pick exhibits a like structure.†

These cases seem very different, but they must have common features, unless, indeed, slaty cleavage is due to essentially diverse causes. Most of the mechanical operations indicated are very complicated, but their common features may be reduced to simple terms by considering a very small cubical portion of the mass before distortion and inquiring how it is affected by strain.

FIGURE 18.—*Origin of Cleavage in Wire.*

If one end of a wire is filed to a flat surface perpendicular to its axis, and the wire is then drawn through two or three successive holes of a draw-plate so that the flat end is the last to come through, it will be found that this end has become concave. If one considers a small cube in the undistorted wire, not on the axis, it is clear that this cube will be converted into an oblique parallelopiped, as is illustrated in the foregoing diagram, showing the wire in section.

The concentric layers of the wire move upon one another much like the joints of a telescope. The little cube is elongated in the direction of the axis, its height is diminished, and its right angles in the plane of the axis are converted into acute or obtuse ones. It is clear that the sphere which might be inscribed in the small cube has been distorted to an ellipsoid, the major axis of which becomes more and more nearly horizontal as the strain increases. The strain is thus a rotational one, and, according to the theory of strain set forth in this paper, a cleavage should be developed nearly in the direction of the axis.

* See p. 74.

† Theoretische Geologie, 1888, p. 577.

If a bar were substituted for a wire, and slots for the circular openings of the draw-plate, the strain would be exactly equivalent to that produced by an inclined pressure acting on a rigidly supported cube. It cannot be doubted that in such a case the end of the bar would also become concave, and that evidences of schistosity would appear.

When a plastic mass is extruded through a small opening, whether circular or rectangular, the action is very similar to that involved in drawing wire, excepting that the external force is a pressure instead of a tension. The friction on the moulding surface delays the motion of the external layers relatively to the internal layers, and so-called fluxion structure results. In the following diagram it is plain that a cube of the plastic mass at *a* would become an oblique parallelopipedon at *b*.*

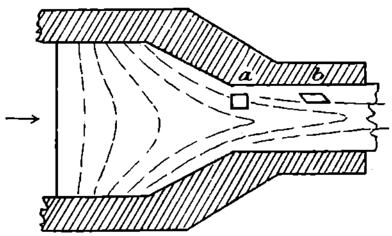

FIGURE 19.—*Development of Cleavage by Extrusion.*

When an oblique blow is struck with a pick the bruise will manifestly show a distortion of a very small cube similar to those already considered.

The case of a direct pressure, such as was employed by Professor Tyndall, seems at first sight very different from the foregoing. To convince myself as to the mechanics of the matter I repeated his experiments, with the following results: † A cake of wax can be compressed to less than half its thickness between glass plates well greased with a heavy oil without bulging of the edges, as shown in figure 20, *a*, *b*. If such cakes are cooled to — 15° C. they show no slaty cleavage, but exhibit a tendency to split at an angle of some 60°, more or less, to the line of pressure. If the plates are not greased, but only wet with water, as in Tyndall's experiments, there is a strong tendency to bulge along the

*M Daubrée, in his Géologie Expérimentale, records striking experiments on this mode of deformation.

† White wax is better than yellow for the purpose of this experiment. To get comparable masses I cast cylindrical cakes at as low a temperature as practicable. These were cooled off and then kept in water at about 40° C. for an hour or more. Below this temperature the wax is too brittle to mould with ease or rapidity. The compressed cakes were cooled in ice and salt. Cakes chilled without preliminary distortion show no cleavage under the hammer or chisel, and crack very like fine-grained basalt.

edges, so that the cake assumes the form of an ordinary American cheese. Cakes compressed to one-quarter of their thickness were very greatly distorted in this sense, as shown in figure 20, *c*. When cooled and struck sharply on the edge with a hammer, they showed slaty cleavage. The character of the distortion of a small included cube follows from the distortion of the mass, and, as appears from the following diagram, it is a distortion similar to that which takes place when a cube is subjected to inclined pressure, as illustrated in figure 6, B, page 56.

The reason for the bulging edges is at once seen to be the frictional resistance between the glass plates and the escaping wax. This resistance, combined with the vertical pressure, gives resultant forces, marked *r r* in the figure, which are not vertical but lie on conical surfaces about the central vertical axis. When this friction is obviated by the use of a lubricant, so that a nearly uniform distribution of pressure is obtained, there is no tendency to relative horizontal motion among the layers, and in a dozen or more trials with lubricators I failed to find any trace of horizontal cleavage. A tendency to cleave is sensible in these cases, but it coincides with the planes of maximum tangential strain as nearly as the imperfection of the surfaces enabled me to judge.

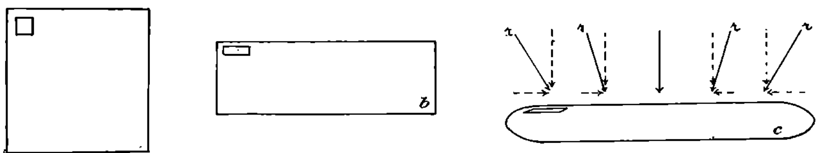

FIGURE 20.—*Development of Cleavage by direct Pressure.*

Thus it appears to me that Professor Tyndall's brilliant experiment has been misinterpreted. He produced slaty cleavage not by a pressure uniformly distributed and vertical to the cleavage planes, but by a system of forces inclined to the cleavage planes.

The effect of rolling metal, clay, or pastry is similar to that of direct pressure combined with lateral friction. A cake of plastic material is reduced to a sheet with bulging edges like figure 20, *c*, and an infinitesimal cubical portion of the mass is distorted as in the other cases.

I am aware of no other ways in which slaty cleavage has been produced artificially. In all of those discussed the distortion attending development of the cleavage is substantially the same. The elementary cube is deformed as it would be by a force inclined to one face of the cube when the opposite face rests upon an inflexible support. In some cases there is lateral constraint; in others there is none. The splinters on rolled metal and pastry seem to show that the cleavage developed is not quite parallel to the surface of the mass.

It might seem as if the varying directions of pressure detected in Tyndall's experiment were geologically unimportant. Granting that the vertical, uniform pressure at first applied to the wax is conically resolved, does it not follow that in orogenic movements also a similar resolution occurs; so that, after all, slaty cleavage is due to a pressure originally uniformly distributed and perpendicular to the cleavage? This query must receive a negative reply.

The reason why the pressure in the experiment is resolved into a conical system of forces is that the bodies between which the wax is squeezed do not themselves yield sensibly. Thus horizontal relative motion attended by friction is brought about. Were these bodies as soft as the wax, they too would extend laterally and the pressure would remain uniformly distributed. It would also produce no slaty cleavage.

In orogenic movements there is seldom any diversity between the resistance of adjoining rock masses approaching the difference between plates of glass and warm wax. Among rocks, therefore, a direct pressure will, as a rule, be distributed with an approach to uniformity, and there will be little or no relative motion between adjoining rock masses in directions perpendicular to the pressure. Hence, also, important masses of slate will not be produced in this way.

Perhaps no combination is entirely wanting in mechanical geology. In artificial cuttings, clay beds underlying harder materials have been known to be squeezed out laterally, and these masses must have been affected like the wax in Tyndall's experiment; but this case scarcely forms an important exception.

In most cases of the geological occurrence of slate there is little direct evidence of the mode of formation, and it is for this reason that the experiments are of so much value. Sometimes, however, the method of formation of natural slate is clear. I refer especially to the slaty selvedges which are not infrequently seen bounding small faults in granite and which have been mentioned under the head of secondary action on ruptured rock. No geologist can doubt that these selvedges are produced by the inclined pressure attending faulting, and it is manifest that the distortion of an elementary cube would be exactly that which so constantly accompanies the artificial production of slaty cleavage. Thus, in some cases at least, natural slate is produced by the same means which are employed in producing artificial slate.

Behavior of included Grit Beds and Fossils.—The theory that slate is produced by a uniformly distributed pressure perpendicular to the planes of cleavage, such as it has been usual to suppose existed in Tyndall's experiment, implies that the strain ellipsoid is an oblate figure of revolution. In such a slate a fossil which lay in the cleavage plane would simply be flattened. From the observed fact that fossils are frequently, though not always, relatively elongated to a sensible degree in one direction in the cleavage plane, Mr Sharpe inferred lateral confinement as well as vertical pressure.

On the theory of inclined pressure, a fossil would always be elongated in the direction of the grain of the slate and contracted across the grain in the cleavage plane, excepting when the pressure made no angle with the fixed plane. A still greater elongation, however, would take place in the direction of the major ellipsoid axis, called A in this paper, which is at right angles to the grain and makes a large angle with the cleavage plane. That such distortions do exist I have convinced myself by the examination of specimens, but I have not had an opportunity of examining any large collection of fossils from slates with reference to this point.

The relations of beds of hard grit occurring in slate bear a close relation to those of fossils. If such a bed were bounded by surfaces parallel to the plane $x\ y$ (or $A\ B$), the bed would behave either to a vertical or to an inclined pressure as an independent mass. On the currently accepted theory it would develop a horizontal cleavage. On the theory of inclined pressure it would develop a cleavage in a direction between that of the pressure and that of the fixed plane; and this would nearly coincide with the cleavage of the surrounding softer mass, because the direction of cleavage lies near that of constant direction and changes but little during strain. The smaller the angle which the force makes with the fixed support, the smaller would the divergence in the two cleavages be.

"Steps" are produced when a grit bed cuts the cleavage across the grain, the plane of the cleavage in the slate and the surface of the grit bed making an acute angle. The grit is a harder material than the slate, and the cleavage developed in the grit makes a larger angle with the bedding than it does in the slate.

To account for steps according to the theory of inclined pressure one may consider the elementary stresses separately. It has been shown that the shear in the plane $B\ C$ does not tend to produce relative motion on the cleavages. One may therefore suppose the stress, minus this shear, to be applied to the rock first, and this shear to come into action later. Figure 22, a represents a cube in the $y\ z$ plane, with a layer of harder material passing diagonally through it. If a shear and a shearing motion or scission in the $x\ y$ plane are impressed upon this mass, both portions must yield simultaneously, because if the force were insufficient to strain the harder layer, this would protect the surrounding mass from the action of the force. Hence these strains would produce in both masses a cleavage, the traces of which on the $y\ z$ plane would be parallel to $o\ z$, and the appearance of the mass would be that shown in figure 22, *b*, where the fine horizontal lines represent mere cleavage, not partings. Now let the final shear at right angles to the $x\,y$ plane be applied. It will elongate the mass in the direction $o\,z$ and contract it in the direction $o\,y$. But since the more rigid layer yields to this stress less freely than that in which it is imbedded, the grit bed will rotate more nearly as if it were a rigid mass, and will assume such a position as is shown in figure 22, *c*. In short, the hard layer will be deflected in just the same way that an imbedded scale of mica parallel to it would be deflected. Thus the cleavage in the hard layer will not be parallel to that in the adjoining mass and will form a larger angle to the bed planes.

It thus seems sufficient to suppose the grit bed to have a greater coëfficient of rigidity to account for the phenomena of steps.*

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*Dr Sorby's theory of this phenomenon, as stated by Mr Harker, is as follows: "Since the grit yields less than the slate to the compressive force, the total voluminal compression is greater for the slate than for the grit. But near the junction of the two rocks the change of dimensions in the direction parallel to the bedding must be the same for both. Consequently, in the direction perpendicular to the bedding, the slate undergoes a less expansion (or greater compression) than the grit; and the cleavage planes, which are in each rock perpendicular to the direction of greatest compression, will therefore be less inclined to the bedding in the slate than they are in the grit."

This is a very ingenious explanation, but I have not been able to convince myself that it is sound. It depends primarily upon the hypothesis that a large cubical condensation is involved in the production of slate. This certainly does not seem to be the case when slaty cleavage is produced in moist clay or wax, for such substances are probably compressible only to a very minute extent. It also implies that there is a very great difference between the cubical compressibility of the grit and the shale. I know of no good reason to suppose that such a difference exists. The difference in hardness does not imply such a relation, for cast iron, though so much harder than gold, is nearly three times as compressible; but even if it be granted that the relations of compressibility are those demanded, it is not clear that any means is provided of changing the direction of the force in the manner required by Sorby's theory of cleavage.

One may suppose a cubical portion of a rock mass to undergo the pressure needful to develop slaty cleavage without change of volume, and that cubical contraction takes place subsequently. If the mass contains a stratum of smaller compressibility than the remainder, the cleavage on the theory now under consideration would be perpendicular to the direction of the force throughout the mass before cubical contraction occurred. In this stage the mass would have the appearance of figure 22, *b*. The effect of the shrinkage would then be to deflect the slaty laminæ close to the contact in curves with points of inflection at the contact, but to leave the direction of the cleavage at a little distance from the contact unchanged. The appearance after cubical contraction would would then resemble that illustrated in the following diagram:

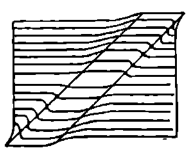

FIGURE 21.—*Effects of Compressibility.*

But this does not represent the phenomenon to be accounted for; so that although the hypothesis of varying cubical compression would explain a change of direction in the surfaces of cleavage at the contact with a gritty bed, it does not, so far as I can see, account for steps.

Conclusion as to Slate.—The fact that slaty structure occurs not only in argillaceous rocks but, though less frequently, in limestone, grit beds, granite and basic eruptives, while it has been artificially produced in wax, clay, metals, dough and glass, throws much doubt on the hypothesis that slaty cleavage is due to re-arrangement under pressure of embedded flakes and grains of matter. This doubt seems confirmed by the fact that although the component grains of many undisturbed shales and sandstones are so arranged that their largest sections lie parallel to the planes of bedding, such rocks do not show any cleavage closely resembling that of slate. Hence a satisfactory explanation must apply to homogeneous matter.

Examination of the experimental methods of producing slaty structure shows that in all cases the distortion of a small portion of the mass is rotational, and is such as would be produced upon a cube resting on a rigid support and affected by an inclined force, with or without the co-operation of lateral forces in the plane of support.

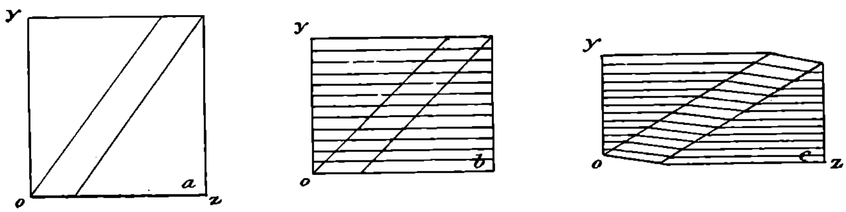

FIGURE 22.—*Deflection of Cleavage by Grit.*

The theory of finite strain in viscous plastic masses shows that rotational strains of this description should be accompanied by the development of a cleavage. The grain of a mass thus distorted should have an absolutely constant direction parallel to the plane of support and perpendicular to the line of force. Elongation should, in general, take place in the direction of the grain, and contraction at right angles to the grain on the cleavage plane. When, however, the force makes no angle with the plane of support there should be no distortion in the plane of cleavage. There should also in all cases be a second direction of elongation perpendicular to the grain and at a considerable angle to the cleavage plane.

This theory explains at least most of the characteristics of slate, including that of steps. The second elongation just mentioned certainly exists in some cases, but I have not data enough to assert its universality. The practical difficulties of fully determining the position of the strain ellipsoid from a fossil are such that the omission of other observers to note the existence of this elongation does not seem to me fatal to the theory. Many observers have obtained satisfactory evidence of elongation in the direction of the grain of the slate, while few, if any, of them appear to have sought for another direction of elongation not in the plane of cleavage.

The theory here advanced has the advantage of being based on some of the best-established facts of natural philosophy and of connecting cleavage in the most intimate and definite manner with schistosity, jointing, faulting, and systems of fissures. It also exhibits the cleavage of slate and the master joints, which usually intersect the cleavage planes at very large angles, as two features of a single strain.

Neither Hooke's law nor any other exterpolated generalization has been employed in reaching conclusions as to the origin of slaty structure. Poisson's hypothetical solid was assumed only in an example in order that the formulas might receive a numerical and geometrical illustration.

Summary.

The studies here presented are an outgrowth of field-work in the Sierra Nevada of California. That range is intersected by faults, joints, schistose and slaty cleavages to such an extent that, on a scale of one mile to the inch, their average separation would be for the most part microscopic. In many areas these dynamic manifestations are very systematic. Such of them as can be considered as concomitants of infinitesimal strain have been treated in a former paper. In a great proportion of cases, however, the strains have been finite. Only such areas are here considered as may be regarded as uniformly affected by finite strains.

In the first portion of the paper finite strain is considered from a purely kinematical standpoint. The subject is treated rather fully because, for the purpose in hand, it is needful to take an extended view of the possibilities. The most important topic is that of the planes of maximum tangential strain and the manner in which they range relatively to the material of a solid which is undergoing strain.

The relations of stress to strain are next sketched, the nature of a finite shear is elucidated, and Hooke's law is examined. Hooke's law is shown to differ from the statement that "stress is proportional to strain" when the deformations are finite. Viscosity, flow, plasticity, ductility and rupture are defined, and the relation of plastic solids to fluids is explained.

The conclusions reached are then applied to cases such as may arise in geology. Large masses of rocks, it is assumed, may be considered as homogeneous. Were it necessary to take into consideration the minute texture of rocks, any general conclusions as to their behavior under orogenic stress would be impracticable. Simple irrotational pressure is first taken up. It is shown that such a pressure will produce two sets of fissures crossing one another at angles approaching 90° if the rock is brittle. If it is plastic, two sets of schistose cleavages will replace the fissures. The line of force bisects the obtuse angles of the cracks or cleavages. Use is made of the theory of this case to prove in a very simple manner why mica scales and flat sand grains tend to arrange themselves parallel to the bedding of sedimentary rocks, and why flat pebbles in water-channels "shingle up stream."

A mass resting on a yielding foundation and subjected to an inclined force is briefly discussed. This case closely approaches that of the simple irrotational pressure. It seems to account for unsymmetrical schistosity.

The most interesting case is that of a mass resting upon a rigid foundation and affected by a force inclined to the foundation at any angle. It really includes the case of the simple irrotational pressure. If the mass is brittle and is strained so gradually as not to bring viscosity into play, the material will rupture in columns, the axes of which are parallel to the fixed plane of support and at right angles to the force. If the strain is so rapidly produced as to excite viscosity, only one set of fissures will form, and these will be intermediate in direction between the line of force and the projection of the force on the fixed plane. If the rock is plastic (or if it is kept strained between the elastic limit and the breaking point sufficiently long to undergo considerable deformation) the fissures intersecting the angle between the line of force and the fixed plane will be replaced to a greater or less extent by cleavage planes; and if the force does not approach the vertical to the fixed plane, these cleavage planes will preserve a nearly constant direction and have a slaty character. In this case the second set of planes of motion, if they receive expression at all, will cut sharply across the cleavage planes as master joints. This seems to be the only way in which slate-like structure can result from the action of force on homogeneous matter.

The spacing of fissures formed by inclined pressures is discussed on the hypothesis that they are so disposed as to lead to the greatest depotentialization of energy. This leads to an exceedingly simple formula for the thickness of a column in a direction perpendicular to either pair of bounding planes. The formation of a single system of parallel fissures and the existence of undistributed faults are shown to arise in particular cases of the formula. This formula is applicable only when the rupture is not brought about by a very rapid strain. When the strain is impulsive it is shown that the interference of vibrations attending rupture may cause further parallel ruptures. The suggestion is made that thick slates and flags may possibly be due to plastic deformation attended by vibrations.

As jointing has been referred to tensile stress, rupture through tension is discussed. It is shown that curved or broken lines, and not plane partings, must result; and the columnar structure of lavas receives a seemingly sufficient explanation.

The last portion of the paper is occupied by a review of the theories and observations on jointing and slaty cleavage. It is maintained that joints are always attended by macroscopic or microscopic faults, and that they are closely allied to slaty cleavage. The ascription of slaty structure to the presence of deflected mica scales and flattened particles is pronounced unsatisfactory. Glass, wax and other substances in which slaty cleavage has been artificially produced can hardly owe their cleavage to such a distribution of flat particles, while sedimentary rocks in which the flat particles are mostly parallel to the bedding do not show slaty cleavage.

Analysis of certain well-known experiments and of some made for this paper shows that artificial slaty cleavage is always attended by rotational strains, such as those to which slaty cleavage is ascribed above. The theory of this paper (that slate is due to pressures inclined at small angles to the cleavage plane and standing at right angles to the grain of the slate) is shown to account for grain, "side" and "end," for elongation of fossils in the direction of the grain, contraction in the cleavage plane at right angles to the grain, and for master joints which intersect the cleavage plane along the grain and make a large angle with this plane.

The most important result of the investigation is that jointing, schistosity and slaty cleavage all imply relative movement, and are thus as truly orogenic as faults of notable throw. They may all be regarded as orogenically equivalent to distributed faults. The great number of joints and planes of slaty cleavage compensates for the minute movement on each, and the sum of their effects is probably at least as important as that of the less numerous faults of sensible throw.

In the light of this conclusion it appears that if one could reproduce the orogenic history of the Sierra in a moderate interval of time on a model made to a scale of one mile to the inch, it would seem to yield to external and bodily forces much like a mass of lard of the same dimensions.*

* I desire to express my thanks to Professor R. S. Woodward, of the Coast and Geodetic Survey, for his kindness in reading this paper in manuscript and for giving me the benefit of his advice. This is not the first time I have had the advantage of Professor Woodward's profound knowledge of physics and keen scientific judgment.

WASHINGTON, D. C., *July 1, 1892.*